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D branes on the conifold and N = 1 gauge/gravity dualities

Igor KLEBANOV
Princeton University
Department of Physics
Jadwin Hall
P.O. Box 708
Princeton, NJ 08544
U.S.A.

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D-Branes on the Conifold and $\mathcal{N}=1$ Gauge/Gravity Dualities*

Christopher P. Herzog, Igor R. Klebanov and Peter Ouyang
Department of Physics, Princeton University
Princeton, NJ 08544, USA

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Abstract

We review extensions of the AdS/CFT correspondence to gauge/ gravity dualities with $\mathcal{N}=1$ supersymmetry. In particular, we describe the gauge/gravity dualities that emerge from placing D3-branes at the apex of the conifold. We consider first the conformal case, with discussions of chiral primary operators and wrapped D-branes. Next, we break the conformal symmetry by adding a stack of partially wrapped D5-branes to the system, changing the gauge group and introducing a logarithmic renormalization group flow. In the gravity dual, the effect of these wrapped D5-branes is to turn on the flux of 3-form field strengths. The associated RR 2-form potential breaks the $U(1)$ R-symmetry to $\mathbb{Z}_{2M}$ and we study this phenomenon in detail. This extra flux also leads to deformation of the cone near the apex, which describes the chiral symmetry breaking and confinement in the dual gauge theory.

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1 Introduction

Comparison of a stack of D3-branes with the geometry it produces leads to formulation of duality between $\mathcal{N} = 4$ supersymmetric Yang-Mills theory and type II strings on $AdS_5 \times \mathbf{S}^5$ [1, 2, 3]. It is of obvious interest to consider more general dualities between gauge theories and string theories where some of the supersymmetry and/or conformal invariance are broken. These notes are primarily devoted to extensions of the AdS/CFT correspondence to theories with $\mathcal{N} = 1$ supersymmetry.

We first show how to break some of the supersymmetry without destroying conformal invariance. This may be accomplished through placing a stack of D3-branes at the apex of a Ricci flat 6-dimensional cone [4, 5, 6, 7]. Then we show how to break the conformal invariance in this set-up and to introduce logarithmic RG flow into the field theory. A convenient way to make the coupling constants run logarithmically is to introduce fractional D3-branes at the apex of the cone [8, 9, 10]; these fractional branes may be thought of as D5-branes wrapped over 2-cycles in the base of the cone. In the gravity dual the effect of these wrapped D5-branes is to turn on the flux of 3-form field strengths. This extra flux may lead to deformation of the cone near the apex, which describes the chiral symmetry breaking and confinement in the dual gauge theory [11]. We will start the notes with a very brief review of some of the basic facts about the AdS/CFT correspondence. For more background the reader may consult, for example, the review papers [12, 13].

To make the discussion more concrete, we consider primarily one particular example of a cone, the conifold. There are two reasons for this focus. The conifold has enough structure that many new aspects of AdS/CFT correspondence emerge that are not immediately visible for the simplest case, where the conifold is replaced with $\mathbb{R}^6$. At the same time, the conifold is simple enough that we can follow the program outlined in the paragraph above in great detail. This program eventually leads to the warped deformed conifold [11], a solution of type IIB supergravity that is dual to a certain $\mathcal{N} = 1$ supersymmetric $SU(N+M) \times SU(N)$ gauge theory in the limit of strong 't Hooft coupling. This solution encodes various interesting gauge theory phenomena in a dual geometrical language, such as the chiral anomaly, the logarithmic running of couplings, the duality cascade in the UV, and chiral symmetry breaking and confinement in the IR.

First, however, we review the original AdS/CFT correspondence. The duality between $\mathcal{N} = 4$ supersymmetric $SU(N)$ gauge theory and the $AdS_5 \times \mathbf{S}^5$ background of type IIB string theory [1, 2, 3] is usually motivated by considering a stack of a large number N of D3-branes. The SYM theory is the low-energy limit of the gauge theory on the stack of D3-branes. On the other hand, the curved background produced by the stack is

$$ds^2 = h^{-1/2}\left(-dt^2 + dx_1^2 + dx_2^2 + dx_3^2\right) + h^{1/2}\left(dr^2 + r^2 d\Omega_5^2\right) \ , \tag{1}$$

where $d\Omega_5^2$ is the metric of a unit 5-sphere and

$$h(r) = 1 + \frac{L^4}{r^4} \ . \tag{2}$$

This 10-dimensional metric may be thought of as a "warped product" of the $\mathbb{R}^{3,1}$ along the branes and the transverse space $\mathbb{R}^6$. Note that the dilaton, $\Phi = 0$, is constant, and the selfdual 5-form field strength is given by

$$F_5 = \mathcal{F}_5 + \star\mathcal{F}_5 \ , \qquad \mathcal{F}_5 = 16\pi(\alpha')^2 N \mathrm{vol}(\mathbf{S}^5) \ . \tag{3}$$

The normalization above is dictated by the quantization of Dp-brane tension which implies

$$\int_{\mathbf{S}^{8-p}} \star F_{p+2} = \frac{2\kappa^2 \tau_p N}{g_s} \ , \tag{4}$$

where

$$\tau_p = \frac{\sqrt{\pi}}{\kappa}(4\pi^2\alpha')^{(3-p)/2} \tag{5}$$

and $\kappa = 8\pi^{7/2} g_s \alpha'^2$ is the 10-dimensional gravitational constant. In particular, for $p = 3$ we have

$$\int_{\mathbf{S}^5} F_5 = (4\pi^2\alpha')^2 N \ , \tag{6}$$

which is consistent with (3) since the volume of a unit 5-sphere is

$$\mathrm{Vol}(\mathbf{S}^5) = \pi^3 \ .$$

Note that the 5-form field strength may also be written as

$$g_s F_5 = d^4x \wedge dh^{-1} - r^5 \frac{dh}{dr} \mathrm{vol}(\mathbf{S}^5) \ . \tag{7}$$

Then it is not hard to see that the Einstein equation

$$R_{MN} = \frac{g_s^2}{96} F_{MPQRS} F_N{}^{PQRS}$$

is satisfied. Since $-r^5 \frac{dh}{dr} = 4L^4$, we find by comparing with (3) that

$$L^4 = 4\pi g_s N \alpha'^2 \ . \tag{8}$$

A related way to determine the scale factor L is to equate the ADM tension of the supergravity solution with N times the tension of a single D3-brane [14]:

$$\frac{2}{\kappa^2} L^4 \mathrm{Vol}(\mathbf{S}^5) = \frac{\sqrt{\pi}}{\kappa} N \ . \tag{9}$$

This way we find

$$L^4 = \frac{\kappa N}{2\pi^{5/2}} = 4\pi g_s N \alpha'^2 \tag{10}$$

in agreement with the preceding paragraph.

