SMR.1557- 1

SPRING SCHOOL ON SUPERSTRING THEORY
AND RELATED TOPICS

15 - 23 March 2004

Topological strings and integrable hierarchies

PART I

Marcos MARINO
CERN
TH Division
CH-1211 Geneva 23
SWITZERLAND

Please note: These are preliminary notes intended for internal distribution only.

TOPOLOGICAL STRINGS & INTEGRABLE HIERARCHIES

1. Introduction
2. Quick review of TFT in 2d
3. Integrability & topological strings in $d<1$
4. B-model & special geometry on CY 3-folds
5. Local B-model & asymptotic deformations of geometry: classical theory
6. Quantization of local B-model and integrability.
7. B-branes & topological strings in $d<1$ revisited.

Integrable systems → origin in <u>classical mechanics</u>

* a dynamical system is integrable if there are enough conserved quantities so as to solve the problem by quadratures

QFT / context: Topological string

$F(t_i)$ generating functional of correlation functions

t_i play the role of "evolution times"

$F(t_i)$ governed by the set of equations of the hierarchy

interesting points:

* encode deep symmetries of the model
* allow to solve explicitly for generating functions

unified framework for looking at these models from the integrable point of view

$\left\{ \begin{array}{l} \text{topological strings in } d\leq 1 \\ \text{" " on noncompact CY's} \end{array} \right.$ →

* allow sometimes to incorporate quantum/stringy effects → <u>exact results on g_s</u>
* philosophical attraction

→ solubility

→ ability to go to all orders in g_s

③ TFT in 2d: general aspects

characterized by a fermionic BRST charge Q

nilpotent $Q^2=0$

and $T_{\mu\nu} = \{Q, b_{\mu\nu}\}$ → guarantees, at least formally, metric independence of partition function & correlators of Q-invariant operators

of course $[Q, S]=0$

$\mathcal{O}$ topological observable $[Q, \mathcal{O}]=0$ and $\mathcal{O} \neq Q\,\Psi$

i.e. we study the cohomology of Q to find the relevant operators

This structure allows to consider families of TFT's:

we have $\{Q, G_\mu\} = P_\mu$ (*)

start with $[Q, \phi^{(0)}]=0$ and construct

$$\phi_\mu^{(1)} = [G_\mu, \phi^{(0)}]$$

$$\phi^{(2)}_{\mu_1\mu_2} = \{G_{\mu_1}, [G_{\mu_2}, \phi^{(0)}]\}$$

define the n-form $\phi^{(n)} = \frac{1}{n!}\phi_{\mu_1\ldots\mu_n} dx^{\mu_1}\wedge\ldots\wedge dx^{\mu_n}$

then (*) implies $d\phi^{(n)} = [Q, \phi^{(n+1)}]$

and we can define $W_\phi^\gamma = \int_\gamma \phi^{(1)}$, $W_\phi^\Sigma = \int_\Sigma \phi^{(2)}$

Q-invariant

therefore, if we consider $S \to S + \sum_n t_n \int_\Sigma \phi_n^{(2)}$
$= S(t)$

in general there is a finite set of operators ϕ_n such that $[Q, \phi_n] = 0$. These operators form a ring, since $\phi_i \phi_j$ is also Q-closed. Therefore,

$$\phi_i \phi_j = \sum C_{ij}^{k} \phi_k$$

C_{ij}^{k} structure constants of the ring

$\boxed{\text{analogy: cohomology ring of a manifold}}$

We also define a "metric" by:

2-point function $\langle \phi_i \phi_j \rangle_0 = \eta_{ij}$ on the sphere

so the 3-point function on the sphere is by:

$$\langle \phi_i \phi_j \phi_k \rangle_0 = \sum_k C_{ij}^{\ell} \underbrace{\langle \phi_\ell \phi_n \rangle}_{\eta_{\ell n}} \equiv C_{ijk}$$

We define the perturbed 3-point function as:

$$C_{ijk}(t) = \langle \phi_i \phi_j \phi_k \, e^{\sum t_n \int_{\Sigma_0} \phi_n^{(2)}} \rangle$$

this is one of the quantities we are interested on & we will see how to compute it in some cases we get a perturbed TFT, ϕ_n a set of observables

How do we construct TFT in 2d?

