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Challenges for string cosmology

Part II

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Lectures on the Theory of Cosmological Perturbations

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Summary. The theory of cosmological perturbations has become a cornerstone of modern quantitative cosmology since it is the framework which provides the link between the models of the very early Universe such as the inflationary Universe scenario (which yield causal mechanisms for the generation of fluctuations) and the wealth of recent high-precision data on the spectrum of density fluctuations and cosmic microwave anisotropies. In these lectures, I provide an overview of the classical and quantum theory of cosmological fluctuations.

Crucial points in both the current inflationary paradigm [1, 2] of the early Universe and in proposed alternatives such as the Pre-Big-Bang [3] and Ekpyrotic [4] scenarios are that, first, the perturbations are generated on microscopic scales as quantum vacuum fluctuations, and, second, that via an accelerated expansion of the background geometry (or by a contraction of the background), the wavelengths of the fluctuations become much larger than the Hubble radius for a long period of cosmic evolution. Hence, both Quantum Mechanics and General Relativity are required in order to understand the generation and evolution of fluctuations.

As a guide to develop the physical intuition for the evolution of inhomogeneities, I begin with a discussion of the Newtonian theory of fluctuations. applicable at late times and on scales smaller than the Hubble radius. The analysis of super-Hubble fluctuations requires a general relativistic analysis. I first review the classical relativistic theory of fluctuations, and then discuss their quantization. I conclude with a brief overview of two applications of the theory of cosmological fluctuations: the trans-Planckian "problem" of inflationary cosmology and the current status of the study of the back-reaction of cosmological fluctuations on the background space-time geometry. Most of this article is based on the review [5] to which the reader is referred to for the details omitted in these lecture notes.

1 Motivation

As described in the lectures by Tegmark at this school [6], observational cosmology is currently in its golden years. Using a variety of observational techniques, physicists and astronomers are exploring the large-scale structure of the Universe. The Cosmic Microwave Background (CMB) is the observational window which in recent years has yielded the most information. The anisotropies in the CMB have now been detected on a wide range of angular scales, giving us a picture of the Universe at the time of recombination, the time that the cosmic photons last scattered. Large-scale galaxy redshift

surveys are providing us with increasingly accurate power spectra of the distribution of objects in the Universe which emit light, which - modulo the question whether light in fact traces mass (this is the issue of the cosmic *bias*) - gives us the distribution of mass at the present time. Analyses of the spectra of quasar absorption line systems and weak gravitational lensing surveys are beginning to give us complementary information about the distribution of matter (independent of whether this matter in fact emits light, thus shedding light on the biasing issue). The analysis of weak gravitational lensing maps is in fact sensitive not only to the baryonic but also to the dark matter, and promises to give a technique which unambiguously reveals where the dark matter is concentrated. X-ray telescopes are providing additional information on the distribution of sources which emit X-rays.

The current data fits astonishingly well with the current paradigm of early Universe cosmology, the *inflationary Universe scenario* [1]. However, it is important to keep in mind that what is tested observationally is the paradigm that the primordial spectrum of inhomogeneities was scale-invariant and predominantly adiabatic (these terms will be explained in the following Section), and that there might exist other scenarios of the very early Universe which do not yield inflation but predict a scale-invariant adiabatic spectrum. For example, within both the Pre-Big-Bang [3] and the Ekpyrotic scenarios [4] there may be models which yield such a spectrum [1] One should also not forget that topological defect models of structure formation (see e.g. [19, 20, 21] for reviews) *naturally yield a scale-invariant spectrum, however of primordial isocurvature nature and thus no longer compatible with the latest CMB anisotropy results.*

The theory of cosmological perturbations is what allows us to connect theories of the very early Universe with the data on the large-scale structure of the Universe at late times and is thus of central importance in modern cosmology. The techniques discussed below are applicable to most scenarios of the very early Universe. Most specific applications mentioned, however, will be within the context of the inflationary Universe scenario. To understand what the key requirements for a viable theory of cosmological perturbations are, recall the basic space-time diagram for inflationary cosmology (Figure 1): Since, during the phase of standard cosmology $t_R < t < t_0$, where t_R cor-

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[1] Note, however, that whereas the simplest inflationary models yield an almost scale-invariant $n = 1$ spectrum of fluctuations, as discussed in detail in these lectures, this is not the case for the simplest models of Pre-Big-Bang type nor for four dimensional descriptions of the Ekpyrotic scenario. In the case of single field realizations of Pre-Big-Bang cosmology, a spectrum with spectral index $n = 4$ emerges [7]. In Ekpyrotic cosmology, the value of the index of the final power spectrum is under active debate. Most studies conclude either that the spectral index is $n = 3$ [8, 9, 10, 11, 12], or that the result is ill-defined because of the singularities at the bounce [13, 14] (see, however, [15, 16, 17] for arguments in support of a final scale-invariant spectrum). See also [18] for criticisms of the basic setup of the Ekpyrotic scenario.
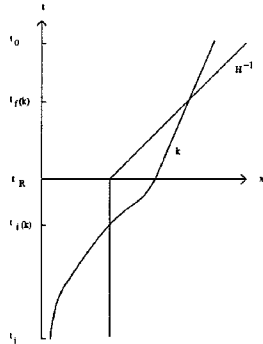
Fig. 1. Space-time diagram (sketch) showing the evolution of scales in inflationary cosmology. The vertical axis is time, and the period of inflation lasts between t_i and t_R, and is followed by the radiation-dominated phase of standard big bang cosmology. During exponential inflation, the Hubble radius H^{-1} is constant in physical spatial coordinates (the horizontal axis), whereas it increases linearly in time after t_R. The physical length corresponding to a fixed comoving length scale labelled by its wavenumber k increases exponentially during inflation but increases less fast than the Hubble radius (namely as $t^{1/2}$), after inflation.

responds to the end of inflation, and t_0 denotes the present time, the Hubble radius $l_H(t) \equiv H^{-1}(t)$ expands faster that the physical wavelength associated with a fixed comoving scale, the wavelength becomes larger than the Hubble radius as we go backwards in time. However, during the phase of accelerated expansion (inflation), the physical wavelength increases much faster than the Hubble radius, and thus at early times the fluctuations emerged at micro-physical sub-Hubble scales. The idea is that micro-physical processes (as we shall see, quantum vacuum fluctuations) are responsible for the origin of the fluctuations. However, during the period when the wavelength is super-Hubble, it is essential to describe the fluctuations using General Relativity. Thus, both Quantum Mechanics and General Relativity are required to successfully describe the generation and evolution of cosmological fluctuations.

A similar conclusion can be reached when considering the space-time diagram in a model of Pre-Big-Bang or Ekpyrotic type, where the Universe starts out in a contracting phase during which the Hubble radius contracts faster than the physical length corresponding to a fixed comoving scale (see Figure 2). The contracting phase ends at a cosmological bounce, after which the Universe is assumed to follow the same evolution history as it does in standard Big Bang cosmology. As in inflationary cosmology, quantum vacuum fluctuation on sub-Hubble scales (in this case in the contracting phase)
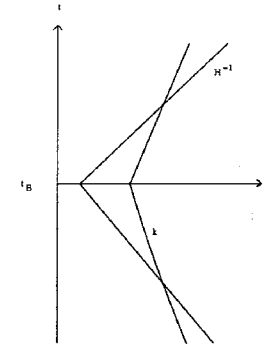
Fig. 2. Space-time diagram (sketch) showing the evolution of scales in a cosmology of PBB or Ekpyrotic type. The axes are as in Figure 1. Times earlier than t_B correspond to the contracting phase, times after describe the post-bounce phase of expansion as described in standard cosmology. The Hubble radius decreases relative to a fixed comoving scale during the contracting phase, and increases faster in the expanding phase. Fluctuations of cosmological interest today are generated sub-Hubble but propagate super-Hubble for a long time interval.

are assumed to be the seeds of the inhomogeneities observed today. For a long time period, the scale of the fluctuation is super-Hubble.

Thus, we see that in inflationary cosmology as well as in Pre-Big-Bang and Ekpyrotic-type models, both Quantum Mechanics and General Relativity are required to understand the generation and evolution of cosmological perturbations.

2 Newtonian Theory of Cosmological Perturbations

2.1 Introduction

The growth of density fluctuations is a consequence of the purely attractive nature of the gravitational force. Imagine (first in a non-expanding background) a density excess $\delta\rho$ localized about some point x in space. This fluctuation produces an attractive force which pulls the surrounding matter towards x. The magnitude of this force is proportional to $\delta\rho$. Hence, by Newton's second law

$$\ddot{\delta\rho} \sim G\delta\rho, \tag{1}$$

where G is Newton's gravitational constant. Hence, there is an exponential instability of flat space-time to the development of fluctuations.

Obviously, in General Relativity it is inconsistent to consider density fluctuations in a non-expanding background. If we consider density fluctuations in an expanding background, then the expansion of space leads to a friction term in (1). Hence, instead of an exponential instability to the development of fluctuations, the growth rate of fluctuations in an expanding Universe will be as a power of time. It is crucial to determine what this power is and how it depends both on the background cosmological expansion rate and on the length scale of the fluctuations.

We will be taking the background space-time to be homogeneous and isotropic, with a metric given by

$$ds^2 = dt^2 - a(t)^2 d\mathbf{x}^2 , \tag{2}$$

where t is physical time, $d\mathbf{x}^2$ is the Euclidean metric of the spatial hypersurfaces (here taken for simplicity to be spatially flat), and $a(t)$ denoting the scale factor, in terms of which the expansion rate is given by $H(t) = \dot{a}/a$. The coordinates $\mathbf{x}$ used above are "comoving" coordinates, coordinates painted onto the expanding spatial hypersurfaces. Note, however, that in the following two subsections $\mathbf{x}$ will denote the physical coordinates, and $\mathbf{q}$ the comoving ones.

The materials covered in this section are discussed in several excellent textbooks on cosmology, e.g. in [22, 23, 24, 25].

2.2 Perturbations about Minkowski Space-Time

To develop some physical intuition, we first consider the evolution of hydrodynamical matter fluctuations in a fixed non-expanding background. Note that in this case the background Einstein equations are not satisfied.

In this context, matter is described by a perfect fluid, and gravity by the Newtonian gravitational potential φ. The fluid variables are the energy density ρ, the pressure p, the fluid velocity $\mathbf{v}$, and the entropy density S. The basic hydrodynamical equations are

$$\dot{\rho} + \nabla \cdot (\rho \mathbf{v}) = 0$$
$$\dot{\mathbf{v}} + (\mathbf{v} \cdot \nabla)\mathbf{v} + \frac{1}{\rho}\nabla p + \nabla\varphi = 0$$
$$\nabla^2\varphi = 4\pi G\rho \tag{3}$$
$$\dot{S} + (\mathbf{v} \cdot \nabla)S = 0$$
$$p = p(\rho, S) .$$

The first equation is the continuity equation, the second is the Euler (force) equation, the third is the Poisson equation of Newtonian gravity, the fourth expresses entropy conservation, and the last describes the equation of state of matter. The derivative with respect to time is denoted by an over-dot.