The radial coordinate r is related to the scale in the dual gauge theory. The low-energy limit corresponds to $r \to 0$. In this limit the metric becomes

$$ds^2 = \frac{L^2}{z^2}\left(-dt^2 + d\vec{x}^2 + dz^2\right) + L^2 d\Omega_5^2 \ , \tag{11}$$

where $z = \frac{L^2}{r}$. This describes the direct product of 5-dimensional Anti-de Sitter space, AdS_5, and the 5-dimensional sphere, $\mathbf{S}^5$, with equal radii of curvature L.

An interesting generalization of the basic AdS/CFT correspondence [1, 2, 3] is found by studying branes at conical singularities [4, 5, 6, 7]. Consider a stack of D3-branes placed at the apex of a Ricci-flat 6-d cone Y_6 whose base is a 5-d Einstein manifold X_5. Comparing the metric with the D-brane description leads one to conjecture that type IIB string theory on $AdS_5 \times X_5$ is dual to the low-energy limit of the world volume theory on the D3-branes at the singularity. The equality of tensions now requires [15]

$$L^4 = \frac{\sqrt{\pi}\kappa N}{2\mathrm{Vol}(X_5)} = 4\pi g_s N \alpha'^2 \frac{\pi^3}{\mathrm{Vol}(X_5)} \ , \tag{12}$$

an important normalization formula which we will use in the following section.

The simplest examples of X_5 are the orbifolds $\mathbf{S}^5/\Gamma$ where Γ is a discrete subgroup of $SO(6)$ [4]. In these cases X_5 has the local geometry of a 5-sphere. The dual gauge theory is the IR limit of the world volume theory on a stack of N D3-branes placed at the orbifold singularity of $\mathbb{R}^6/\Gamma$. Such theories typically involve product gauge groups $SU(N)^k$ coupled to matter in bifundamental representations [16].

Constructions of the dual gauge theories for Einstein manifolds X_5 which are not locally equivalent to $\mathbf{S}^5$ are also possible. The simplest example is the Romans compactification on $X_5 = T^{1,1} = (SU(2) \times SU(2))/U(1)$ [17, 6]. The dual gauge theory is the conformal limit of the world volume theory on a stack of N D3-branes placed at the singularity of a Calabi-Yau manifold known as the conifold [6], which is a cone over $T^{1,1}$. Let us explain this connection in more detail.

2 D3-branes on the Conifold

The conifold may be described by the following equation in four complex variables,

$$\sum_{a=1}^{4} z_a^2 = 0 \ . \tag{13}$$

Since this equation is invariant under an overall real rescaling of the coordinates, this space is a cone. Remarkably, the base of this cone is precisely the space $T^{1,1}$ [18, 6]. In fact, the metric on the conifold may be cast in the form [18]

$$ds_6^2 = dr^2 + r^2 ds_{T^{1,1}}^2 \ , \tag{14}$$

where

$$ds_{T^{1,1}}^2 = \frac{1}{9}\Big(d\psi + \sum_{i=1}^{2} \cos\theta_i d\phi_i\Big)^2 + \frac{1}{6}\sum_{i=1}^{2}\left(d\theta_i^2 + \sin^2\theta_i d\phi_i^2\right) \tag{15}$$

is the metric on $T^{1,1}$. Here ψ is an angular coordinate which ranges from 0 to 4π, while (θ_1, ϕ_1) and (θ_2, ϕ_2) parametrize two $\mathbf{S}^2$s in a standard way. Therefore, this form of the metric shows that $T^{1,1}$ is an $\mathbf{S}^1$ bundle over $\mathbf{S}^2 \times \mathbf{S}^2$.

Now placing N D3-branes at the apex of the cone we find the metric

$$\begin{aligned} ds^2 &= \left(1 + \frac{L^4}{r^4}\right)^{-1/2} \left(-dt^2 + dx_1^2 + dx_2^2 + dx_3^2\right) \\ &+ \left(1 + \frac{L^4}{r^4}\right)^{1/2} \left(dr^2 + r^2 ds_{T^{1,1}}^2\right) \end{aligned} \tag{16}$$

whose near-horizon limit is $AdS_5 \times T^{1,1}$. Using the metric (15) it is not hard to find that the volume of $T^{1,1}$ is $\frac{16\pi^3}{27}$ [8]. From (12) it then follows that

$$L^4 = 4\pi g_s N (\alpha')^2 \frac{27}{16} = \frac{27\kappa N}{32\pi^{5/2}} \ . \tag{17}$$

The same logic that leads us to the maximally supersymmetric version of the AdS/CFT correspondence now shows that the type IIB string theory on this space should be dual to the infrared limit of the field theory on N D3-branes placed at the singularity of the conifold. Since Calabi-Yau spaces preserve 1/4 of the original supersymmetries we find that this should be an $\mathcal{N} = 1$ superconformal field theory. This field theory was constructed in [6]: it is $SU(N) \times SU(N)$ gauge theory coupled to two chiral superfields, A_i, in the $(\mathbf{N}, \overline{\mathbf{N}})$ representation and two chiral superfields, B_j, in the $(\overline{\mathbf{N}}, \mathbf{N})$ representation. The A's transform as a doublet under one of the global $SU(2)$s while the B's transform as a doublet under the other $SU(2)$.

A simple way to motivate the appearance of the fields A_i, B_j is to rewrite the defining equation of the conifold, (13), as

$$\det_{i,j} z_{ij} = 0 \ , \qquad z_{ij} = \frac{1}{\sqrt{2}} \sum_n \sigma^n_{ij} z_n \tag{18}$$

where σ^n are the Pauli matrices for $n = 1, 2, 3$ and σ^4 is i times the unit matrix. This quadratic constraint may be "solved" by the substitution

$$z_{ij} = A_i B_j \ , \tag{19}$$

where A_i, B_j are unconstrained variables. If we place a single D3-brane at the singularity of the conifold, then we find a $U(1) \times U(1)$ gauge theory coupled to fields A_1, A_2 with charges $(1, -1)$ and B_1, B_2 with charges $(-1, 1)$.

In constructing the generalization to the non-abelian theory on N D3-branes, cancellation of the anomaly in the $U(1)$ R-symmetry requires that the A's and the B's each have R-charge 1/2. For consistency of the duality it is necessary that we add an exactly marginal superpotential which preserves the $SU(2) \times SU(2) \times U(1)_R$ symmetry of the theory (this superpotential produces a critical line related to the radius of $AdS_5 \times T^{1,1}$). Since a marginal superpotential has R-charge equal to 2 it must be quartic, and the symmetries fix it uniquely up to overall normalization:

$$W = \epsilon^{ij} \epsilon^{kl} \mathrm{tr} A_i B_k A_j B_l \ . \tag{20}$$

Therefore, it was proposed in [6] that the $SU(N) \times SU(N)$ SCFT with this superpotential is dual to type IIB strings on $AdS_5 \times T^{1,1}$.