Twist of $N=2$ SUSY (Witten) of 2d Euclidean QFT

SUSY algebra: $\underbrace{Q_\alpha, \bar{Q}_\alpha}_{\text{doublet } SO(2)_R}$ $\quad \alpha = +,-$ fermion chirality

Superspace: $\theta^\pm, \bar{\theta}^\pm$, $F(x, \theta^\pm, \bar{\theta}^\pm)$ superfields

<u>dimensional reduction of $N=1$ in 4d</u>

U(1) symmetries:

$$U(1)_L \qquad\qquad U(1)_R$$
$$\left.\begin{array}{l} \theta^+ \to e^{-i\alpha}\theta^+ \\ \bar{\theta}^+ \to e^{i\alpha}\bar{\theta}^+ \end{array}\right\} \quad \begin{array}{l} \theta^- \to e^{-i\alpha}\theta^- \\ \bar{\theta}^- \to e^{i\alpha}\bar{\theta}^- \end{array}$$

so we have two currents:

$J_V = \frac{1}{2}(J_L + J_R)$ vectorial, anomaly-free

$J_A = \frac{1}{2}(J_L - J_R)$ axial, may be anomalous

<u>twist</u>: the idea is to obtain scalar supercharges out of the fermionic supercharges $Q_\pm, \bar{Q}_\pm$ by redefining the Lorentz generators

$$SO(2)_E \longrightarrow SO(2)'_E$$

$$\qquad\qquad\qquad\qquad \parallel$$

A twist $\longrightarrow$ $(SO(2)_E \times SO(2)_V)$diag

B twist <u>or</u> $(SO(2)_E \times SO(2)_A)$diag

consistency of B-twist requires J_A to be anomaly-free

A-twist: Q_-, $\bar{Q}_+$ are scalars wrt $SO(2)_E'$

$$Q_A = Q_- + \bar{Q}_+ \quad \text{and} \quad Q_A^2 = 0$$ (in the absence of central charge)

B-twist: $\bar{Q}_+$, $\bar{Q}_-$ scalars

$$Q_B = \bar{Q}_+ + \bar{Q}_-$$

also, in the A-twist we can construct descent operators

$$G_z = G_1 - iG_2 = Q_+$$

$$G_{\bar{z}} = G_1 + iG_2 = \bar{Q}_-$$

while in the B-twist $G_z = Q_+$, $G_{\bar{z}} = Q_-$

Exercise: verify nilpotency of $Q_{A,B}$ and $\{Q, G_\mu\} = P_\mu$ by using the susy algebra

hint / ref: Labastida / Llatas hep-th/9112051

A particular case of TFT in 2d occurs when the theory is conformal, so we start with an $N=2$ SCFT with algebra:

	L	R
	$T, J_L, G^\pm$	$\bar{T}, \bar{J}_R, \bar{G}^\pm$
charges NS:	$Q_+ = G^-_{-1/2}$	$\bar{Q}_- = \bar{G}^-_{-1/2}$
	$\bar{Q}_+ = G^+_{-1/2}$	$Q_- = \bar{G}^+_{-1/2}$

in these theories are defines four rings

chiral states $G^+_{-1/2}|\phi\rangle = 0$ (c, c)

(a, a)

antichiral " $G^-_{-1/2}|\phi\rangle = 0$ (a, c)

(c, a)

In this context, the twisting involves the redefinition:

$$T(z) \to T'(z) = T(z) + \frac{1}{2}\partial J$$

$$h \to h - \frac{1}{2}q = h'$$

$$G^\pm \begin{array}{l} \nearrow Q(z) \quad h' = 1 \\ \searrow G(z) \quad h' = 2 \end{array}$$

The A & the B twist are here

A: $T \to T + \frac{1}{2}\partial J$ B: $T \to T + \frac{1}{2}\partial J$

$\overline{T} \to \overline{T} + \frac{1}{2}\overline{\partial}\overline{J}$ $\overline{T} \to \overline{T} - \frac{1}{2}\overline{\partial}\overline{J}$

Since for chiral primaries $h = q/2$ we see that.