The background is given by the background energy density ρ_o, the background pressure p_0, vanishing velocity, constant gravitational potential φ_0 and constant entropy density S_0. As mentioned above, it does not satisfy the background Poisson equation.

The equations for cosmological perturbations are obtained by perturbing the fluid variables about the background,

$$\rho = \rho_0 + \delta\rho$$
$$\mathbf{v} = \delta\mathbf{v}$$
$$p = p_0 + \delta p \tag{4}$$
$$\varphi = \varphi_0 + \delta\varphi$$
$$S = S_0 + \delta S ,$$

where the fluctuating fields $\delta\rho, \delta\mathbf{v}, \delta p, \delta\varphi$ and δS are functions of space and time, by inserting these expressions into the basic hydrodynamical equations (3), by linearizing, and by combining the resulting equations which are of first order in time to obtain the following second order differential equations for the energy density fluctuation $\delta\rho$ and the entropy perturbation δS

$$\ddot{\delta\rho} - c_s^2 \nabla^2 \delta\rho - 4\pi G\rho_0\delta\rho = \sigma\nabla^2\delta S \tag{5}$$
$$\dot{\delta S} = 0 ,$$

where the variables c_s^2 and σ describe the equation of state

$$\delta p = c_s^2\delta\rho + \sigma\delta S \tag{6}$$

with

$$c_s^2 = \left(\frac{\delta p}{\delta\rho}\right)_{|S} \tag{7}$$

denoting the square of the speed of sound.

What can we learn from these equations? First of all, since the equations are linear, we can work in Fourier space. Each Fourier component $\delta\rho_k(t)$ of the fluctuation field $\delta\rho(\mathbf{x}, t)$

$$\delta\rho(\mathbf{x}, t) = \int e^{i\mathbf{k}\cdot\mathbf{x}}\delta\rho_k(t) \tag{8}$$

evolves independently.

There are various types of fluctuations. If the entropy fluctuation δS vanishes, we have **adiabatic** fluctuations. If the entropy fluctuation δS is non-vanishing but $\dot{\delta\rho} = 0$, we speak on an **entropy** fluctuation.

The first conclusions we can draw from the basic perturbation equations (5) are that

1) entropy fluctuations do not grow,
2) adiabatic fluctuations are time-dependent, and
3) entropy fluctuations seed an adiabatic mode.

Taking a closer look at the equation of motion (5) for $\delta\rho$, we see that the third term on the left hand side represents the force due to gravity, a purely attractive force yielding an instability of flat space-time to the development of density fluctuations (as discussed earlier, see (1)). The second term on the left hand side of (5) represents a force due to the fluid pressure which tends to set up pressure waves. In the absence of entropy fluctuations, the evolution of $\delta\rho$ is governed by the combined action of both pressure and gravitational forces.

Restricting our attention to adiabatic fluctuations, we see from (5) that there is a critical wavelength, the Jeans length, whose wavenumber k_J is given by

$$k_J = \left(\frac{4\pi G\rho_0}{c_s^2}\right)^{1/2} .$$ (9)

Fluctuations with wavelength longer than the Jeans length ($k \ll k_J$) grow exponentially

$$\delta\rho_k(t) \sim e^{\omega_k t} \text{ with } \omega_k \sim 4(\pi G\rho_0)^{1/2}$$ (10)

whereas short wavelength modes ($k \gg k_J$) oscillate with frequency $\omega_k \sim c_s k$. Note that the value of the Jeans length depends on the equation of state of the background. For a background dominated by relativistic radiation, the Jeans length is large (of the order of the Hubble radius $H^{-1}(t)$), whereas for pressure-less matter the Jeans length goes to zero.

2.3 Perturbations about an Expanding Background

Let us now improve on the previous analysis and study Newtonian cosmological fluctuations about an expanding background. In this case, the background equations are consistent (the non-vanishing average energy density leads to cosmological expansion). However, we are still neglecting general relativistic effects (the fluctuations of the metric). Such effects turn out to be dominant on length scales larger than the Hubble radius $H^{-1}(t)$, and thus the analysis of this section is applicable only to scales smaller than the Hubble radius.

The background cosmological model is given by the energy density $\rho_0(t)$, the pressure $p_0(t)$, and the recessional velocity $\mathbf{v}_0 = H(t)\mathbf{x}$, where $\mathbf{x}$ is the Euclidean spatial coordinate vector ("physical coordinates"). The space- and time-dependent fluctuating fields are defined as in the previous section:

$$\begin{aligned}
\rho(t, \mathbf{x}) &= \rho_0(t)\big(1 + \delta_\epsilon(t, \mathbf{x})\big) \\
\mathbf{v}(t, \mathbf{x}) &= \mathbf{v}_0(t, \mathbf{x}) + \delta\mathbf{v}(t, \mathbf{x}) \\
p(t, \mathbf{x}) &= p_0(t) + \delta p(t, \mathbf{x}) ,
\end{aligned}$$ (11)

where δ_ϵ is the fractional energy density perturbation (we are interested in the fractional rather than in the absolute energy density fluctuation!), and the pressure perturbation δp is defined as in (6). In addition, there is the possibility of a non-vanishing entropy perturbation defined as in (4).

We now insert this ansatz into the basic hydrodynamical equations (3), linearize in the perturbation variables, and combine the first order differential equations for δ_ϵ and δp into a single second order differential equation for $\delta\rho_\epsilon$. The result simplifies if we work in "comoving coordinates" q which are the coordinates painted onto the expanding background, i.e.

$$\mathbf{x}(t) = a(t)\mathbf{q}(t) .$$ (12)

After a substantial amount of algebra, we obtain the following equation which describes the time evolution of density fluctuations:

$$\ddot{\delta}_\epsilon + 2H\dot{\delta}_\epsilon - \frac{c_s^2}{a^2}\nabla_q^2\delta_\epsilon - 4\pi G\rho_0\delta_\epsilon = \frac{\sigma}{\rho_0 a^2}\delta S ,$$ (13)

where the subscript q on the ∇ operator indicates that derivatives with respect to comoving coordinates are used. In addition, we have the equation of entropy conservation

$$\dot{\delta S} = 0 .$$ (14)

Comparing with the equations (5) obtained in the absence of an expanding background, we see that the only difference is the presence of a Hubble damping term in the equation for δ_ϵ. This term will moderate the exponential instability of the background to long wavelength density fluctuations. In addition, it will lead to a damping of the oscillating solutions on short wavelengths. More specifically, for physical wavenumbers $k_p \ll k_J$ (where k_J is again given by (9)), and in a matter-dominated background cosmology, the general solution of (13) in the absence of any entropy fluctuations is given by

$$\delta_k(t) = c_1 t^{2/3} + c_2 t^{-1} ,$$ (15)

where c_1 and c_2 are two constants determined by the initial conditions, and we have dropped the subscript ϵ in expressions involving δ_ϵ. There are two fundamental solutions, the first is a growing mode with $\delta_k(t) \sim a(t)$, the second a decaying mode with $\delta_k(t) \sim t^{-1}$. On short wavelength, one obtains damped oscillatory motion:

$$\delta_k(t) \sim a^{-1/2}(t)exp\big(\pm ic_s k \int dt' a^{-1}(t')\big) .$$ (16)

As a simple application of the Newtonian equations for cosmological perturbations derived above, let us compare the predicted cosmic microwave background (CMB) anisotropies in a spatially flat Universe with only baryonic matter - Model A - to the corresponding anisotropies in a flat Universe with mostly cold dark matter (pressure-less non-baryonic dark matter) - Model B. We start with the observationally known amplitude of the relative density fluctuations today (time t_0), and we use the fact that the amplitude of the CMB anisotropies on the angular scale $\theta(k)$ corresponding to the comoving wavenumber k is set by the value of the primordial gravitational potential

ϕ - introduced in the following section - which in turn is related to the value of the primordial density fluctuations at Hubble radius crossing (and **not** to its value of the time t_{rec}). See e.g. Chapter 17 of [5]).

In Model A, the dominant component of the pressure-less matter is coupled to radiation between t_{eq} and t_{rec}, the time of last scattering. Thus, the Jeans length is comparable to the Hubble radius. Therefore, for comoving galactic scales, $k \gg k_J$ in this time interval, and thus the fractional density contrast decreases as $a(t)^{-1/2}$. In contrast, in Model B, the dominant component of pressure-less matter couples only weakly to radiation, and hence the Jeans length is negligibly small. Thus, in Model B, the relative density contrast grows as $a(t)$ between t_{eq} and t_{rec}. In the time interval $t_{rec} < t < t_0$, the fluctuations scale identically in Models A and B. Summarizing, we conclude, working backwards in time from a fixed amplitude of δ_k today, that the amplitudes of $\delta_k(t_{eq})$ in Models A and B (and thus their primordial values) are related by

$$\delta_k(t_{eq})|_A \simeq \left(\frac{a(t_{rec})}{a(t_{eq})}\right)\delta_k(t_{eq})|_B . \tag{17}$$

Hence, in Model A (without non-baryonic dark matter) the CMB anisotropies are predicted to be a factor of about 30 larger [26] than in Model B, way in excess of the recent observational results. This is one of the strongest arguments for the existence of non-baryonic dark matter.

2.4 Characterizing Perturbations

Let us consider perturbations on a fixed comoving length scale given by a comoving wavenumber k. The corresponding physical length increases as $a(t)$. This is to be compared to the Hubble radius $H^{-1}(t)$ which scales as t provided $a(t)$ grows as a power of t. In the late time Universe, $a(t) \sim t^{1/2}$ in the radiation-dominated phase (i.e. for $t < t_{eq}$, and $a(t) \sim t^{2/3}$ in the matter-dominated period ($t_{eq} < t < t_0$). Thus, we see that at sufficiently early times, all comoving scales had a physical length larger than the Hubble radius. If we consider large cosmological scales (e.g. those corresponding to the observed CMB anisotropies or to galaxy clusters), the time $t_H(k)$ of "Hubble radius crossing" (when the physical length was equal to the Hubble radius) was in fact later than t_{eq}. As we will see in later sections, the time of Hubble radius crossing plays an important role in the evolution of cosmological perturbations.

Cosmological fluctuations can be described either in position space or in momentum space. In position space, we compute the root mean square mass fluctuation $\delta M/M(k,t)$ in a sphere of radius $l = 2\pi/k$ at time t. A scale-invariant spectrum of fluctuations is defined by the relation

$$\frac{\delta M}{M}(k, t_H(k)) = \text{const.} . \tag{18}$$

Such a spectrum was first suggested by Harrison [27] and Zeldovich [28] as a reasonable choice for the spectrum of cosmological fluctuations. We can introduce the "spectral index" n of cosmological fluctuations by the relation

$$\left(\frac{\delta M}{M}\right)^2(k, t_H(k)) \sim k^{n-1} , \tag{19}$$

and thus a scale-invariant spectrum corresponds to $n = 1$.