This proposal can be checked in an interesting way by comparing to a certain $AdS_5 \times \mathbf{S}^5/\mathbb{Z}_2$ background. If $\mathbf{S}^5$ is described by an equation

$$\sum_{i=1}^{6} x_i^2 = 1, \tag{21}$$

with real variables $x_1, \ldots, x_6$, then the $\mathbb{Z}_2$ acts as -1 on four of the x_i and as $+1$ on the other two. The importance of this choice is that this particular $\mathbb{Z}_2$ orbifold of $AdS_5 \times \mathbf{S}^5$ has $\mathcal{N} = 2$ superconformal symmetry. Using orbifold results for D-branes [16], this model has been identified [4] as an AdS dual of a $U(N) \times U(N)$ theory with hypermultiplets transforming in $(\mathbf{N}, \overline{\mathbf{N}}) \oplus (\overline{\mathbf{N}}, \mathbf{N})$. From an $\mathcal{N} = 1$ point of view, the hypermultiplets correspond to chiral multiplets A_k, B_l, $k, l = 1, 2$ in the $(\mathbf{N}, \overline{\mathbf{N}})$ and $(\overline{\mathbf{N}}, \mathbf{N})$ representations respectively. The model also contains, from an $\mathcal{N} = 1$ point of view, chiral multiplets Φ and $\tilde{\Phi}$ in the adjoint representations of the two $U(N)$'s. The superpotential is

$$g\text{Tr}\Phi(A_1B_1 - A_2B_2) + g\text{Tr}\tilde{\Phi}(B_1A_1 - B_2A_2) \ .$$

Now, let us add to the superpotential of this $\mathbb{Z}_2$ orbifold a relevant term,

$$\frac{m}{2}(\text{Tr}\Phi^2 - \text{Tr}\tilde{\Phi}^2) \ . \tag{22}$$

It is straightforward to see what this does to the field theory. We simply integrate out Φ and $\tilde{\Phi}$, to find the superpotential

$$-\frac{g^2}{m}\left[\text{Tr}(A_1B_1A_2B_2) - \text{Tr}(B_1A_1B_2A_2)\right] \ .$$

This expression is the same as (20), so the $\mathbb{Z}_2$ orbifold with relevant perturbation (22) apparently flows to the $T^{1,1}$ model associated with the conifold.

Let us try to understand why this works from the point of view of the geometry of $\mathbf{S}^5/\mathbb{Z}_2$. The perturbation in (22) is odd under exchange of the two $U(N)$'s. The exchange of the two $U(N)$'s is the quantum symmetry of the $AdS_5 \times \mathbf{S}^5/\mathbb{Z}_2$ orbifold – the symmetry that acts as -1 on string states in the twisted sector and $+1$ in the untwisted sector. Therefore we associate this perturbation with a twisted sector mode of string theory on $AdS_5 \times \mathbf{S}^5/\mathbb{Z}_2$. The twisted sector mode which is a relevant perturbation of the field theory is the blowup of the orbifold singularity of $\mathbf{S}^5/\mathbb{Z}_2$ into the smooth space $T^{1,1}$. A somewhat different derivation of the field theory on D3-branes at the conifold singularity, which is based on blowing up a $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold, was given in [7].

It is interesting to examine how various quantities change under the RG flow from the $\mathbf{S}^5/\mathbb{Z}_2$ theory to the $T^{1,1}$ theory. The behavior of the conformal anomaly (which is equal to the $U(1)_R^3$ anomaly) was studied in [15]. Using the fact that the chiral superfields carry R-charge equal to 1/2, on the field theory side it was found that

$$\frac{c_{IR}}{c_{UV}} = \frac{27}{32} \ . \tag{23}$$

On the other hand, all 3-point functions calculated from supergravity on $AdS_5 \times X_5$ carry normalization factor inversely proportional to Vol(X_5). Thus, on the supergravity side

$$\frac{c_{IR}}{c_{UV}} = \frac{\text{Vol }(\mathbf{S}^5/\mathbb{Z}_2)}{\text{Vol }(T^{1,1})} = \frac{27}{32} \ . \tag{24}$$

Thus, the supergravity calculation is in exact agreement with the field theory result (23) [15]. This is a striking and highly sensitive test of the $\mathcal{N} = 1$ dual pair constructed in [6, 7].

2.1 Dimensions of Chiral Operators

There are a number of further convincing checks of the duality between this field theory and type IIB strings on $AdS_5 \times T^{1,1}$. Here we discuss the supergravity modes which correspond to chiral primary operators. (For a more extensive analysis of the spectrum of the model, see [19].) For the $AdS_5 \times \mathbf{S}^5$ case, these modes are mixtures of the conformal factors of the AdS_5 and $\mathbf{S}^5$ and the 4-form field. The same has been shown to be true for the $T^{1,1}$ case [15, 20, 19]. In fact, we may keep the discussion of such modes quite general and consider $AdS_5 \times X_5$ where X_5 is any Einstein manifold.

The diagonalization of such modes carried out in [22] for the $\mathbf{S}^5$ case is easily generalized to any X_5. The mixing of the conformal factor and 4-form modes results in the following mass-squared matrix,

$$m^2 = \begin{pmatrix} E+32 & 8E \\ 4/5 & E \end{pmatrix} \tag{25}$$

where $E \geq 0$ is the eigenvalue of the Laplacian on X_5. The eigenvalues of this matrix are

$$m^2 = 16 + E \pm 8\sqrt{4+E}\ . \tag{26}$$

We will be primarily interested in the modes which correspond to picking the minus branch: they turn out to be the chiral primary fields. For such modes there is a possibility of m^2 falling in the range

$$-4 < m^2 < -3 \tag{27}$$

where there is a two-fold ambiguity in defining the corresponding operator dimension [21].

First, let us recall the $\mathbf{S}^5$ case where the spherical harmonics correspond to traceless symmetric tensors of $SO(6)$, $d^{(k)}_{i_1\ldots i_k}$. Here $E = k(k+4)$, and it seems that the bound (27) is satisfied for $k = 1$. However, this is precisely the special case where the corresponding mode is missing: for $k = 1$ one of the two mixtures is the singleton [22]. Thus, all chiral primary operators in the $\mathcal{N} = 4$ $SU(N)$ theory correspond to the conventional branch of dimension, Δ_+. It is now well-known that this family of operators with dimensions $\Delta = k$, $k = 2, 3, \ldots$ is $d^{(k)}_{i_1\ldots i_k}\mathrm{Tr}(X^{i_1}\ldots X^{i_k})$. The absence of $k = 1$ is related to the gauge group being $SU(N)$ rather than $U(N)$. Thus, in this case we do not encounter operator dimensions lower than 2.

The situation is different for $T^{1,1}$. Here there is a family of wave functions labeled by non-negative integer k, transforming under $SU(2) \times SU(2)$ as $(k/2, k/2)$, and with

$U(1)_R$ charge k [15, 20, 19]. The corresponding eigenvalues of the Laplacian are

$$E(k) = 3\left(k(k+2) - \frac{k^2}{4}\right) . \tag{28}$$

In [6] it was argued that the dual chiral operators are

$$\mathrm{tr}(A_{i_1} B_{j_1} \ldots A_{i_k} B_{j_k}) . \tag{29}$$

Since the F-term constraints in the gauge theory require that the i and the j indices are separately symmetrized, we find that their $SU(2) \times SU(2) \times U(1)$ quantum numbers agree with those given by the supergravity analysis. In the field theory the A's and the B's have dimension 3/4, hence the dimensions of the chiral operators are $3k/2$.