A model: observables are (c, c) $(h', \bar{h}') = (0, 0)$

and correspond to <u>marginal</u> perturbations

B model: observables are (c, a)

We will focus on the <u>B model</u> in these lectures

For TCFT's the existence of a conformal symmetry previous to twisting gives further conditions on $C_{ijk}(t)$:

$$\frac{\partial C_{ijk}}{\partial t_l} = \frac{\partial C_{ijl}}{\partial t_k}$$

consequences:

1) locally $C_{ijk} = \dfrac{\partial^3 F_0(t)}{\partial t_i \partial t_j \partial t_k}$ $\quad F_0(t)$ perturbed free energy at $g=0$

2) $\dfrac{\partial C_{ijk}}{\partial t_0} = 0$ but $= \dfrac{\partial C_{ij0}}{\partial t_k} = \dfrac{\partial \eta_{ij}}{\partial t_k}$

t_0 coupling to the identity operator

$\Rightarrow$ η_{ij} independent of t_k

$$F_0(t) = \langle \exp \sum_i t_i \int_{\Sigma_0} \phi^{(2)}$$

Examples of TFT in 2d

X_i chiral field

$$S = \int d^2z\, d^4\theta\, \underbrace{K(X_i, \bar{X}_i)}_{\text{Kähler potential}} + \Big[\int d^2z\, d^2\theta\, \underbrace{W(X_i)}_{\text{superpotential}} + \text{h.c.} \Big]$$

notice that $d^2\theta = d\theta^+ d\theta^-$ is not Lorentz invariant under the A-twist, but is for the B-twist

$\Rightarrow$ superpotential terms are not allowed in A-model
yes in B-model

We will consider two special examples (for simplicity):

* $W = 0$, $K(X_i, \bar{X}_i)$ Kähler potential of a CY 3-fold
  topological σ-model of B-type

* $K(X_i, \bar{X}_i) = \sum_{i=1}^{n} X_i \bar{X}_i$, $W(X_i)$ quasihomogeneous polynomial
  LG model

actually we will mostly restrict to $n=1$, a single field Φ

homogeneous $\rightarrow$ guarantees the existence of classical symmetries $U(1)_{L,R}$:

$$W(\lambda^{\omega} \Phi) = \lambda W(\Phi) \qquad \text{then} \quad q_L(\Phi) = q_R(\Phi) = \omega$$

$U(1)_{L,R}$ anomaly-free in the untwisted theory.

in general: $W(\lambda^{\omega_i} \Phi_i) = \lambda W(\Phi_i) \qquad q_L(\Phi_i) = q_R(\Phi_i) = \omega_i$

Let us write up the action and Q-transformations

$$S = \int_\Sigma d^2z \sqrt{g} \left\{ G_{I\bar J} \left(\partial_z \phi^I \partial_{\bar z} \phi^{\bar J} + \partial_{\bar z} \phi^I \partial_z \phi^{\bar J} \right) \right.$$
$$- \rho^I_z \left(G_{I\bar J} D_{\bar z} \eta^{\bar J} + D_{\bar z} \theta_I \right) - \rho^I_{\bar z} \left(G_{I\bar J} D_z \eta^{\bar J} - D_z \theta_I \right)$$
$$- R^I{}_{\bar J L K} \eta^L \rho^J_z \rho^K_{\bar z} \theta_I + G^{I\bar J} \partial_I W \partial_{\bar J} W + (D_{\bar J} \partial_{\bar J} \overline{W})$$
$$+ \dots$$