To make the transition to the (more frequently used) momentum space representation, we Fourier decompose the fractional spatial density contrast

$$\delta_\epsilon(\mathbf{x}, t) = \int d^3k \tilde{\delta}_\epsilon(\mathbf{k}, t) e^{i\mathbf{k}\cdot\mathbf{x}} . \tag{20}$$

The **power spectrum** P_δ of density fluctuations is defined by

$$P_\delta(k) = k^3 |\tilde{\delta}_\epsilon(k)|^2 , \tag{21}$$

where k is the magnitude of $\mathbf{k}$, and we have assumed for simplicity a Gaussian distribution of fluctuations in which the amplitude of the fluctuations only depends on k.

We can also define the power spectrum of the gravitational potential φ:

$$P_\varphi(k) = k^3 |\tilde{\delta\varphi}(k)|^2 . \tag{22}$$

These two power spectra are related by the Poisson equation (3)

$$P_\varphi(k) \sim k^{-4} P_\delta(k) . \tag{23}$$

In general, the condition of scale-invariance is expressed in momentum space in terms of the power spectrum evaluated at a fixed time. To obtain this condition, we first use the time dependence of the fractional density fluctuation from (15) to determine the mass fluctuations at a fixed time $t > t_H(k) > t_{eq}$ (the last inequality is a condition on the scales considered)

$$\left(\frac{\delta M}{M}\right)^2(k, t) = \left(\frac{t}{t_H(k)}\right)^{4/3}\left(\frac{\delta M}{M}\right)^2(k, t_H(k)) . \tag{24}$$

The time of Hubble radius crossing is given by

$$a(t_H(k))k^{-1} = 2t_H(k) , \tag{25}$$

and thus

$$t_H(k)^{1/2} \sim k^{-1} . \tag{26}$$

Inserting this result into (24) making use of (19) we find

$$\left(\frac{\delta M}{M}\right)^2(k, t) \sim k^{n+3} . \tag{27}$$

Since, for reasonable values of the index of the power spectrum, $\delta M/M(k,t)$ is dominated by the Fourier modes with wavenumber k, we find that (27) implies

$$|\tilde{\delta}_\epsilon|^2 \sim k^n,\tag{28}$$

or, equivalently,

$$P_\varphi(k) \sim k^{n-1}.\tag{29}$$

2.5 Matter Fluctuations in the Radiation Era

Let us now briefly consider fluctuations in the radiation dominated epoch. We are interested in both the fluctuations in radiation and in matter (cold dark matter). In the Newtonian treatment, Eq. (13) is replaced by separate equations for each matter fluid component (these components are designated by the labels A or B):

$$\ddot{\delta}_A + 2H\dot{\delta}_A - v_A^2 a^{-2}\nabla^2\delta_A = 4\pi G\sum_B \rho_B\delta_B,\tag{30}$$

where ρ_B indicate the background densities, and δ_B the fractional density fluctuations. The velocities of the respective fluid components are denoted by v_B, with $v_r^2 = 1/3$ for radiation and $v_m = 0$ for cold dark matter.

In the radiation dominated epoch, the evolution of the fluctuations in radiation is to a first approximation (in the ratio of the background densities) independent of the cold matter content. Inserting the expansion rate for this epoch, we thus immediately obtain

$$\delta_r(t) \sim a(t)^2\tag{31}$$

on scales much larger than the Hubble scale, i.e. $k \ll k_H$, whereas δ_r undergoes damped oscillatory motion on smaller scales.

The evolution of the matter fluctuation δ_m is more complicated. Its equation of motion is dominated by the source term coming from δ_r. What results is logarithmic growth of the amplitude of δ_m, instead of the growth proportional to $a(t)$ which would occur on these scales in the absence of radiation. This damping effect on matter fluctuations due to the presence of radiation is called the "Meszaros effect". It leads to a turnover in the spectrum of cosmological fluctuations at a scale k_{eq} which crosses the Hubble radius at the time of equal matter and radiation. On larger scales ($k < k_{eq}$), one has the primordial power spectrum with spectral index n, on smaller scales, to a first approximation, the spectral index changes to $n - 4$. The details of the power spectrum on small scales depend largely on the specifics of the matter content in the Universe. One can write

$$P_{final}(k,t) = T(k,t)P_0(k,t)\tag{32}$$

where P_0 is the primordial power spectrum extrapolated to late times with unchanged spectral index, and P_{final} denotes the actual power spectrum which depends on effects such as the ones mentioned above. For more details see e.g. [23, 24].

3 Relativistic Theory of Cosmological Fluctuations

3.1 Introduction

The Newtonian theory of cosmological fluctuations discussed in the previous section breaks down on scales larger than the Hubble radius because it neglects perturbations of the metric, and because on large scales the metric fluctuations dominate the dynamics.

Let us begin with a heuristic argument to show why metric fluctuations are important on scales larger than the Hubble radius. For such inhomogeneities, one should be able to approximately describe the evolution of the space-time by applying the first Friedmann-Lemaître-Robertson-Walker (FLRW) equation of homogeneous and isotropic cosmology to the local Universe (this approximation is made more rigorous in [29]):

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho.\tag{33}$$

Based on this equation, a large-scale fluctuation of the energy density will lead to a fluctuation ("δa") of the scale factor a which grows in time. This is due to the fact that self gravity amplifies fluctuations even on length scales λ greater than the Hubble radius.

This argument is made rigorous in the following analysis of cosmological fluctuations in the context of general relativity, where both metric and matter inhomogeneities are taken into account. We will consider fluctuations about a homogeneous and isotropic background cosmology, given by the metric (2), which can be written in conformal time η (defined by $dt = a(t)d\eta$) as

$$ds^2 = a(\eta)^2\left(d\eta^2 - d\mathbf{x}^2\right).\tag{34}$$

The evolution of the scale factor is determined by the two FLRW equations, (33) and

$$\dot{\rho} = -3H(\rho + p),\tag{35}$$

which determine the expansion rate and its time derivative in terms of the equation of state of the matter, whose background stress-energy tensor can be written as

$$T_\nu^\mu = \begin{pmatrix} \rho & 0 & 0 & 0 \\ 0 & -p & 0 & 0 \\ 0 & 0 & -p & 0 \\ 0 & 0 & 0 & -p \end{pmatrix}.\tag{36}$$

$$\Delta Q(p) = \tilde{\delta Q}(p) - \delta Q(p) \qquad (44)$$

is obviously a gauge artifact and carries no physical significance.

Some of the metric perturbation degrees of freedom introduced in the first subsection are gauge artifacts. To isolate these, we must study how coordinate transformations act on the metric. There are four independent gauge degrees of freedom corresponding to the coordinate transformation

$$x^\mu \rightarrow \tilde{x}^\mu = x^\mu + \xi^\mu \ . \qquad (45)$$

The zero (time) component ξ^0 of ξ^μ leads to a scalar metric fluctuation. The spatial three vector ξ^i can be decomposed

$$\xi^i = \xi^i_{tr} + \gamma^{ij}\xi_{,j} \qquad (46)$$

(where γ^{ij} is the spatial background metric) into a transverse piece ξ^i_{tr} which has two degrees of freedom which yield vector perturbations, and the second term (given by the gradient of a scalar ξ) which gives a scalar fluctuation. To summarize this paragraph, there are two scalar gauge modes given by ξ^0 and ξ, and two vector modes given by the transverse three vector ξ^i_{tr}. Thus, there remain two physical scalar and two vector fluctuation modes. The gravitational waves are gauge-invariant.

Let us now focus on how the scalar gauge transformations (i.e. the transformations given by ξ^0 and ξ) act on the scalar metric fluctuation variables ϕ, B, E, and ψ. An immediate calculation yields:

$$\begin{aligned}
\tilde{\phi} &= \phi - \frac{a'}{a}\xi^0 - (\xi^0)' \\
\tilde{B} &= B + \xi^0 - \xi' \\
\tilde{E} &= E - \xi \\
\tilde{\psi} &= \psi + \frac{a'}{a}\xi^0 \ ,
\end{aligned} \qquad (47)$$

where a prime indicates the derivative with respect to conformal time η.

There are two approaches to deal with the gauge ambiguities. The first is to fix a gauge, i.e. to pick conditions on the coordinates which completely eliminate the gauge freedom, the second is to work with a basis of gauge-invariant variables.

If one wants to adopt the gauge-fixed approach, there are many different gauge choices. Note that the often used synchronous gauge determined by $\delta g^{0\mu} = 0$ does not totally fix the gauge. A convenient system which completely fixes the coordinates is the so-called **longitudinal** or **conformal Newtonian gauge** defined by $B = E = 0$.

If one prefers a gauge-invariant approach, there are many choices of gauge-invariant variables. A convenient basis first introduced by [31] is the basis Φ, Ψ given by

$$\Phi = \phi + \frac{1}{a}\left[(B - E')a\right]' \qquad (48)$$

$$\Psi = \psi - \frac{a'}{a}(B - E') \ . \qquad (49)$$

It is obvious from the above equations that the gauge-invariant variables Φ and Ψ coincide with the corresponding diagonal metric perturbations ϕ and ψ in longitudinal gauge.

Note that the variables defined above are gauge-invariant only under linear space-time coordinate transformations. Beyond linear order, the structure of perturbation theory becomes much more involved. In fact, one can show [38] that the only fluctuation variables which are invariant under all coordinate transformations are perturbations of variables which are constant in the background space-time.

3.4 Equation of Motion

We begin with the Einstein equations

$$G_{\mu\nu} = 8\pi G T_{\mu\nu} \ , \qquad (50)$$

where $G_{\mu\nu}$ is the Einstein tensor associated with the space-time metric $g_{\mu\nu}$, and $T_{\mu\nu}$ is the energy-momentum tensor of matter, insert the ansatz for metric and matter perturbed about a FLRW background $(g^{(0)}_{\mu\nu}(\eta), \varphi^{(0)}(\eta))$:

$$g_{\mu\nu}(\mathbf{x}, \eta) = g^{(0)}_{\mu\nu}(\eta) + \delta g_{\mu\nu}(\mathbf{x}, \eta) \qquad (51)$$

$$\varphi(\mathbf{x}, \eta) = \varphi_0(\eta) + \delta\varphi(\mathbf{x}, \eta) \ , \qquad (52)$$

(where we have for simplicity replaced general matter by a scalar matter field φ) and expand to linear order in the fluctuating fields, obtaining the following equations:

$$\delta G_{\mu\nu} = 8\pi G \delta T_{\mu\nu} \ . \qquad (53)$$

In the above, $\delta g_{\mu\nu}$ is the perturbation in the metric and $\delta\varphi$ is the fluctuation of the matter field φ.