In studying the dimensions from the supergravity point of view, one encounters an interesting subtlety discussed in [21]. While for $k > 1$ only the dimension Δ_+ is admissible, for $k = 1$ one could pick either branch. Indeed, from (28) we have $E(1) = 33/4$ which falls within the range (27). Here we find that $\Delta_- = 3/2$, while $\Delta_+ = 5/2$. Since the supersymmetry requires the corresponding dimension to be 3/2, in this case we have to pick the unconventional Δ_- branch [21]. Choosing this branch for $k = 1$ and Δ_+ for $k > 1$ we indeed find following [15, 20, 19] that the supergravity analysis based on (26), (28) reproduces the dimensions $3k/2$ of the chiral operators (29). Thus, the conifold theory provides a simple example of AdS/CFT duality where the Δ_- branch has to be chosen for certain operators.

Let us also note that substituting $E(1) = 33/4$ into (26) we find $m^2 = -15/4$ which corresponds to a conformally coupled scalar in AdS_5 [22]. In fact, the short chiral supermultiplet containing this scalar includes another conformally coupled scalar and a massless fermion [19]. One of these scalar fields corresponds to the lower component of the superfield $\mathrm{Tr}(A_i B_j)$, which has dimension 3/2, while the other corresponds to the upper component which has dimension 5/2. Thus, the supersymmetry requires that we pick dimension Δ_+ for one of the conformally coupled scalars, and Δ_- for the other.

2.2 Wrapped D3-branes as "dibaryons"

It is of further interest to consider various branes wrapped over the cycles of $T^{1,1}$ and attempt to identify these states in the field theory [8]. For example, wrapped D3-branes turn out to correspond to baryon-like operators A^N and B^N where the indices of both $SU(N)$ groups are fully antisymmetrized. For large N the dimensions of such operators calculated from the supergravity are found to be $3N/4$ [8]. This is in complete agreement with the fact that the dimension of the chiral superfields at the fixed point is 3/4 and may be regarded as a direct supergravity calculation of an anomalous dimension in the dual gauge theory.

To show how this works in detail, we need to calculate the mass of a D3-brane wrapped over a minimal volume 3-cycle. An example of such a 3-cycle is the subspace at a constant value of (θ_2, ϕ_2), and its volume is found to be $V_3 = 8\pi^2 L^3/9$ [8]. The mass of the D3-brane wrapped over the 3-cycle is, therefore,

$$m = V_3 \frac{\sqrt{\pi}}{\kappa} = \frac{8\pi^{5/2} L^3}{9\kappa} \ . \tag{30}$$

For large mL, the corresponding operator dimension Δ approaches

$$mL = \frac{8\pi^{5/2} L^4}{9\kappa} = \frac{3}{4} N \ , \tag{31}$$

where in the last step we used (17).

Let us construct the corresponding operators in the dual gauge theory. Since the fields $A^\alpha_{k\beta}$, $k = 1, 2$, carry an index α in the $\mathbf{N}$ of $SU(N)_1$ and an index β in the $\overline{\mathbf{N}}$ of $SU(N)_2$, we can construct color-singlet "dibaryon" operators by antisymmetrizing completely with respect to both groups:

$$\mathcal{B}_{1l} = \epsilon_{\alpha_1 \ldots \alpha_N} \epsilon^{\beta_1 \ldots \beta_N} D_l^{k_1 \ldots k_N} \prod_{i=1}^{N} A^{\alpha_i}_{k_i \beta_i} \ , \tag{32}$$

where $D_l^{k_1 \ldots k_N}$ is the completely symmetric $SU(2)$ Clebsch-Gordon coefficient corresponding to forming the $\mathbf{N+1}$ of $SU(2)$ out of N 2's. Thus the $SU(2) \times SU(2)$ quantum numbers of $\mathcal{B}_{1l}$ are $(\mathbf{N+1}, \mathbf{1})$. Similarly, we can construct "dibaryon" operators which transform as $(\mathbf{1}, \mathbf{N+1})$,

$$\mathcal{B}_{2l} = \epsilon^{\alpha_1 \ldots \alpha_N} \epsilon_{\beta_1 \ldots \beta_N} D_l^{k_1 \ldots k_N} \prod_{i=1}^{N} B^{\beta_i}_{k_i \alpha_i} \ . \tag{33}$$

Under the duality these operators map to D3-branes classically localized at a constant (θ_1, ϕ_1). Thus, the existence of two types of "dibaryon" operators is related on the supergravity side to the fact that the base of the $U(1)$ bundle is $\mathbf{S}^2 \times \mathbf{S}^2$. At the quantum level, the collective coordinate for the wrapped D3-brane has to be quantized, and this explains its $SU(2) \times SU(2)$ quantum numbers [8]. The most basic check on the operator identification is that, since the exact dimension of the A's and the B's is 3/4, the dimension of the "dibaryon" operators agrees exactly with the supergravity calculation.

2.3 Other ways of wrapping D-branes over cycles of $T^{1,1}$

There are many other admissible ways of wrapping branes over cycles of $T^{1,1}$ (for a complete list, see [23]). For example, a D3-brane may be wrapped over a 2-cycle, which produces a string in AdS_5. The tension of such a "fat" string scales as $L^2/\kappa \sim N(g_sN)^{-1/2}/\alpha'$. The non-trivial dependence of the tension on the 't Hooft coupling g_sN indicates that such a string is not a BPS saturated object. This should be contrasted with the tension of a BPS string obtained in [24] by wrapping a D5-brane over $\mathbf{RP}^4$: $T \sim N/\alpha'$.

In discussing wrapped 5-branes, we will limit explicit statements to D5-branes: since a (p,q) 5-brane is an $SL(2,\mathbb{Z})$ transform of a D5-brane, our discussion may be generalized to wrapped (p,q) 5-branes using the $SL(2,\mathbb{Z})$ symmetry of the Type IIB string theory. If a D5-brane is wrapped over the entire $T^{1,1}$ then, according to the arguments in [24, 25], it serves as a vertex connecting N fundamental strings. Since each string ends on a charge in the fundamental representation of one of the $SU(N)$'s, the resulting field theory state is a baryon built out of external quarks.

If a D5-brane is wrapped over an $\mathbf{S}^3$, with its remaining two dimensions parallel to $\mathbf{R}^{3,1}$, then we find a domain wall in the dual field theory. Consider positioning a "fat" string made of a wrapped D3-brane orthogonally to the domain wall. As the string is brought through the membrane, a fundamental string stretched between them is created. The origin of this effect is creation of fundamental strings by crossing D5 and D3 branes, as shown in [26, 27].