Q-transformations:

$$\boxed{\begin{array}{lll} [Q, \phi^I] = 0 & [Q, \rho^I_z] = 2\partial_z \phi^I & [Q, \rho^I_{\bar z}] = 2\partial_{\bar z}\phi^I \\ {[Q, \phi^{\bar I}]} = \eta^{\bar I} & \{Q, \theta_I\} = \partial_I W & \end{array}}$$

where

$$-\eta^{\bar I} = \bar\psi^{\bar I}_- + \bar\psi^{\bar I}_+ \quad , \quad \theta_J G^{\bar I J} = \bar\psi^{\bar I}_- - \bar\psi^{\bar I}_+$$

$$\rho^I_z = \psi^I_- , \quad \rho^I_{\bar z} = \psi^I_+$$

and the chiral fields are expanded as

$$\Phi^I = \phi^I + \theta^\pm \psi^I_\pm + \dots \qquad \text{chiral}$$
$$\bar\Phi^{\bar I} = \phi^{\bar I} + \bar\theta^\pm \bar\psi^{\bar I}_\pm + \dots \qquad \text{antichiral}$$

Let us first consider a LG theory

W polynomial in ϕ_i

Vafa-Warner-Martinec:

they represent universality class of CFT with $N=2$

A flow in the IR to an $N=2$ SCFT with same W

after twisting, we have an anomaly in the $U(1)_{L,R}$ currents:

$U(1)_L$: $\theta^+ \to e^{-i\alpha}\theta_+$, $\theta^- \to \theta^-$

$$\Rightarrow \psi_+ \to e^{i(\omega-1)\alpha}\psi_+ , \quad \psi_- \to e^{i\omega\alpha}\psi_-$$

$$\text{anomaly} = (\#\text{ fermionic zero modes}) \times (1-2\omega)$$
$$= (1-g)(1-2\omega)$$

from here one can read $\hat{c} = 1-2\omega$ the central charge of the theory.

in general : $\hat{c} = \sum_{i=1}^{n} (1-2\omega_i)$

simple case $n=1$: $\boxed{W(x) = \frac{x^{k+2}}{k+2} , \quad k \geq 1}$

an example of <u>$N=2$ minimal model</u> (classified by ADE, and this is the <u>A series</u>)

$\omega(x) = \frac{1}{k+2}$ <u>charge of</u> x

$\hat{c} = \frac{k}{k+2} < 1$

<u>observables</u>: $[Q, x^i] = 0$ but $\partial_x W = x^{k+1}$ is Q-exact

$\Rightarrow \phi_i = x^i$, $i = 0, \dots, k$ $\quad q_i = \frac{i}{k+2}$

<u>chiral ring</u> $\phi_i \phi_j = \phi_{i+j}$ if $i+j \leq k$
$\qquad\qquad\qquad 0 \qquad > k$

in general : $\frac{\mathbb{C}[\phi_i]}{\partial_i W}$

$$\eta_{ij} = \delta_{i+j,k}$$

notice that the anomaly in $U(1)_L, U(1)_R$ leads to a selection rule: for the correlator

$$\langle \phi_{i_1} \cdots \phi_{i_s} \int \phi_{j_1} \cdots \int \phi_{j_r} \rangle_{\Sigma_g}$$

namely $\displaystyle \sum_{i=1}^{s} q_i + \sum_{j=1}^{r} (q_j - 1) = \hat{c}(1-g)$

therefore $\langle \phi_k \rangle = 1$, $\langle \phi_i \rangle = 0$ if $i < k$

problem: solve for $C_{ijk}(t)$ in these models.

idea: introduce an "effective" superpotential

$$W(x,t) = \frac{x^{k+2}}{k+2} - \sum_{i=0}^{k} g_i(t)\, x^i$$

s.t. $$\phi_i(x,t) = -\frac{\partial W(x,t)}{\partial t_i}$$

and $$\phi_i(x,t)\,\phi_j(x,t) = \sum_{\ell} C_{ij}^{\ell}\, \phi_\ell(x,t)$$

using various inputs, we can solve for $W(x,t)$:

define M thing $\displaystyle \frac{M^{k+2}}{k+2} = W(x,t)$

i.e. $$M = x + \sum_{j=1}^{\infty} b_j x^{-j}$$

claim: $$\phi_i(x,t) = \frac{1}{i+1}\frac{\partial}{\partial x}\left[\ \ ^{i+1}\right]_+$$

key facts: the solution $\phi_i(x,t)$ can be uniquely determined by

* $\phi_i(x,t) = x^i + \mathcal{O}(x^{i-2})$

* $\langle \phi_i | \phi_j \rangle \equiv \operatorname{res}\left(\frac{\phi_i \phi_j}{W'}\right) = \eta_{ij}$ (t-independent)