Note that the components δG^μ_ν and δT^μ_ν are not gauge invariant. If we want to use the gauge-invariant approach, we note [5] that it is possible to construct a gauge-invariant tensor $\delta G^{(gi)\,\mu}_\nu$ via

$$\begin{aligned}
\delta G^{(gi)\,0}_0 &\equiv \delta G^0_0 + {}^{(0)}G'^0_0(B - E') \\
\delta G^{(gi)\,0}_i &\equiv \delta G^0_i + ({}^{(0)}G^0_i - \frac{1}{3}{}^{(0)}G^k_k)(B - E')_{,i} \\
\delta G^{(gi)\,i}_j &\equiv \delta G^i_j + ({}^{(0)}G'^i_j)(B - E') \ ,
\end{aligned} \qquad (54)$$

where ${}^{(0)}G^\mu_\nu$ denote the background values of the Einstein tensor. Analogously, a gauge-invariant linearized stress-energy tensor $\delta T^{(gi)\,\mu}_\nu$ can be defined. In terms of these tensors, the gauge-invariant form of the equations of motion for linear fluctuations reads

$$\delta G^{(gi)}_{\mu\nu} = 8\pi G \delta T^{(gi)}_{\mu\nu}. \tag{55}$$

If we insert into this equation the ansatz for the general metric and matter fluctuations (which depend on the gauge), only gauge-invariant combinations of the fluctuation variables will appear.

In a gauge-fixed approach, one can start with the metric in longitudinal gauge

$$ds^2 = a^2 \big[(1 + 2\phi)d\eta^2 - (1 - 2\psi)\gamma_{ij}dx^i dx^j \big] \tag{56}$$

and insert this ansatz into the general perturbation equations (53). The short-cut of inserting a restricted ansatz for the metric into the action and deriving the full set of variational equations is justified in this case.

Both approaches yield the following set of equations of motion:

$$-3\mathcal{H}\big(\mathcal{H}\Phi + \Psi'\big) + \nabla^2\Psi = 4\pi G a^2 \delta T^{(gi)\,0}_0$$

$$\big(\mathcal{H}\Phi + \Psi'\big)_{,i} = 4\pi G a^2 \delta T^{(gi)\,0}_i \tag{57}$$

$$\big[(2\mathcal{H}' + \mathcal{H}^2)\Phi + \mathcal{H}\Phi' + \Psi'' + 2\mathcal{H}\Psi' \big]\delta^i_j$$

$$+ \frac{1}{2}\nabla^2 D\delta^i_j - \frac{1}{2}\gamma^{ik}D_{,kj} = -4\pi G a^2 \delta T^{(gi)\,i}_j,$$

where $D \equiv \Phi - \Psi$ and $\mathcal{H} = a'/a$. If we work in longitudinal gauge, then $\delta T^{(gi)\,i}_j = \delta T^i_j$, $\Phi = \phi$ and $\Psi = \psi$.

The first conclusion we can draw is that if no anisotropic stress is present in the matter at linear order in fluctuating fields, i.e. $\delta T^i_j = 0$ for $i \neq j$, then the two metric fluctuation variables coincide:

$$\Phi = \Psi. \tag{58}$$

This will be the case in most simple cosmological models, e.g. in theories with matter described by a set of scalar fields with canonical form of the action, and in the case of a perfect fluid with no anisotropic stress.

Let us now restrict our attention to the case of matter described in terms of a single scalar field φ with action

$$S = \int d^4x \sqrt{-g}\big[\frac{1}{2}\varphi^{,\alpha}\varphi_{,\alpha} - V(\varphi) \big] \tag{59}$$

(where g denotes the determinant of the metric) and we expand the matter field as

$$\varphi(\mathbf{x}, \eta) = \varphi_0(\eta) + \delta\varphi(\mathbf{x}, \eta) \tag{60}$$

in terms of background matter φ_0 and matter fluctuation $\delta\varphi(\mathbf{x}, \eta)$, then in longitudinal gauge (57) reduce to the following set of equations of motion (making use of (58))

$$\nabla^2\phi - 3\mathcal{H}\phi' - (\mathcal{H}' + 2\mathcal{H}^2)\phi = 4\pi G\big(\varphi_0'\delta\varphi' + V'a^2\delta\varphi\big)$$

$$\phi' + \mathcal{H}\phi = 4\pi G\varphi_0'\delta\varphi \tag{61}$$

$$\phi'' + 3\mathcal{H}\phi' + (\mathcal{H}' + 2\mathcal{H}^2)\phi = 4\pi G\big(\varphi_0'\delta\varphi' - V'a^2\delta\varphi\big),$$

where V' denotes the derivative of V with respect to φ. These equations can be combined to give the following second order differential equation for the relativistic potential ϕ:

$$\phi'' + 2\left(\mathcal{H} - \frac{\varphi_0''}{\varphi_0'}\right)\phi' - \nabla^2\phi + 2\left(\mathcal{H}' - \mathcal{H}\frac{\varphi_0''}{\varphi_0'}\right)\phi = 0. \tag{62}$$

Let us now discuss this final result for the classical evolution of cosmological fluctuations. First of all, we note the similarities with the equation (13) obtained in the Newtonian theory. The final term in (62) is the force due to gravity leading to the instability, the second to last term is the pressure force leading to oscillations (relativistic since we are considering matter to be a relativistic field), and the second term is the Hubble friction term. For each wavenumber there are two fundamental solutions. On small scales $(k > H)$, the solutions correspond to damped oscillations, on large scales $(k < H)$ the oscillations freeze out and the dynamics is governed by the gravitational force competing with the Hubble friction term. Note, in particular, how the Hubble radius naturally emerges as the scale where the nature of the fluctuating modes changes from oscillatory to frozen.

Considering the equation in a bit more detail, observe that if the equation of state of the background is independent of time (which will be the case if $\mathcal{H}' = \varphi_0'' = 0$), that then in an expanding background, the dominant mode of (62) is constant, and the sub-dominant mode decays. If the equation of state is not constant, then the dominant mode is not constant in time. Specifically, at the end of inflation $\mathcal{H}' < 0$, and this leads to a growth of ϕ (see the following subsection).

To study the quantitative implications of the equation of motion (62), it is convenient to introduce [39, 40] the variable ζ (which, up to correction terms of the order $\nabla^2\phi$ which are unimportant for large-scale fluctuations is equal to the curvature perturbation $\mathcal{R}$ in comoving gauge [41]) by

$$\zeta \equiv \phi + \frac{2}{3}\frac{(H^{-1}\dot{\phi} + \phi)}{1 + w}, \tag{63}$$

where

$$w = \frac{p}{\rho} \tag{64}$$

characterizes the equation of state of matter. In terms of ζ, the equation of motion (62) takes on the form

$$\frac{3}{2}\dot{\zeta}H(1 + w) = 0 + \mathcal{O}(\nabla^2\phi). \tag{65}$$

On large scales, the right hand side of the equation is negligible, which leads to the conclusion that large-scale cosmological fluctuations satisfy

$$\dot{\zeta}(1 + w) = 0. \tag{66}$$

This implies that except possibly if $1 + w = 0$ at some points in time during cosmological evolution (which occurs during reheating in inflationary cosmology if the inflaton field undergoes oscillations - see [42] and [43, 44] for discussions of the consequences in single and double field inflationary models, respectively) ζ is constant. In single matter field models it is indeed possible to show that $\zeta = 0$ on super-Hubble scales independent of assumptions on the equation of state [45, 46]. This "conservation law" makes it easy to relate initial fluctuations to final fluctuations in inflationary cosmology, as will be illustrated in the following subsection.

3.5 Application to Inflationary Cosmology

Let us now return to the space-time sketch of the evolution of fluctuations in inflationary cosmology (Figure 1) and use the conservation law (66) - in the form $\zeta =$ const on large scales - to relate the amplitude of ϕ at initial Hubble radius crossing during the inflationary phase (at $t = t_i(k)$) with the amplitude at final Hubble radius crossing at late times (at $t = t_f(k)$). Since both at early times and at late times $\dot{\phi} = 0$ on super-Hubble scales as the equation of state is not changing, (66) leads to

$$\phi(t_f(k)) \simeq \frac{(1+w)(t_f(k))}{(1+w)(t_i(k))}\phi(t_i(k))\,. \tag{67}$$

This equation will allow us to evaluate the amplitude of the cosmological perturbations when they re-enter the Hubble radius at time $t_f(k)$, under the assumption (discussed in detail in the following section) that the origin of the primordial fluctuations is quantum vacuum oscillations.

The time-time perturbed Einstein equation (the first equation of (57)) relates the value of ϕ at initial Hubble radius crossing to the amplitude of the relative energy density fluctuations. This, together with the fact that the amplitude of the scalar matter field quantum vacuum fluctuations is of the order H, yields

$$\phi(t_i(k)) \sim H\frac{V^{'}}{V}(t_i(k))\,. \tag{68}$$

In the late time radiation dominated phase, $w = 1/3$, whereas during slow-roll inflation

$$1 + w(t_i(k)) \simeq \frac{\dot{\varphi}_0^2}{V}(t_i(k))\,. \tag{69}$$

Making, in addition, use of the slow roll conditions satisfied during the inflationary period

$$H\dot{\varphi}_0 \simeq -V^{'}$$
$$H^2 \simeq \frac{8\pi G}{3}V\,, \tag{70}$$

we arrive at the final result

$$\phi(t_f(k)) \sim \frac{V^{3/2}}{V^{'}}(t_i(k))\,, \tag{71}$$

which gives the position space amplitude of cosmological fluctuations on a scale labelled by the comoving wavenumber k at the time when the scale re-enters the Hubble radius at late times, a result first obtained in the case of the Starobinsky model [47] of inflation in [48], and later in the context of scalar field-driven inflation in [49, 50, 51, 39].

In the case of slow roll inflation, the right hand side of (71) is, to a first approximation, independent of k, and hence the resulting spectrum of fluctuations is scale-invariant.

4 Quantum Theory of Cosmological Fluctuations

4.1 Overview

As already mentioned in the last subsection of the previous section, in many models of the very early Universe, in particular in inflationary cosmology, but also in the Pre-Big-Bang and in the Ekpyrotic scenarios, primordial inhomogeneities emerge from quantum vacuum fluctuations on microscopic scales (wavelengths smaller than the Hubble radius). The wavelength is then stretched relative to the Hubble radius, becomes larger than the Hubble radius at some time and then propagates on super-Hubble scales until re-entering at late cosmological times. In the context of a Universe with a de Sitter phase, the quantum origin of cosmological fluctuations was first discussed in [48] - see [52] for a more general discussion of the quantum origin of fluctuations in cosmology, and also [53, 54] for earlier ideas. In particular, Mukhanov [48] and Press [53] realized that in an exponentially expanding background, the curvature fluctuations would be scale-invariant, and Mukhanov provided a quantitative calculation which also yielded the logarithmic deviation from exact scale-invariance.