We should note, however, that the domain wall positioned at some arbitrary AdS_5 radial coordinate r is not stable: its energy scales as r^3. Therefore, the only stable position is at $r = 0$ which is the horizon. The domain wall is tensionless there, and it is unlikely that this object really exists in the dual CFT. We will see, however, that the domain wall made of a wrapped D5-brane definitely exists in the $SU(N)\times SU(N+M)$ generalization of the gauge theory. This theory is confining and, correspondingly, the dual background does not have a horizon. In this case the wrapped D5-brane again falls to the minimum value of the radial coordinate, but its tension there is non-vanishing. This is the BPS domain wall which separates adjacent inequivalent vacua distinguished by the phase of the gluino condensate.

Finally, we show how to construct the $SU(N) \times SU(N+M)$ theories mentioned above. Consider a D5-brane wrapped over the 2-cycle, with its remaining directions filling $\mathbf{R}^{3,1}$. If this object is located at some fixed r, then it is a domain walls in AdS_5. The simplest domain wall is a D3-brane which is not wrapped over the compact manifold. Through an analysis of the five-form flux carried over directly from [24] one can conclude that when one crosses the domain wall, the effect in field theory is to change the gauge group from $SU(N) \times SU(N)$ to $SU(N+1) \times SU(N+1)$.

The field theory interpretation of a D5-brane wrapped around $\mathbf{S}^2$ is more interesting: if on one side of the domain wall we have the original $SU(N) \times SU(N)$ theory, then on the other side the theory is $SU(N) \times SU(N+1)$ [8]. The matter fields A_k and B_k are still bifundamentals, filling out $2(\mathbf{N}, \overline{\mathbf{N}+\mathbf{1}}) \oplus 2(\overline{\mathbf{N}}, \mathbf{N}+1)$. One piece of evidence for this claim is the way the D3-branes wrapped over the $\mathbf{S}^3$ behave when crossing the D5-brane domain wall. In homology there is only one $\mathbf{S}^3$, but for definiteness let us wrap the D3-brane around a particular three-sphere $\mathbf{S}^3_{(1)}$ which is invariant under the group $SU(2)_B$ under which the fields B_k transform. The corresponding state in the $SU(N) \times SU(N)$ field theory is $\mathcal{B}_1$ of (33). In the $SU(N) \times SU(N+1)$ theory, one has instead

$$\epsilon_{\alpha_1 \dots \alpha_N} \epsilon^{\beta_1 \dots \beta_{N+1}} A^{\alpha_1}_{\beta_1} \dots A^{\alpha_N}_{\beta_N} \qquad \text{or}$$

$$\epsilon_{\alpha_1 \dots \alpha_N} \epsilon^{\beta_1 \dots \beta_{N+1}} A^{\alpha_1}_{\beta_1} \dots A^{\alpha_N}_{\beta_N} A^{\alpha_{N+1}}_{\beta_{N+1}} \tag{34}$$

where we have omitted $SU(2)$ indices. Either the upper index β_{N+1}, indicating a fundamental of $SU(N+1)$, or the upper index α_{N+1}, indicating a fundamental of $SU(N)$, is free.

How can this be in supergravity? The answer is simple: the wrapped D3-brane must have a string attached to it. Indeed, after a wrapped D3-brane has passed through the wrapped D5-brane domain wall, it emerges with a string attached to it due to the string creation by crossing D-branes which together span 8 dimensions [26, 27]. Calculating the tension of a wrapped D5-brane as a function of r shows that it scales as r^4/L^2. Hence, the domain wall is not stable, but in fact wants to move towards $r = 0$. We will assume that the wrapped D5-branes "fall" behind the horizon and are replaced by their flux in the SUGRA background. This gives a well-defined way of constructing the SUGRA duals of the $SU(N) \times SU(N+M)$ gauge theories.

The D5-branes wrapped over 2-cycles are examples of a more general phenomenon. For many singular spaces Y_6 there are fractional D3-branes which can exist only within the singularity [28, 29, 8, 9]. These fractional D3-branes are D5-branes wrapped over (collapsed) 2-cycles at the singularity. In the case of the conifold, the singularity is a point. The addition of M fractional branes at the singular point changes the gauge group to $SU(N+M) \times SU(N)$; the four chiral superfields remain, now in the representation $(\mathbf{N}+\mathbf{M}, \overline{\mathbf{N}})$ and its conjugate, as does the superpotential [8, 9]. The theory is no longer conformal. Instead, the relative gauge coupling $g_1^{-2} - g_2^{-2}$ runs logarithmically, as pointed out in [9], where the supergravity equations corresponding to this situation were solved to leading order in M/N. In [10] this solution was completed to all orders; the conifold suffers logarithmic warping, and the relative gauge coupling runs logarithmically at all scales. The D3-brane charge, i.e. the 5-form flux, decreases logarithmically as well. However, the logarithm in the solution is not cut off at small radius; the D3-brane charge eventually becomes negative and the metric becomes singular.

In [10] it was conjectured that this solution corresponds to a flow in which the gauge group factors repeatedly drop in size by M units, until finally the gauge groups are perhaps $SU(2M) \times SU(M)$ or simply $SU(M)$. It was further suggested that the strong dynamics of this gauge theory would resolve the naked singularity in the metric. The flow is in fact an infinite series of Seiberg duality transformations — a "duality cascade" — in which the number of colors repeatedly drops by M units [11]. Once the number of colors in the smaller gauge group is fewer than M, non-perturbative effects become essential. We will show that these gauge theories have an exact anomaly-free $\mathbb{Z}_{2M}$ R-symmetry, which is broken dynamically, as in pure $\mathcal{N} = 1$ Yang-Mills theory, to $\mathbb{Z}_2$. In the supergravity, this occurs through the deformation of the conifold. In short, the resolution of the naked singularity found in [10] occurs through the chiral symmetry breaking of the gauge theory. The resulting space, *a warped deformed conifold*, is completely nonsingular and without a horizon, leading to confinement [11].

3 The RG cascade

The addition of M fractional 3-branes (wrapped D5-branes) at the singular point changes the gauge group to $SU(N+M) \times SU(N)$. Let us consider the effect on the dual supergravity background of adding M wrapped D5-branes. The D5-branes serve as sources of the magnetic RR 3-form flux through the $\mathbf{S}^3$ of $T^{1,1}$. Therefore, the supergravity dual of this field theory involves M units of the 3-form flux, in addition to N units of the 5-form flux:

$$\frac{1}{4\pi^2\alpha'}\int_{\mathbf{S}^3} F_3 = M \ , \qquad \frac{1}{(4\pi^2\alpha')^2}\int_{T^{1,1}} F_5 = N \ . \qquad (35)$$

The coefficients above follow from the quantization rule (4). The warped conifold solution with such fluxes was constructed in [10].