Exercise: verify the above satisfies

so we are just solving a Gram-Schmidt problem.

connection to integrable systems:

$i = 0, \dots, k$

we have a flow equation for W: $\boxed{\frac{\partial W}{\partial t_i} = -\frac{1}{i+1}\frac{\partial}{\partial x} M^{i+1}}$

and this in fact is a particular case of (dispersionless) KdV

CRASH COURSE on hierarchies of the KP type

Consider formal power series $A(x,p) = \sum a_i(x) p^i$

$B(x,p) = \sum b_i(x) p^i$

and define the $*_\hbar$ product

$$A *_\hbar B = \sum_{k \geq 0} \frac{\hbar^k}{k!} \frac{\partial^k A}{\partial p^k} \frac{\partial^k B}{\partial x^k} = A\, e^{\hbar \overleftarrow{\frac{\partial}{\partial p}} \overrightarrow{\frac{\partial}{\partial x}}} B$$

(x,p) quantum phase space

also notice that $[A,B]_{*_\hbar} = \hbar \{A,B\} + \mathcal{O}(\hbar^2)$

where $\{A,B\} = \frac{\partial A}{\partial p}\frac{\partial B}{\partial x} - \frac{\partial B}{\partial p}\frac{\partial A}{\partial x}$

Consider now: $L = p + u_2 p^{-1} + u_3 p^{-2} + \dots$

and the flow equations $\hbar \frac{\partial L}{\partial t_m} = [B_m, L]_{*_\hbar}$ $\qquad B_m = (L_*^m)_+$

KP hierarchy

the products is taken to be $*_\hbar$

when $\hbar \to 0$ the flows become simply:

$$\frac{\partial L}{\partial t_n} = \{ B_n, L \} \qquad \underline{\text{dispersionless}} \text{ limit}$$

Another case: $\underset{\sim}{m\text{-KdV}}$

take $\quad L = p^m + u_2 p^{m-2} + \dots + u_m.$

$$\hbar \frac{\partial L}{\partial t_p} = [B_p, L] \qquad B_p = \left(L^{p/m}\right)_+$$

again, the dispersionless limit is $\boxed{\hbar \frac{\partial L}{\partial t_p} = \{B_p, L\}}$

notice that if $p = km$, the flow is $\underline{\text{trivial}}$

Now we can see the relation between m-KdV and the LG model:

write $\quad L = (k+2)\, W(p, t) \qquad$ i.e. put $x \to p$

identify $\quad x = t_0$

$\qquad\qquad p =$ LG field x

then,
$$\frac{\partial L}{\partial t_{i+1}} = \{ B_{i+1}, L \} = \frac{\partial B_{i+1}}{\partial x} \frac{\partial L}{\partial t_0} - \frac{\partial B_{i+1}}{\partial t_0} \cdot \frac{\partial L}{\partial x}$$

$$B_i = \left(L^{i+1/k+2}\right)_+$$

but one can see that $L^{\frac{i+1}{k+2}}_+$ does not depend on t_0,

so we find
$$\frac{\partial L}{\partial t_{i+1}} = -(k+2) \frac{\partial L^{\frac{i+1}{k+2}}_+}{\partial x} \qquad \text{since } L = -(k+2) t_0 + \dots$$

which, up to a redefinition $t_{i+1} \to \frac{t_i}{i+1}$ is the flow equation which determines the effective superpotential