To understand the role of the Hubble radius, consider the equation of a free scalar matter field φ on an unperturbed expanding background:

$$\ddot{\varphi} + 3H\dot{\varphi} - \frac{\nabla^2}{a^2}\varphi = 0\,. \tag{72}$$

The second term on the left hand side of this equation leads to damping of φ with a characteristic decay rate given by H. As a consequence, in the absence of the spatial gradient term, $\dot{\varphi}$ would be of the order of magnitude $H\varphi$. Thus, comparing the second and the third term on the left hand side, we immediately see that the microscopic (spatial gradient) term dominates on length scales smaller than the Hubble radius, leading to oscillatory motion, whereas this term is negligible on scales larger than the Hubble radius, and the evolution of φ is determined primarily by gravity.

To understand the generation and evolution of fluctuations in current models of the very early Universe, we thus need both Quantum Mechanics and General Relativity, i.e. quantum gravity. At first sight, we are thus faced with an intractable problem, since the theory of quantum gravity is not yet established. We are saved by the fact that today on large cosmological scales the fractional amplitude of the fluctuations is smaller than 1. Since gravity is a purely attractive force, the fluctuations had to have been - at least in the context of an eternally expanding background cosmology - very small in the early Universe. Thus, a linearized analysis of the fluctuations (about a classical cosmological background) is self-consistent.

From the classical theory of cosmological perturbations discussed in the previous section, we know that the analysis of scalar metric inhomogeneities can be reduced - after extracting gauge artifacts - to the study of the evolution of a single fluctuating variable. Thus, we conclude that the quantum theory of cosmological perturbations must be reducible to the quantum theory of a single free scalar field which we will denote by v. Since the background in which this scalar field evolves is time-dependent, the mass of v will be time-dependent. The time-dependence of the mass will lead to quantum particle production over time if we start the evolution in the vacuum state for v. As we will see, this quantum particle production corresponds to the development and growth of the cosmological fluctuations. Thus, the quantum theory of cosmological fluctuations provides a consistent framework to study both the generation and the evolution of metric perturbations. The following analysis is based on Part II of [5].

4.2 Outline of the Analysis

In order to obtain the action for linearized cosmological perturbations, we expand the action to quadratic order in the fluctuating degrees of freedom. The linear terms cancel because the background is taken to satisfy the background equations of motion.

We begin with the Einstein-Hilbert action for gravity and the action of a scalar matter field (for the more complicated case of general hydrodynamical fluctuations the reader is referred to [5])

$$ S = \int d^4x \sqrt{-g} \left[-\frac{1}{16\pi G} R + \frac{1}{2} \partial_\mu \varphi \partial^\mu \varphi - V(\varphi) \right], \tag{73} $$

where g is the determinant of the metric.

The simplest way to proceed is to work in a fixed gauge, longitudinal gauge, in which the metric and matter take the form

$$ ds^2 = a^2(\eta) \left[(1 + 2\phi(\eta, \mathbf{x})) d\eta^2 - (1 - 2\psi(t, \mathbf{x})) d\mathbf{x}^2 \right] \tag{74} $$
$$ \varphi(\eta, \mathbf{x}) = \varphi_0(\eta) + \delta\varphi(\eta, \mathbf{x}). $$

The next step is to reduce the number of degrees of freedom. First, as already mentioned in the previous section, the off-diagonal spatial Einstein

equations force $\psi = \phi$ since $\delta T_j^i = 0$ for scalar field matter (no anisotropic stresses to linear order). The two remaining fluctuating variables ϕ and φ must be linked by the Einstein constraint equations since there cannot be matter fluctuations without induced metric fluctuations.

The two nontrivial tasks of the lengthy [5] computation of the quadratic piece of the action is to find out what combination of φ and ϕ gives the variable v in terms of which the action has canonical form, and what the form of the time-dependent mass is. This calculation involves inserting the ansatz (74) into the action (73), expanding the result to second order in the fluctuating fields, making use of the background and of the constraint equations, and dropping total derivative terms from the action. In the context of scalar field matter, the quantum theory of cosmological fluctuations was developed by Mukhanov [55, 56] (see also [57]). The result is the following contribution $S^{(2)}$ to the action quadratic in the perturbations:

$$ S^{(2)} = \frac{1}{2} \int d^4x \left[v'^2 - v_{,i}v_{,i} + \frac{z''}{z} v^2 \right], \tag{75} $$

where the canonical variable v (the "Mukhanov variable" introduced in [56] - see also [52]) is given by

$$ v = a \left[\delta\varphi + \frac{\varphi_0'}{\mathcal{H}} \phi \right], \tag{76} $$

with $\mathcal{H} = a'/a$, and where

$$ z = \frac{a\varphi_0'}{\mathcal{H}}. \tag{77} $$

In both the cases of power law inflation and slow roll inflation, $\mathcal{H}$ and φ_0' are proportional and hence

$$ z(\eta) \sim a(\eta). \tag{78} $$

Note that the variable v is related to the curvature perturbation $\mathcal{R}$ in comoving coordinates introduced in [41] and closely related to the variable ζ used in [39, 40]:

$$ v = z\mathcal{R}. \tag{79} $$

The equation of motion which follows from the action (75) is

$$ v'' - \nabla^2 v - \frac{z''}{z} v = 0, \tag{80} $$

or, in momentum space

$$ v_k'' + k^2 v_k - \frac{z''}{z} v_k = 0, \tag{81} $$

where v_k is the k'th Fourier mode of v. As a consequence of (78), the mass term in the above equation is given by the Hubble scale

$$k_H^2 \equiv \frac{z''}{z} \simeq H^2 \,. \tag{82}$$

Thus, it immediately follows from (81) that on small length scales, i.e. for $k > k_H$, the solutions for v_k are constant amplitude oscillations . These oscillations freeze out at Hubble radius crossing, i.e. when $k = k_H$. On longer scales ($k \ll k_H$), the solutions for v_k increase as z:

$$v_k \sim z \,, \quad k \ll k_H \,. \tag{83}$$

Given the action (75), the quantization of the cosmological perturbations can be performed by canonical quantization (in the same way that a scalar matter field on a fixed cosmological background is quantized [58]).

The final step in the quantum theory of cosmological perturbations is to specify an initial state. Since in inflationary cosmology, all pre-existing classical fluctuations are red-shifted by the accelerated expansion of space, one usually assumes (we will return to a criticism of this point when discussing the trans-Planckian problem of inflationary cosmology) that the field v starts out at the initial time t_i mode by mode in its vacuum state. Two questions immediately emerge: what is the initial time t_i, and which of the many possible vacuum states should be chosen. It is usually assumed that since the fluctuations only oscillate on sub-Hubble scales, that the choice of the initial time is not important, as long as it is earlier than the time when scales of cosmological interest today cross the Hubble radius during the inflationary phase. The state is usually taken to be the Bunch-Davies vacuum (see e.g. [58]), since this state is empty of particles at t_i in the coordinate frame determined by the FLRW coordinates (see e.g. [59] for a discussion of this point), and since the Bunch-Davies state is a local attractor in the space of initial states in an expanding background (see e.g. [60]). Thus, we choose the initial conditions

$$v_k(\eta_i) = \frac{1}{\sqrt{2\omega_k}} \tag{84}$$

$$v_k'(\eta_i) = \frac{\sqrt{\omega_k}}{\sqrt{2}}$$

where here $\omega_k = k$, and η_i is the conformal time corresponding to the physical time t_i.

Let us briefly summarize the quantum theory of cosmological perturbations. In the linearized theory, fluctuations are set up at some initial time t_i mode by mode in their vacuum state. While the wavelength is smaller than the Hubble radius, the state undergoes quantum vacuum fluctuations. The accelerated expansion of the background redshifts the length scale beyond the Hubble radius. The fluctuations freeze out when the length scale is equal to the Hubble radius. On larger scales, the amplitude of v_k increases as the scale factor. This corresponds to the squeezing of the quantum state present

at Hubble radius crossing (in terms of classical general relativity, it is self-gravity which leads to this growth of fluctuations). As discussed e.g. in [61], the squeezing of the quantum vacuum state leads to the emergence of the classical nature of the fluctuations.

4.3 Application to Inflationary Cosmology

In this subsection we will use the quantum theory of cosmological perturbations developed in this section to calculate the spectrum of curvature fluctuations in inflationary cosmology.

We need to compute the power spectrum $\mathcal{P}_\mathcal{R}(k)$ of the curvature fluctuation $\mathcal{R}$ defined in (79), namely

$$\mathcal{R} = z^{-1}v = \phi + \delta\varphi \frac{\mathcal{H}}{\varphi_0'} \tag{85}$$

The idea in calculating the power spectrum at a late time t is to first relate the power spectrum via the growth rate (83) of v on super-Hubble scales to the power spectrum at the time $t_H(k)$ of Hubble radius crossing, and to then use the constancy of the amplitude of v on sub-Hubble scales to relate it to the initial conditions (84). Thus

$$\begin{aligned}
\mathcal{P}_\mathcal{R}(k,t) &\equiv k^3 \mathcal{R}_k^2(t) = k^3 z^{-2}(t)|v_k(t)|^2 \\
&= k^3 z^{-2}(t)\left(\frac{z(t)}{z(t_H(k))}\right)^2 |v_k(t_H(k))|^2 \\
&= k^3 z^{-2}(t_H(k))|v_k(t_H(k))|^2 \\
&\sim k^3 a^{-2}(t_H(k))|v_k(t_i)|^2 \,,
\end{aligned} \tag{86}$$

where in the final step we have used (78) and the constancy of the amplitude of v on sub-Hubble scales. Making use of the condition

$$a^{-1}(t_H(k))k = H \tag{87}$$

for Hubble radius crossing, and of the initial conditions (84), we immediately see that

$$\mathcal{P}_\mathcal{R}(k,t) \sim k^3 k^{-2} k^{-1} H^2 \,, \tag{88}$$

and that thus a scale invariant power spectrum with amplitude proportional to H^2 results, in agreement with what was argued on heuristic grounds in Section (3.5).

4.4 Quantum Theory of Gravitational Waves

The quantization of gravitational waves parallels the quantization of scalar metric fluctuations, but is more simple because there are no gauge ambiguities. Note again that at the level of linear fluctuations, scalar metric fluctuations and gravitational waves are independent. Both can be quantized on the

same cosmological background determined by background scale factor and background matter. However, in contrast to the case of scalar metric fluctuations, the tensor modes are also present in pure gravity (i.e. in the absence of matter).