It will be useful to employ the following basis of 1-forms on the compact space [30]:

$$g^1 = \frac{e^1 - e^3}{\sqrt{2}} \ , \qquad g^2 = \frac{e^2 - e^4}{\sqrt{2}} \ ,$$

$$g^3 = \frac{e^1 + e^3}{\sqrt{2}} \ , \qquad g^4 = \frac{e^2 + e^4}{\sqrt{2}} \ ,$$
$$g^5 = e^5 \ , \tag{36}$$

where

$$\begin{aligned} e^1 &\equiv -\sin\theta_1 d\phi_1 \ , \qquad e^2 \equiv d\theta_1 \ , \\ e^3 &\equiv \cos\psi \sin\theta_2 d\phi_2 - \sin\psi d\theta_2 \ , \\ e^4 &\equiv \sin\psi \sin\theta_2 d\phi_2 + \cos\psi d\theta_2 \ , \\ e^5 &\equiv d\psi + \cos\theta_1 d\phi_1 + \cos\theta_2 d\phi_2 \ . \end{aligned} \tag{37}$$

In terms of this basis, the Einstein metric on $T^{1,1}$ assumes the form

$$ds^2_{T^{1,1}} = \frac{1}{9}(g^5)^2 + \frac{1}{6}\sum_{i=1}^{4}(g^i)^2 \ . \tag{38}$$

Keeping track of the normalization factors, in order to be consistent with the quantization conditions (35),

$$F_3 = \frac{M\alpha'}{2}\omega_3 \ , \qquad B_2 = \frac{3g_s M\alpha'}{2}\omega_2 \ln(r/r_0) \ , \tag{39}$$

$$H_3 = dB_2 = \frac{3g_s M\alpha'}{2r} dr \wedge \omega_2 \ , \tag{40}$$

where

$$\omega_2 = \frac{1}{2}(g^1 \wedge g^2 + g^3 \wedge g^4) = \frac{1}{2}(\sin\theta_1 d\theta_1 \wedge d\phi_1 - \sin\theta_2 d\theta_2 \wedge d\phi_2) \ , \tag{41}$$

$$\omega_3 = \frac{1}{2} g^5 \wedge (g^1 \wedge g^2 + g^3 \wedge g^4) \ . \tag{42}$$

One can show that [31]

$$\int_{S^2} \omega_2 = 4\pi \ , \qquad \int_{S^3} \omega_3 = 8\pi^2 \tag{43}$$

where the S^2 is parametrized by $\psi = 0$, $\theta_1 = \theta_2$ and $\phi_1 = -\phi_2$, and the S^3 by $\theta_2 = \phi_2 = 0$. As a result, the quantization condition for RR 3-form flux is obeyed.

Both ω_2 and ω_3 are closed. Note also that

$$g_s \star_6 F_3 = H_3 \ , \qquad g_s F_3 = - \star_6 H_3 \ , \tag{44}$$

where $\star_6$ is the Hodge dual with respect to the metric ds_6^2. Thus, the complex 3-form G_3 satisfies the self-duality condition

$$\star_6 G_3 = iG_3 \ , \qquad G_3 = F_3 - \frac{i}{g_s} H_3 \ . \tag{45}$$

Note that the self-duality fixes the relative factor of 3 in (39) (see (14), (15)). We will see that this geometrical factor is crucial for reproducing the well-known factor of 3 in the $\mathcal{N} = 1$ beta functions.

It follows from (44) that

$$g_s^2 F_3^2 = H_3^2 \ , \tag{46}$$

which implies that the dilaton is constant, $\Phi = 0$. Since $F_{3\mu\nu\lambda} H_3^{\mu\nu\lambda} = 0$, the RR scalar vanishes as well.

The 10-d metric found in [10] has the structure of a "warped product" of $\mathbb{R}^{3,1}$ and the conifold:

$$ds_{10}^2 = h^{-1/2}(r) dx_n dx_n + h^{1/2}(r)(dr^2 + r^2 ds_{T^{1,1}}^2) \ . \tag{47}$$

The solution for the warp factor h may be determined from the trace of the Einstein equation:

$$R = \frac{1}{24}(H_3^2 + g_s^2 F_3^2) = \frac{1}{12} H_3^2 \ . \tag{48}$$

This implies

$$- h^{-3/2} \frac{1}{r^5} \frac{d}{dr}(r^5 h') = \frac{1}{6} H_3^2 \ . \tag{49}$$

Integrating this differential equation, we find that

$$h(r) = \frac{27\pi(\alpha')^2 [g_s N + a(g_s M)^2 \ln(r/r_0) + a(g_s M)^2/4]}{4r^4} \tag{50}$$

with $a = 3/(2\pi)$.

An important feature of this background is that $\tilde{F}_5$ acquires a radial dependence [10]. This is because

$$\tilde{F}_5 = F_5 + B_2 \wedge F_3 \ , \qquad F_5 = dC_4 \ , \tag{51}$$

and $\omega_2 \wedge \omega_3 = 54 \mathrm{vol}(T^{1,1})$. Thus, we may write

$$\tilde{F}_5 = \mathcal{F}_5 + \star \mathcal{F}_5 \ , \qquad \mathcal{F}_5 = 27\pi \alpha'^2 N_{eff}(r) \mathrm{vol}(\mathrm{T}^{1,1}) \ , \tag{52}$$

and

$$N_{eff}(r) = N + \frac{3}{2\pi} g_s M^2 \ln(r/r_0) \ . \tag{53}$$

The novel phenomenon in this solution is that the 5-form flux present at the UV scale $r = r_0$ may completely disappear by the time we reach a scale where $N_{eff} = 0$. The non-conservation of the flux is due to the type IIB SUGRA equation

$$d\tilde{F}_5 = H_3 \wedge F_3 \ . \tag{54}$$

A related fact is that $\int_{\mathbf{S}^2} B_2$ is no longer a periodic variable in the SUGRA solution once the M fractional branes are introduced: as the B_2 flux goes through a period, $N_{eff}(r) \to N_{eff}(r) - M$ which has the effect of decreasing the 5-form flux by M units. Note from (53) that for a single cascade step $N_{eff}(r) \to N_{eff}(r) - M$ the radius changes by a factor $r_2/r_1 = \exp(-2\pi/3g_sM)$, agreeing with a result of [32].

Due to the non-vanishing RHS of (54), $\frac{1}{(4\pi^2\alpha')^2}\int_{T^{1,1}} \tilde{F}_5$ is not quantized. We may identify this quantity with N_{eff} defining the gauge group $SU(N_{eff}+M) \times SU(N_{eff})$ only at special radii $r_k = r_0 \exp(-2\pi k/3g_sM)$ where k is an integer. Thus, $N_{eff} = N - kM$. Furthermore, we believe that the continuous logarithmic variation of $N_{eff}(r)$ is related to continuous reduction in the number of degrees of freedom as the theory flows to the IR. Some support for this claim comes from studying the high-temperature phase of this theory using black holes embedded into asymptotic KT geometry [33]. The effective number of degrees of freedom computed from the Bekenstein–Hawking entropy grows logarithmically with the temperature, in agreement with (53).