B-model & special geometry

We focus on now on the topological σ model of B-type w. target X

First, since the twist is performed with the J_A current,

there is a restriction on X: $\quad c_1(X) = 0 \quad$ since J_A anomalous if $c_1(X) \neq 0$

→ X CY manifold

Observables: since $[Q, \phi^I] = 0$, $[Q, \phi^{\bar I}] = \eta^{\bar I}$ ⇒ $Q = \bar\partial$, $\eta^{\bar I} \equiv d\bar z^{\bar I}$, $\theta_I \equiv \partial/\partial z^I$

the Q-cohomology depends on a choice of cplx structure on X

One can check that

$$\omega^{J_1 \dots J_q}_{\bar I_1 \dots \bar I_p} \eta^{\bar I_1} \dots \eta^{\bar I_p} \theta_{J_1} \dots \theta_{J_q} \quad \text{is } Q\text{-closed}$$

if $\omega^{J_1 \dots J_q}_{\bar I_1 \dots \bar I_p} d\bar z^{\bar I_1} \wedge \dots d\bar z^{\bar I_p} \wedge \frac{\partial}{\partial z^{J_1}} \wedge \dots \wedge \frac{\partial}{\partial z^{J_q}}$ is $\bar\partial$ closed

as an element in $\Omega^{0,p}(X, \underbrace{\Lambda^q T_X}_{})$ — q-th ext power of the holomorphic tangent bundle

→ observables live in $H^{0,p}(X, \Lambda^q T_X)$

It is useful now to point out that, since X is a CY of complex dimension n there is a holomorphic (n,0)-form

$$\Omega = \frac{1}{n!} \Omega_{I_1 \dots I_n} dz^{I_1} \wedge \dots \wedge dz^{I_n}$$

which is nowhere vanishing

This form gives a map

$$\Omega^{(0,p)}(\Lambda^q T_X) \longrightarrow \Omega^{(n-q,p)}(X)$$

$$\varphi^{I_1 \dots I_q}_{\bar{J}_1 \dots \bar{J}_p} \longmapsto \Omega_{I_1 \dots I_q \dots I_n} \varphi^{I_1 \dots I_q}_{\bar{J}_1 \dots \bar{J}_p}$$

contract q indices

Let us now consider correlation functions at $g=0$. For the B-model, the moduli space of vacua is X itself:

$$\text{vacua} \Leftrightarrow \{Q, \text{fermi}\} = 0 \Leftrightarrow \partial_z \phi^I = \partial_{\bar{z}} \phi^I = 0$$

$\Rightarrow \phi^I$ is a constant map

We can then evaluate correlation functions by just evaluating on zero modes, i.e. in this case integrating the forms over X.

so if we have $\mathcal{O}_1 \in H^{0,p_1}(X, \Lambda^{q_1} T_X)$

$\vdots$

$\mathcal{O}_r \in H^{0,p_r}(X, \Lambda^{q_r} T_X)$

we will need $\sum p_j = n, \ \sum q_j = n$

since $\mathcal{O}_1 \dots \mathcal{O}_r \in H^{0,n}(X, \Lambda^n T_X) \xrightarrow[\Omega]{\sim} H^{n,n}(X)$ which we can integrate

This also follows from selection rules

Consider now $n=3$ CY threefold

and observables of the form $\mathcal{O} = A_{\bar I}{}^{J} \eta^{\bar I} \theta_J$

in $H^{(0,1)}(X, T_X)$ with $\partial_{[\bar K} A_{\bar I]}{}^{J} = 0$

there are $h^{2,1}$ observables of this type

(since using Ω we map them to $H^{(2,1)}(X)$)

and they are in one-to-one correspondence with deformations of complex structures of X.

(KS theory + Tian-Todorov)

3-point functions:

$$C_{abc} = \langle \mathcal{O}_a \mathcal{O}_b \mathcal{O}_c \rangle = \int_X (A_a)_{\bar J_1}{}^{I_1} (A_b)_{\bar J_2}{}^{I_2} (A_c)_{\bar J_3}{}^{I_3} \Omega_{I_1 I_2 I_3} \wedge \Omega$$

a, b, c in $\{1, \dots, h^{2,1}\}$

Recall that we can perturb the theory with these observables $\sum t^a \int_\Sigma \mathcal{O}_a^{(2)}$. Since $\mathcal{O}_a$ are defs. of complex structures, the family of TFT's we obtain is B-model on the CY X with varying complex structures.