Starting point is the action (73). Into this action we insert the metric which corresponds to a classical cosmological background plus tensor metric fluctuations:

$$ds^2 = a^2(\eta)\left[d\eta^2 - (\delta_{ij} + h_{ij})dx^i dx^j\right], \tag{89}$$

where the second rank tensor $h_{ij}(\eta, \mathbf{x})$ represents the gravitational waves, and in turn can be decomposed as

$$h_{ij}(\eta, \mathbf{x}) = h_+(\eta, \mathbf{x})e_{ij}^+ + h_x(\eta, \mathbf{x})e_{ij}^x \tag{90}$$

into the two polarization states. Here, e_{ij}^+ and e_{ij}^x are two fixed polarization tensors, and h_+ and h_x are the two coefficient functions.

To quadratic order in the fluctuating fields, the action separates into separate terms involving h_+ and h_x. Each term is of the form

$$S^{(2)} = \int d^4x \frac{a^2}{2}\left[h'^2 - (\nabla h)^2\right], \tag{91}$$

leading to the equation of motion

$$h_k'' + 2\frac{a'}{a}h_k' + k^2 h_k = 0. \tag{92}$$

The variable in terms of which the action (91) has canonical kinetic term is

$$\mu_k \equiv a h_k, \tag{93}$$

and its equation of motion is

$$\mu_k'' + \left(k^2 - \frac{a''}{a}\right)\mu_k = 0. \tag{94}$$

This equation is very similar to the corresponding equation (81) for scalar gravitational inhomogeneities, except that in the mass term the scale factor $a(\eta)$ is replaced by $z(\eta)$, which leads to a very different evolution of scalar and tensor modes during the reheating phase in inflationary cosmology during which the equation of state of the background matter changes dramatically.

Based on the above discussion we have the following theory for the generation and evolution of gravitational waves in an accelerating Universe (first developed by Grishchuk [62]): waves exit as quantum vacuum fluctuations at the initial time on all scales. They oscillate until the length scale crosses the Hubble radius. At that point, the oscillations freeze out and the quantum state of gravitational waves begins to be squeezed in the sense that

$$\mu_k(\eta) \sim a(\eta), \tag{95}$$

which, from (93) corresponds to constant amplitude of h_k. The squeezing of the vacuum state leads to the emergence of classical properties of this state, as in the case of scalar metric fluctuations.

5 The Trans-Planckian Window

Whereas the contents of the previous sections are well established, this and the following section deal with aspects of the theory of cosmological perturbations which are currently under investigation and are at the present time rather controversial. First, we consider the trans-Planckian issue (this section is adapted from [63]).

The same background dynamics which yields the causal generation mechanism for cosmological fluctuations, the most spectacular success of inflationary cosmology, bears in it the nucleus of the "trans-Planckian problem". This can be seen from Fig. 3. If inflation lasts only slightly longer than the minimal time it needs to last in order to solve the horizon problem and to provide a causal generation mechanism for CMB fluctuations, then the corresponding physical wavelength of these fluctuations is smaller than the Planck length at the beginning of the period of inflation. The theory of cosmological perturbations is based on classical general relativity coupled to a weakly coupled scalar field description of matter. Both the theories of gravity and of matter will break down on trans-Planckian scales, and this immediately leads to the trans-Planckian problem: are the predictions of standard inflationary cosmology robust against effects of trans-Planckian physics [64]?

The simplest way of modeling the possible effects of trans-Planckian physics, while keeping the mathematical analysis simple, is to replace the linear dispersion relation $\omega_{phys} = k_{phys}$ of the usual equation for cosmological perturbations by a non standard dispersion relation $\omega_{phys} = \omega_{phys}(k)$ which differs from the standard one only for physical wavenumbers larger than the Planck scale. This method was introduced [65, 66] in the context of studying the dependence of the thermal spectrum of black hole radiation on trans-Planckian physics. In the context of cosmology, it has been shown [67, 68, 69] that this amounts to replacing k^2 appearing in (81) with $k_{\text{eff}}^2(n, \eta)$ defined by

$$k^2 \rightarrow k_{\text{eff}}^2(k, \eta) \equiv a^2(\eta)\omega_{phys}^2\left(\frac{k}{a(\eta)}\right). \tag{96}$$

For a fixed comoving mode, this implies that the dispersion relation becomes time-dependent. Therefore, the equation of motion of the quantity $v_k(\eta)$ takes the form (with $z(\eta) \propto a(\eta)$)

$$v_k'' + \left[k_{\text{eff}}^2(k, \eta) - \frac{a''}{a}\right]v_k = 0. \tag{97}$$

A more rigorous derivation of this equation, based on a variational principle, has been provided [70] (see also Ref. [71]).

The evolution of modes thus must be considered separately in three phases, see Fig. 3. In Phase I the wavelength is smaller than the Planck scale, and trans-Planckian physics can play an important role. In Phase II,
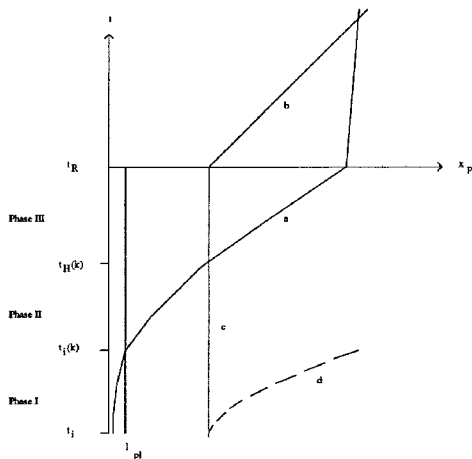
Fig. 3. Space-time diagram (physical distance vs. time) showing the origin of the trans-Planckian problem of inflationary cosmology: at very early times, the wavelength is smaller than the Planck scale ℓ_{Pl} (Phase I), at intermediate times it is larger than ℓ_{Pl} but smaller than the Hubble radius H^{-1} (Phase II), and at late times during inflation it is larger than the Hubble radius (Phase III). The line labeled a) is the physical wavelength associated with a fixed comoving scale k. The line b) is the Hubble radius or horizon in SBB cosmology. Curve c) shows the Hubble radius during inflation. The horizon in inflationary cosmology is shown in curve d).

the wavelength is larger than the Planck scale but smaller than the Hubble radius. In this phase, trans-Planckian physics will have a negligible effect (this statement can be quantified [72]). Hence, by the analysis of the previous section, the wave function of fluctuations is oscillating in this phase,

$$v_k = B_1 \exp(-ik\eta) + B_2 \exp(ik\eta) \tag{98}$$

with constant coefficients B_1 and B_2. In the standard approach, the initial conditions are fixed in this region and the usual choice of the vacuum state leads to $B_1 = 1/\sqrt{2k}$, $B_2 = 0$. Phase III starts at the time $t_H(k)$ when the mode crosses the Hubble radius. During this phase, the wave function is squeezed.

One source of trans-Planckian effects [67, 68] on observations is the possible non-adiabatic evolution of the wave function during Phase I. If this occurs, then it is possible that the wave function of the fluctuation mode is not in its vacuum state when it enters Phase II and, as a consequence, the

coefficients B_1 and B_2 are no longer given by the standard expressions above. In this case, the wave function will not be in its vacuum state when it crosses the Hubble radius, and the final spectrum will be different. In general, B_1 and B_2 are determined by the matching conditions between Phase I and II. If the dynamics is adiabatic throughout (in particular if the a''/a term is negligible), the WKB approximation holds and the solution is always given by

$$v_k(\eta) \simeq \frac{1}{\sqrt{2k_{\text{eff}}(k,\eta)}} \exp\left(-i \int_{\eta_i}^{\eta} k_{\text{eff}} d\tau\right), \tag{99}$$

where η_i is some initial time. Therefore, if we start with a positive frequency solution only and use this solution, we find that no negative frequency solution appears. Deep in Region II where $k_{\text{eff}} \simeq k$ the solution becomes

$$v_k(\eta) \simeq \frac{1}{\sqrt{2k}} \exp(-i\phi - ik\eta), \tag{100}$$

i.e. the standard vacuum solution times a phase which will disappear when we calculate the modulus. To obtain a modification of the inflationary spectrum, it is sufficient to find a dispersion relation such that the WKB approximation breaks down in Phase I.

A concrete class of dispersion relations for which the WKB approximation breaks down is

$$k_{\text{eff}}^2(k,\eta) = k^2 - k^2 |b_m| \left[\frac{\ell_{Pl}}{\lambda(\eta)}\right]^{2m}, \tag{101}$$

where $\lambda(\eta) = 2\pi a(\eta)/k$ is the wavelength of a mode. If we follow the evolution of the modes in Phases I, II and III, matching the mode functions and their derivatives at the junction times, the calculation [67, 68, 73] demonstrates that the final spectral index is modified and that superimposed oscillations appear.

However, the above example suffers from several problems. First, in inflationary models with a long period of inflationary expansion, the dispersion relation (101) leads to complex frequencies at the beginning of inflation for scales which are of current interest in cosmology. Furthermore, the initial conditions for the Fourier modes of the fluctuation field have to be set in a region where the evolution is non-adiabatic and the use of the usual vacuum prescription can be questioned. These problems can be avoided in a toy model in which we follow the evolution of fluctuations in a bouncing cosmological background which is asymptotically flat in the past and in the future. The analysis [74] shows that even in this case the final spectrum of fluctuations depends on the specific dispersion relation used.

An example of a dispersion relation which breaks the WKB approximation in the trans-Planckian regime but does not lead to the problems mentioned in the previous paragraph was investigated in [70]. It is a dispersion relation which is linear for both small and large wavenumbers, but has an intermediate

interval during which the frequency decreases as the wavenumber increases, much like what happens in (101). The violation of the WKB condition occurs for wavenumbers near the local minimum of the $\omega(k)$ curve.

A justified criticism against the method summarized in the previous analysis is that the non-standard dispersion relations used are completely ad hoc, without a clear basis in trans-Planckian physics. There has been a lot of recent work [75, 76, 77, 78] on the implication of space-space uncertainty relations [79, 80] on the evolution of fluctuations. The application of the uncertainty relations on the fluctuations lead to two effects [81, 82]. Firstly, the equation of motion of the fluctuations in modified. Secondly, for fixed comoving length scale k, the uncertainty relation is saturated before a critical time $t_i(k)$. Thus, in addition to a modification of the evolution, trans-Planckian physics leads to a modification of the boundary condition for the fluctuation modes. The upshot of this work is that the spectrum of fluctuations is modified.

In [83], the implications of the stringy space-time uncertainty relation [84, 85]

$$\Delta x_{\mathrm{phys}} \Delta t \geq l_s^2 \tag{102}$$

on the spectrum of cosmological fluctuations was studied. Again, application of this uncertainty relation to the fluctuations leads to two effects. Firstly, the coupling between the background and the fluctuations is nonlocal in time, thus leading to a modified dynamical equation of motion (a similar modification also results [86] from quantum deformations, another example of a consequence of non-commutative basic physics). Secondly, the uncertainty relation is saturated at the time $t_i(k)$ when the physical wavelength equals the string scale l_s. Before that time it does not make sense to talk about fluctuations on that scale. By continuity, it makes sense to assume that fluctuations on scale k are created at time $t_i(k)$ in the local vacuum state (the instantaneous WKB vacuum state).