The metric (47) has a naked singularity at $r = r_s$ where $h(r_s) = 0$. Writing

$$h(r) = \frac{L^4}{r^4}\ln(r/r_s) \ , \qquad L^2 = \frac{9g_sM\alpha'}{2\sqrt{2}} \ , \tag{55}$$

we find a purely logarithmic RG cascade:

$$ds^2 = \frac{r^2}{L^2\sqrt{\ln(r/r_s)}}dx_n dx_n + \frac{L^2\sqrt{\ln(r/r_s)}}{r^2}dr^2 + L^2\sqrt{\ln(r/r_s)}ds^2_{T^{1,1}} \ . \tag{56}$$

Since $T^{1,1}$ expands slowly toward large r, the curvatures decrease there so that corrections to the SUGRA become negligible. Therefore, even if g_sM is very small, this SUGRA solution is reliable for sufficiently large radii where $g_sN_{eff}(r) \gg 1$. In this regime the separation between the cascade steps is very large, so that the SUGRA calculation of the β-functions may be compared with $SU(N_{eff}+M) \times SU(N_{eff})$ gauge theory. We will work near $r = r_0$ where N_{eff} may be replaced by N.

3.1 Matching of the β-functions

In order to match the two gauge couplings to the moduli of the type IIB theory on $AdS_5 \times T^{1,1}$, one notes that the integrals over the $\mathbf{S}^2$ of $T^{1,1}$ of the NS-NS and R-R

2-form potentials, B_2 and C_2, are moduli. In particular, the two gauge couplings are determined as follows [6, 7]:[1]

$$\frac{4\pi^2}{g_1^2} + \frac{4\pi^2}{g_2^2} = \frac{\pi}{g_s e^\Phi} , \tag{57}$$

$$\left[\frac{4\pi^2}{g_1^2} - \frac{4\pi^2}{g_2^2}\right] g_s e^\Phi = \frac{1}{2\pi\alpha'}\left(\int_{\mathbf{S}^2} B_2\right) - \pi \pmod{2\pi} . \tag{58}$$

From the quantization condition on H_3, $\frac{1}{2\pi\alpha'}(\int_{\mathbf{S}^2} B_2)$ must be a periodic variable with period 2π. This periodicity is crucial for the cascade phenomenon. These equations are crucial for relating the SUGRA background to the field theory β-functions when the theory is generalized to $SU(N+M)\times SU(N)$ [9, 10].

In gauge/gravity duality the 5-dimensional radial coordinate defines the RG scale of the dual gauge theory [1, 2, 3, 36, 35]. There are different ways of establishiing the precise relation. The simplest one is to identify the field theory energy scale Λ with the energy of a stretched string ending on a probe brane positioned at radius r. For all metrics of the form (47) this gives

$$\Lambda \sim r . \tag{59}$$

In this section we adopt this UV/IR relation, which typically corresponds to the Wilsonian renormalization group.

Now we are ready to interpret the solution of [10] in terms of RG flow in the dual $SU(N+M)\times SU(N)$ gauge theory. The constancy of the dilaton translates into the vanishing of the β-function for $\frac{8\pi^2}{g_1^2}+\frac{8\pi^2}{g_2^2}$. Substituting the solution for B_2 into (58) we find

$$\frac{8\pi^2}{g_1^2} - \frac{8\pi^2}{g_2^2} = 6M\ln(r/r_s) + \text{const} . \tag{60}$$

Since $\ln(r/r_s) = \ln(\Lambda/\mu)$, (60) implies a logarithmic running of $\frac{1}{g_1^2}-\frac{1}{g_2^2}$ in the $SU(N+M)\times SU(N)$ gauge theory. As we mentioned earlier, this SUGRA result is reliable for any value of g_sM provided that $g_sN \gg 1$. We may consider, for instance, $g_sM \ll 1$ so that the cascade jumps are well-separated.

[1]Exactly the same relations apply to the $\mathcal{N}=2$ supersymmetric $\mathbb{Z}_2$ orbifold theory [4, 34].

Let us compare with the Shifman–Vainshtein β-functions [37]:[2]

$$\frac{d}{d\log(\Lambda/\mu)}\frac{8\pi^2}{g_1^2} = 3(N+M) - 2N(1-\gamma)\ , \tag{61}$$

$$\frac{d}{d\log(\Lambda/\mu)}\frac{8\pi^2}{g_2^2} = 3N - 2(N+M)(1-\gamma)\ , \tag{62}$$

where γ is the anomalous dimension of operators $\mathrm{Tr}A_iB_j$. The conformal invariance of the field theory for $M=0$, and symmetry under $M\to -M$, require that $\gamma = -\frac{1}{2}+O[(M/N)^{2n}]$ where n is a positive integer [11]. Taking the difference of the two equations in (61) we then find

$$\begin{aligned}\frac{8\pi^2}{g_1^2}-\frac{8\pi^2}{g_2^2} &= M\ln(\Lambda/\mu)[3+2(1-\gamma)] \\ &= 6M\ln(\Lambda/\mu)(1+O[(M/N)^{2n}])\ .\end{aligned} \tag{63}$$

Remarkably, the coefficient $6M$ is in *exact* agreement with the result (60) found on the SUGRA side. This consitutes a geometrical explanation of a field theory β-function, including its normalization.

We may also trace the jumps in the rank of the gauge group to a well-known phenomenon in the dual $\mathcal{N}=1$ field theory, namely, Seiberg duality [40]. The essential observation is that $1/g_1^2$ and $1/g_2^2$ flow in opposite directions and, according to (61), there is a scale where the $SU(N+M)$ coupling, g_1, diverges. To continue past this infinite coupling, we perform a $\mathcal{N}=1$ duality transformation on this gauge group factor. The $SU(N+M)$ gauge factor has $2N$ flavors in the fundamental representation. Under a Seiberg duality transformation, this becomes an $SU(2N-[N+M]) = SU(N-M)$ gauge group. Thus we obtain an $SU(N)\times SU(N-M)$ theory which resembles closely the theory we started with [11].

As the theory flows to the IR, the cascade must stop, however, because negative N is physically nonsensical. Thus, we should not be able to continue the solution (56) to the region where N_{eff} is negative. To summarize, the fact that the solution of [10] is singular tells us that it has to be modified in the IR. The necessary modification proceeds via the deformation of the conifold, and is discussed in section 5.

[2] These expressions for the β-functions differ from the standard NSVZ form [38] by a factor of $1/(1-g^2N_c/8\pi^2)$. The difference comes from the choice of normalization of the vector superfields. We choose the normalization so that the relevant kinetic term in the field theory action is $\frac{1}{2g^2}\int d^4xd^2\theta Tr(W^\alpha W_\alpha)+$ h.c.; this choice is dictated by the form of the supergravity action and differs from the canonical normalization by a factor of $1/g^2$. With this convention the additional factor in the β-function does not appear. A nice review of the derivation of the exact β-functions is in [39].