Notice that Ω depends itself on the complex structure. For example, suppose X is defined by the zero locus of a polynomial P in $\mathbb{P}^4$:

$P(X) = 0$ $X_1, \dots, X_5$ homogeneous coordinates on $\mathbb{P}^4$

Consider $$\Xi = \frac{1}{4!} \sum_{\kappa} \varepsilon_{i_1 \dots i_5} X^{i_1} dX^{i_2} \wedge \dots \wedge dX^{i_5}$$

and let γ_P be a loop around $P=0$. Then, one has:

$$\Omega = \int_{\gamma_P} \frac{\Xi}{P}$$

if, for example, $\frac{\partial P}{\partial X_4} \neq 0$, we rewrite $dX_4 = \frac{\partial X_4}{\partial P} \cdot dP$ and integrate the above as

$$\Omega = \frac{X_5 \, dX_1 \wedge dX_2 \wedge dX_3}{(\partial P / \partial X_4)} \longrightarrow \text{encodes dependence on complex structure } \tau$$

A convenient parametrization of complex coordinates are indeed the <u>periods</u> of Ω:

choose a symplectic basis for $H_3(X)$, (A, B_a)

$a = 0, 1, \dots, h^{2,1}$

then $z^a = \int_{A^a} \Omega$, $F_a = \int_{B_a} \Omega$

and it turns out that z^a are (local) complex projective coords. for the moduli space. One goes to inhomogeneous coordinates when say $z^0 \neq 0$ by taking $t^a = \frac{z^a}{z^0}$ $a = 1, \dots, h^{2,1}$

z^a: special projective coordinates

in particular, $F_a = F_a(z^a)$

One important result on the variation of complex structures is that:

$$\frac{\partial \Omega}{\partial z^a} = \underset{(3,0)}{k_a \Omega} + \underset{(2,1)}{\hat{\mathcal{O}}_a}$$

and that $\dfrac{\partial^2 \Omega}{\partial z^a \partial z^b}$ has only degrees $(2,1)$, $(1,2)$ and $(3,0)$

This is because under a change of complex structure dz mixes to dz and $d\bar{z}$. Degree considerations give then:

$$\int_X \Omega \wedge \frac{\partial \Omega}{\partial z^a} = \int_X \Omega \wedge \frac{\partial^2 \Omega}{\partial z^a \partial z^b} = 0$$

From the first equality follows that:

$$\mathcal{F}_a = \frac{\partial \mathcal{F}}{\partial z^a} \quad \text{with } \mathcal{F} = \frac{1}{2} z^a \mathcal{F}_a \quad \text{homogenous of degree 2 in } z^a.$$

from the second equality one finds that:

$$\int \Omega \wedge \frac{\partial^3 \Omega}{\partial z^a \partial z^b \partial z^c} = \frac{\partial^3 \mathcal{F}}{\partial z^a \partial z^b \partial z^c} = \frac{1}{z_0} \frac{\partial^3 F}{\partial t^a \partial t^b \partial t^c}$$

where $\mathcal{F} = z_0^2 F$

therefore, if we redefine $\Omega \to \frac{1}{z_0} \Omega$ the periods become:

$$\left(1, t^a, \frac{\partial F}{\partial t^a}, 2F - t^a \frac{\partial F}{\partial t^a}\right) \quad \text{and}$$

$$\int \Omega \wedge \frac{\partial^3 \Omega}{\partial t^a \partial t^b \partial t^c} = \frac{\partial^3 F}{\partial t^a \partial t^b \partial t^c} = \langle \mathcal{O}_a \mathcal{O}_b \mathcal{O}_c \rangle (t)$$

which is the relation that hold in general