Let us for the moment neglect the nonlocal coupling between background and fluctuation, and thus consider the usual equation of motion for fluctuations in an accelerating background cosmology. We distinguish two ranges of scales. Ultraviolet modes are generated at late times when the Hubble radius is larger than l_s. On these scales, the spectrum of fluctuations does not differ from what is predicted by the standard theory, since at the time of Hubble radius crossing the fluctuation mode will be in its vacuum state. However, the evolution of infrared modes which are created when the Hubble radius is smaller than l_s is different. The fluctuations undergo *less* squeezing than they do in the absence of the uncertainty relation, and hence the final amplitude of fluctuations is lower. From the equation (86) for the power spectrum of fluctuations, and making use of the condition

$$a(t_i(k)) = kl_s \tag{103}$$

for the time $t_i(k)$ when the mode is generated, it follows immediately that the power spectrum is scale-invariant

$$\mathcal{P}_{\mathcal{R}}(k) \sim k^0 . \tag{104}$$

In the standard scenario of power-law inflation the spectrum is red ($\mathcal{P}_{\mathcal{R}}(k) \sim k^{n-1}$ with $n < 1$). Taking into account the effects of the nonlocal coupling between background and fluctuation mode leads [83] to a modification of this result: the spectrum of fluctuations in a power-law inflationary background is in fact blue ($n > 1$).

Note that, if we neglect the nonlocal coupling between background and fluctuation mode, the result of (104) also holds in a cosmological background which is NOT accelerating. Thus, we have a method of obtaining a scale-invariant spectrum of fluctuations without inflation. This result has also been obtained in [87], however without a micro-physical basis for the prescription for the initial conditions.

A key problem with the method of modified dispersion relations is the issue of back-reaction [88, 89]. If the mode occupation numbers of the fluctuations at Hubble radius crossing are significant, the danger arises that the back-reaction of the fluctuations will in fact prevent inflation. Another constraint arises from the observational limits on the flux of ultra-high-energy cosmic rays. Such cosmic rays would be produced [90] in the present Universe if Trans-Planckian effects of the type discussed in this section were present.

An approach to the trans-Planckian issue pioneered by Danielsson [91] which has recently received a lot of attention is to avoid the issue of the unknown trans-Planckian physics and to start the evolution of the fluctuation modes at the mode-dependent time when the wavelength equals the limiting scale. Obviously, the resulting spectrum will depend sensitively on which state is taken to be the initial state. The vacuum state is not unambiguous, and the choice of a state minimizing the energy density depends on the space-time splitting [92]. The signatures of this prescription are typically oscillations superimposed on the usual spectrum. The amplitude of this effect depends sensitively on the prescription of the initial state, and for a fixed prescription also on the background cosmology. For a discussion of these issues and a list of references on this approach the reader is referred to [93].

In summary, due to the exponential red-shifting of wavelengths, present cosmological scales originate at wavelengths smaller than the Planck length early on during the period of inflation. Thus, Planck physics may well encode information in these modes which can now be observed in the spectrum of microwave anisotropies. Two examples have been shown to demonstrate the existence of this "window of opportunity" to probe trans-Planckian physics in cosmological observations. The first method makes use of modified dispersion relations to probe the robustness of the predictions of inflationary cosmology, the second applies the stringy space-time uncertainty relation on the fluctuation modes. Both methods yield the result that trans-Planckian physics may lead to measurable effects in cosmological observables. An important issue which must be studied more carefully is the back-reaction of the cosmological fluctuations (see e.g. [94] for a possible formalism).

6 Back-Reaction of Cosmological Fluctuations

The presence of cosmological fluctuations influences the background cosmology in which the perturbations evolve. This back-reaction arises as a second order effect in the cosmological perturbation expansion. The effect is cumulative in the sense that all fluctuation modes contribute to the change in the background geometry, and as a consequence the back-reaction effect can be large even if the amplitude of the fluctuation spectrum is small. In this section (based on the review [95]) we discuss two approaches used to quantify back-reaction. In the first approach [96, 94], the effect of the fluctuations on the background is expressed in terms of an effective energy-momentum tensor. We show that in the context of an inflationary background cosmology, the long wavelength contributions to the effective energy-momentum tensor take the form of a negative cosmological constant, whose absolute value increases as a function of time since the phase space of infrared modes is increasing. This then leads to the speculation [97, 98] that gravitational back-reaction may lead to a dynamical cancellation mechanism for a bare cosmological constant, and yield a scaling fixed point in the asymptotic future in which the remnant cosmological constant satisfies $\Omega_\Lambda \sim 1$. We then discuss [99] how infrared modes effect local observables (as opposed to mathematical background quantities) and find that the leading infrared back-reaction contributions cancel in single field inflationary models. However, we expect non-trivial back-reaction of infrared modes in models with more than one matter field.

It is well known that gravitational waves propagating in some background space-time affect the dynamics of the background. This back-reaction can be described in terms of an effective energy-momentum tensor $\tau_{\mu\nu}$. In the short wave limit, when the typical wavelength of the waves is small compared with the curvature of the background space-time, $\tau_{\mu\nu}$ has the form of a radiative fluid with an equation of state $p = \rho/3$ (where p and ρ denote pressure and energy density, respectively). As we have seen in previous section, in inflationary cosmology it is the long wavelength scalar metric fluctuations which are more important. Like short wavelength gravitational waves, these cosmological fluctuations will contribute to the effective energy-momentum tensor $\tau_{\mu\nu}$. The work of [96, 94] is closely related to work by Woodard and Tsamis [100, 101] who considered the back-reaction of long wavelength gravitational waves in pure gravity with a bare cosmological constant. The recent paper [99] is related to the work of Abramo and Woodard [102, 103] who initiated the study of back-reaction of infrared modes on local observables.

We first review the derivation of the effective energy-momentum tensor $\tau_{\mu\nu}$ which describes the back-reaction of linear cosmological fluctuations on the background cosmology, and summarize the evaluation of this tensor in an inflationary cosmological background. This gravitational back-reaction calculation is related to the early work on the back-reaction of gravitational waves by Brill, Hartle and Isaacson [104], among others. The idea is to expand the Einstein equations to second order in the perturbations, to assume that the

first order terms satisfy the equations of motion for linearized cosmological perturbations discussed in previous section (hence these terms cancel), to take the spatial average of the remaining terms, and to regard the resulting equations as equations for a new homogeneous metric $g_{\mu\nu}^{(0,br)}$ which includes the effect of the perturbations to quadratic order:

$$G_{\mu\nu}(g_{\alpha\beta}^{(0,br)}) = 8\pi G \left[T_{\mu\nu}^{(0)} + \tau_{\mu\nu} \right] \tag{105}$$

where the effective energy-momentum tensor $\tau_{\mu\nu}$ of gravitational back-reaction contains the terms resulting from spatial averaging of the second order metric and matter perturbations:

$$\tau_{\mu\nu} = < T_{\mu\nu}^{(2)} - \frac{1}{8\pi G} G_{\mu\nu}^{(2)} >, \tag{106}$$

where pointed brackets stand for spatial averaging, and the superscripts indicate the order in perturbation theory.

As analyzed in detail in [96, 94], the back-reaction equation (105) is covariant under linear space-time coordinate transformations even though $\tau_{\mu\nu}$ is not invariant [2]. In the following, we will work in longitudinal gauge.

For simplicity, we shall take matter to be described in terms of a single scalar field. By expanding the Einstein and matter energy-momentum tensors to second order in the metric and matter fluctuations ϕ and $\delta\varphi$, respectively, it can be shown that the non-vanishing components of the effective back-reaction energy-momentum tensor $\tau_{\mu\nu}$ become

$$\begin{aligned}
\tau_{00} = \frac{1}{8\pi G} &\left[+12H\langle\phi\dot\phi\rangle - 3\langle(\dot\phi)^2\rangle + 9a^{-2}\langle(\nabla\phi)^2\rangle \right] \\
&+ \frac{1}{2}\langle(\delta\dot\varphi)^2\rangle + \frac{1}{2}a^{-2}\langle(\nabla\delta\varphi)^2\rangle \\
&+ \frac{1}{2}V''(\varphi_0)\langle\delta\varphi^2\rangle + 2V'(\varphi_0)\langle\phi\delta\varphi\rangle \quad,
\end{aligned} \tag{107}$$

and

$$\begin{aligned}
\tau_{ij} = a^2\delta_{ij} \bigg\{ \frac{1}{8\pi G} &\left[(24H^2 + 16\dot H)\langle\phi^2\rangle + 24H\langle\phi\dot\phi\rangle \right. \\
&+ \langle(\dot\phi)^2\rangle + 4\langle\phi\ddot\phi\rangle - \frac{4}{3}a^{-2}\langle(\nabla\phi)^2\rangle \bigg] + 4\dot\varphi_0^2\langle\phi^2\rangle \\
&+ \frac{1}{2}\langle(\delta\dot\varphi)^2\rangle - \frac{1}{6}a^{-2}\langle(\nabla\delta\varphi)^2\rangle - 4\dot\varphi_0\langle\delta\dot\varphi\phi\rangle \\
&- \frac{1}{2}V''(\varphi_0)\langle\delta\varphi^2\rangle + 2V'(\varphi_0)\langle\phi\delta\varphi\rangle \bigg\} \quad,
\end{aligned} \tag{108}$$

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[2] See [105], however, for important questions concerning the covariance of the analysis under higher order coordinate transformations.

where H is the Hubble expansion rate.

The metric and matter fluctuation variables ϕ and $\delta\varphi$ are linked via the Einstein constraint equations, and hence all terms in the above formulas for the components of $\tau_{\mu\nu}$ can be expressed in terms of two point functions of ϕ and its derivatives. The two point functions, in turn, are obtained by integrating over all of the Fourier modes of ϕ, e.g.

$$\langle \phi^2 \rangle \sim \int_{k_i}^{k_u} dk k^2 |\phi_k|^2 \,, \tag{109}$$

where ϕ_k denotes the amplitude of the k'th Fourier mode. The above expression is divergent both in the infrared and in the ultraviolet. The ultraviolet divergence is the usual divergence of a free quantum field theory and can be "cured" by introducing an ultraviolet cutoff k_u. In the infrared, we will discard all modes $k < k_i$ with wavelength larger than the Hubble radius at the beginning of inflation, since these modes are determined by the pre-inflationary physics. We take these modes to contribute to the background.

At any time t we can separate the integral in (109) into the contribution of infrared and ultraviolet modes, the separation being defined by setting the physical wavelength equal to the Hubble radius. Thus, in an inflationary Universe the infrared phase space is continually increasing since comoving modes are stretched beyond the Hubble radius, while the ultraviolet phase space is either constant (if the ultraviolet cutoff corresponds to a fixed physical wavelength), or decreasing (if the ultraviolet cutoff corresponds to fixed comoving wavelength). In either case, unless the spectrum of the initial fluctuations is extremely blue, two point functions such as (109) will at later stages of an inflationary Universe be completely dominated by the infrared sector. In the following, we will therefore restrict our attention to this sector, i.e. to wavelengths larger than the Hubble radius.

In order to evaluate the two point functions which enter into the expressions for $\tau_{\mu\nu}$, we make use of the known time evolution of the linear fluctuations ϕ_k discussed in previous section. On scales larger than the Hubble radius, and for a time-independent equation of state, ϕ_k is constant in time. From the Einstein constraint equations relating the metric and matter fluctuations, and making use of the inflationary slow roll approximation conditions we find

$$\delta\varphi = -\frac{2V}{V'}\phi \,. \tag{110}$$

Hence, in the expressions (107) and (108) for $\tau_{\mu\nu}$, all terms with space and time derivatives can be neglected, and we obtain

$$\rho_{br} \equiv \tau_0^0 \cong \left(2\frac{V''V^2}{V'^2} - 4V \right) < \phi^2 > \tag{111}$$

and

$$p_{br} \equiv -\frac{1}{3}\tau_i^i \cong -\rho_{br} \,, \tag{112}$$

The main result which emerges from this analysis is that the equation of state of the dominant infrared contribution to the energy-momentum tensor $\tau_{\mu\nu}$ which describes back-reaction takes the form of a *negative cosmological constant*

$$p_{br} = -\rho_{br} \text{ with } \rho_{br} < 0 \,. \tag{113}$$

The second crucial result is that the magnitude of ρ_{br} increases as a function of time. This is due in part to the fact that, in an inflationary Universe, as time increases more and more wavelengths become longer than the Hubble radius and begin to contribute to ρ_{br}.

How large is the magnitude of back-reaction? The basic point is that since the amplitude of each fluctuation mode is small, we need a very large phase space of infrared modes in order to induce any interesting effects. In models with a very short period of primordial inflation, the back-reaction of long-wavelength cosmological fluctuations hence will not be important. However, in many single field models of inflation, in particular in those of chaotic inflation type [106], inflation lasts so long that the infrared back-reaction effects can build up to become important for the cosmological background dynamics.

To give an example, consider chaotic inflation with a potential

$$V(\varphi) = \frac{1}{2}m^2\varphi^2 \,. \tag{114}$$

In this case, the values of ϕ_k for long wavelength modes are well known (see e.g. [5]), and the integral in (109) can be easily performed, thus yielding explicit expressions for the dominant terms in the effective energy-momentum tensor. Comparing the resulting back-reaction energy density ρ_{br} with the background density ρ_0, we find

$$\frac{\rho_{br}(t)}{\rho_0} \simeq \frac{3}{4\pi}\frac{m^2\varphi_0^2(t_i)}{M_P^4}\left[\frac{\varphi_0(t_i)}{\varphi_0(t)}\right]^4 \,, \tag{115}$$

where M_P denotes the Planck mass. Without back-reaction, inflation would end [106] when $\varphi_0(t) \sim M_P$. Inserting this value into (115), we see that if $\varphi_0(t_i) > \varphi_{br} \sim m^{-1/3}M_P^{4/3}$, then back-reaction will become important before the end of inflation and may shorten the period of inflation. It is interesting to compare this value with the scale $\varphi_0(t_i) \sim \varphi_{sr} = m^{-1/2}M_P^{3/2}$ above which the stochastic terms in the scalar field equation of motion arising in the context of the stochastic approach to chaotic inflation [107, 108] are dominant. Notice that since $\varphi_{sr} \gg \varphi_{br}$ (recall that $m \ll M_P$), back-reaction effects can be very important in the entire range of field values relevant to stochastic inflation.

Since the back-reaction of cosmological fluctuations in an inflationary cosmology acts (see (113)) like a negative cosmological constant, and since the magnitude of the back-reaction effect increases in time, one may speculate

[97] that back-reaction will lead to a dynamical relaxation of the cosmological constant (see Tsamis & Woodard [100] for similar speculations based on the back-reaction of long wavelength gravitational waves).

The background metric $g_{\mu\nu}^{(0,br)}$ including back-reaction evolves as if the cosmological constant at time t were

$$\Lambda_{\text{eff}}(t) = \Lambda_0 + 8\pi G\rho_{br}(t) \tag{116}$$

and not the bare cosmological constant Λ_0. Hence one might hope to identify (116) with a time dependent effective cosmological constant. Since $|\rho_{br}(t)|$ increases as t grows, the effective cosmological constant will decay. Note that even if the initial magnitude of the perturbations is small, eventually (if inflation lasts a sufficiently long time) the back-reaction effect will become large enough to cancel any bare cosmological constant.

Furthermore, one might speculate that this dynamical relaxation mechanism for Λ will be self-regulating. As long as $\Lambda_{\text{eff}}(t) > 8\pi G\rho_m(t)$, where $\rho_m(t)$ stands for the energy density in ordinary matter and radiation, the evolution of $g_{\mu\nu}^{(0,br)}$ is dominated by $\Lambda_{\text{eff}}(t)$. Hence, the Universe will be undergoing accelerated expansion, more scales will be leaving the Hubble radius and the magnitude of the back-reaction term will increase. However, once $\Lambda_{\text{eff}}(t)$ falls below $\rho_m(t)$, the background will start to decelerate, scales will enter the Hubble radius, and the number of modes contributing to the back-reaction will decrease, thus reducing the strength of back-reaction. Hence, it is likely that there will be a scaling solution to the effective equation of motion for $\Lambda_{\text{eff}}(t)$ of the form

$$\Lambda_{\text{eff}}(t) \sim 8\pi G\rho_m(t). \tag{117}$$

Such a scaling solution would correspond to a contribution to the relative closure density of $\Omega_\Lambda \sim 1$.

There are important concerns about the above formalism, and even more so about the resulting speculations (many of these were first discussed in print in [105]). On a formal level, since our back-reaction effect is of second order in cosmological perturbation theory, it is necessary to demonstrate covariance of the proposed back-reaction equation (105) beyond linear order, and this has not been done. Next, it might be argued that by causality super-Hubble fluctuations cannot affect local observables. Thirdly, from an observational perspective one is not interested in the effect of fluctuations on the background metric (since what the background is cannot be determined precisely using local observations). Instead, one should compute the back-reaction of cosmological fluctuations on observables describing the local Hubble expansion rate. One might then argue that even if long-wavelength fluctuations have an effect on the background metric, they do not influence local observables. Finally, it is clear that the speculations in the previous section involve the extrapolation of perturbative physics deep into the non-perturbative regime.

These important issues have now begun to be addressed. Good physical arguments can be given [102, 103] supporting the idea that long-wavelength

fluctuations can effect local physics. Consider, for example, a black hole of mass M absorbing a particle of mass m. Even after this particle has disappeared beyond the horizon, its gravitational effects (in terms of the increased mass of the black hole) remain measurable to an external observer. A similar argument can be given in inflationary cosmology: consider an initial localized mass fluctuation with a characteristic physical length scale λ in an exponentially expanding background. Even after the length scale of the fluctuation redshifts to be larger than the Hubble radius, the gravitational potential associated with this fluctuation remains measurable. On a more technical level, it has recently been shown that super-Hubble scale (but sub-horizon-scale) metric fluctuations can be parametrically amplified during inflationary reheating [109, 42, 43, 44]. This clearly demonstrates a coupling between local physics and super-Hubble-scale fluctuations.

These arguments, however, make it even more important to focus on back-reaction effects of cosmological fluctuations on local physical observables rather than on the mathematical background metric. In recent work [99], the leading infrared back-reaction effects on a local observable measuring the Hubble expansion rate were calculated.

Consider a perfect fluid with velocity four vector u^α in an inhomogeneous cosmological geometry, then the local expansion rate which generalizes the Hubble expansion rate $H(t)$ of homogeneous isotropic Friedmann-Robertson-Walker cosmology is given by $\frac{1}{3}\Theta$, where Θ is the four divergence of u^α:

$$\Theta = u^\alpha_{;\alpha}, \tag{118}$$

the semicolon indicating the covariant derivative. In [99], the effects of cosmological fluctuations on this variable were computed to second order in perturbation theory. To leading order in the infrared expansion, the result is

$$\Theta = 3\frac{a'}{a^2}\left(1 - \phi + \frac{3}{2}\phi^2\right) - 3\frac{\phi'}{a}, \tag{119}$$

where the prime denotes the derivative with respect to conformal time. If we now calculate the spatial average of Θ, the term linear in ϕ vanishes, and - as expected - we are left with a quadratic back-reaction contribution.

Superficially, it appears from (119) that there is a non-vanishing back-reaction effect at quadratic order which is not suppressed for super-Hubble modes. However, we must be careful and evaluate Θ not at a constant value of the background coordinates, but rather at a fixed value of some physical observable. For example, if we work out the value of Θ in the case of a matter-dominated Universe, and express the result as a function of the proper time τ given by $d\tau^2 = a(\eta)^2(1+2\phi)d\eta^2$ instead of as a function of conformal time η, then we find that the leading infrared terms proportional to ϕ^2 exactly cancel, and that thus there is no un-suppressed infrared back-reaction on the local measure of the Hubble expansion rate.

A more relevant example with respect to the discussion in earlier sections is a model in which matter is given by a single scalar field. In this case, the

leading infrared back-reaction terms in Θ are again given by (119) which looks different from the background value $3H$. However, once again it is important to express Θ in terms of a physical background variable. If we choose the value of the matter field φ as this variable, we find after easy manipulations that, including only the leading infrared back-reaction terms,

$$\Theta(\varphi) = \sqrt{3}\sqrt{V(\varphi)}\,. \qquad (120)$$

Hence, once again the leading infrared back-reaction contributions vanish, as already found in the work of [103] which considered the leading infrared back-reaction effects on a local observable different than the one we have used, and applied very different methods [3].

However, in a model with two matter fields, it is clear that if we e.g. use the second matter field as a physical clock, then the leading infrared back-reaction terms will not cancel in Θ, and that thus in such models infrared back-reaction will be physically observable. The situation will be very much analogous to what happens in the case of parametric resonance of gravitational fluctuations during inflationary reheating. This process is a gauge artifact in single field models of inflation [42] (see also [45, 110, 46]), but it is real and unsuppressed in certain two field models [43, 44]. In the case of two field models, work on the analysis of the back-reaction effects of infrared modes on the observable representing the local Hubble expansion rate is in progress.

Provided that it can indeed be shown that infrared modes have a nontrivial gravitational back-reaction effect in interesting models at second order in perturbation theory, it then becomes important to extend the analysis beyond perturbation theory. For initial attempts in this direction see [111, 112].

Acknowledgements

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[3] For a different approach which also leads to the conclusion that there can be no back-reaction effects from infrared modes on local observables in models with a single matter component see [29].

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