4 The Chiral Anomaly

In theories with $\mathcal{N} = 1$ supersymmetry, β-functions are related to chiral anomalies [37]. The essential mechanism is the β-functions contribute to the trace anomaly, $\langle T_i^i \rangle$, which is related by supersymmetry to the divergence of the $U(1)_R$ current, $\partial_i J^i$. In the previous section we showed how the logarithmic running of the gauge couplings manifests itself in the dual supergravity solution of [10]. Here we show that the chiral anomaly can be read off the solution as well. Although the metric has a continuous $U(1)_R$ symmetry, the full supergravity solution is only invariant under a $\mathbb{Z}_{2M}$ subgroup of this $U(1)$. In the dual quantum field theory there are chiral fermions charged under the $U(1)_R$, and so we can understand the R-symmetry breaking as an effect of the chiral anomaly. Anomalies are especially interesting creatures for the gauge/gravity duality, because the Adler-Bardeen theorem [41] guarantees that anomaly coefficients computed at one loop are exact, with no radiative corrections; the significance of this fact is that we can compute anomaly coefficients in the field theory at weak coupling, then extrapolate the results to strong coupling, where we can use dual gravity methods to check the calculation. In this section we will study some aspects of the anomaly in detail for the cascading gauge theory.

There are three lessons that we can take away from this analysis [42]. First, the anomaly coefficients computed on each side of the duality agree exactly, even for our non-conformal cascading theory with only $\mathcal{N} = 1$ supersymmetry; although this result is hardly surprising, it is a nice check of the duality. Second, the symmetry breaking is a classical effect on the gravity side. There is no need to appeal to instantons, which is a good thing as they do not appear anywhere explicitly in the gravity dual. Finally, the R-symmetry is broken *spontaneously* in the supergravity solution – the bulk vector field dual to the R-current of the gauge theory acquires a mass. The symmetry breaking then appears "anomalous" if one insists on a four-dimensional description.

4.1 The Anomaly as a Classical Effect in Supergravity

The asymptotic UV metric (47,15) has a $U(1)$ symmetry associated with the rotations of the angular coordinate $\beta = \psi/2$, normalized so that β has period 2π. This is the R-symmetry of the dual gauge theory. It is crucial, however, that the background value of the R-R 2-form C_2 does not have this continuous symmetry. Indeed, although F_3 is $U(1)$ symmetric, there is no smooth global expression for C_2. Locally, we may write

$$C_2 \to M\alpha'\beta\omega_2 \ . \tag{64}$$

This expression is not single-valued as a function of the angular variable β, but it is single-valued up to a gauge transformation, so that $F_3 = dC_2$ is single-valued. In fact, F_3 is completely independent of β. Because of the explicit β dependence, C_2 is not $U(1)$-invariant. Under the transformation $\beta \to \beta + \epsilon$,

$$C_2 \to C_2 + M\alpha' \epsilon \omega_2 \ . \tag{65}$$

Since $\int_{\mathbf{S}^2} C_2$ is defined modulo $4\pi^2\alpha'$, a gauge transformation can shift $C_2/(4\pi^2\alpha')$ by an arbitrary integer multiple of $\omega_2/(4\pi)$, so $\beta \to \beta + \epsilon$ is a symmetry precisely if ϵ is an integer multiple of π/M. Because ϵ is anyway only defined mod 2π, a $\mathbb{Z}_{2M}$ subgroup of the $U(1)$ leaves fixed the asymptotic values of the fields, and thus corresponds to a symmetry of the system. This $\mathbb{Z}_{2M}$ is a symmetry since it respects the asymptotic values of the fields.

Let us compare the above analysis with the gauge theory. (A similar comparison for the case of an $\mathcal{N} = 2$ orbifold theory appeared in [43].) As pointed out in [6], the integral of the RR 2-form potential C_2 over the $\mathbf{S}^2$ of $T^{1,1}$ is a modulus. Because the integral of B_2 was dual to the difference of gauge couplings for the two gauge groups, it is natural that the integral of C_2 is dual to the difference of Θ-angles (it is possible to check this statement explicitly in orbifold backgrounds). The Θ-angles are given by

$$\Theta_1 - \Theta_2 = \frac{1}{\pi\alpha'} \int_{S^2} C_2 \ , \qquad \Theta_1 + \Theta_2 \sim C \ , \tag{66}$$

where C is the RR scalar, which vanishes for the case under consideration. Using the fact that $\int_{\mathbf{S}^2} \omega_2 = 4\pi$, we find that the small $U(1)$ rotation $\beta \to \beta + \epsilon$ induces

$$\Theta_1 = -\Theta_2 = 2M\epsilon \ . \tag{67}$$

With a conventional normalization, the Θ terms appear in the gauge theory action as

$$\int d^4x \left(\frac{\Theta_1}{32\pi^2} F^a_{ij} \tilde{F}^{aij} + \frac{\Theta_2}{32\pi^2} G^b_{ij} \tilde{G}^{bij}\right) . \tag{68}$$

If we assume that ϵ is a function of the 4 world volume coordinates x^i, then under the $U(1)$ rotation (67) the terms linear in ϵ in the dual gauge theory (68) are

$$\int d^4x [-\epsilon \partial_i J^i + \frac{M\epsilon}{16\pi^2} (F^a_{ij} \tilde{F}^{aij} - G^b_{ij} \tilde{G}^{bij})] \ , \tag{69}$$

where J^i is the chiral R-current. The appearance of the second term is due to the non-invariance of C_2 under the $U(1)$ rotation. Varying with respect to ϵ, we therefore obtain

$$\partial_i J^i = \frac{M}{16\pi^2} \left(F^a_{ij} \tilde{F}^{aij} - G^b_{ij} \tilde{G}^{bij}\right) \ . \tag{70}$$

This anomaly equation, derived from supergravity, agrees exactly with our expectations from the gauge theory. A standard result of quantum field theory is that in a theory with chiral fermions charged under a global $U(1)$ symmetry of the classical Lagrangian, the Noether current associated with that symmetry is not generally conserved but instead obeys the equation

$$\partial_i J^i = \frac{1}{32\pi^2} \sum_m n_m R_m F^a_{ij} \tilde{F}^{aij} \tag{71}$$

where n_m is the number of chiral fermions with R-charge R_m circulating in the loop of the relevant triangle diagram. In the case of interest, there are two gauge groups, so let us define F^a_{ij} and G^b_{ij} to be the field strengths of $SU(N+M)$ and $SU(N)$ respectively. Now, the chiral superfields A_i, B_j contribute $2N$ flavors to the gauge group $SU(N+M)$, and each one carries R-charge $1/2$. The chiral fermions which are their superpartners have R-charge $-1/2$ while the gluinos have R-charge 1. Therefore, the anomaly coefficient is $\frac{M}{16\pi^2}$. An equivalent calculation for the $SU(N)$ gauge group with $2(N+M)$ flavors produces the opposite anomaly, so the anomaly equation as computed from field theory is just (70).

The upshot of the calculation presented above is that the chiral anomaly of the $SU(N+M) \times SU(N)$ gauge theory is encoded in the ultraviolet (large r) behavior of the dual classical supergravity solution. No additional fractional D-instanton effects are needed to explain the anomaly. Thus, as often occurs in the gauge/gravity duality, a quantum effect on the gauge theory side turns into a classical effect in supergravity. Similar methods have been used to describe chiral anomalies in other supersymmetric gauge theories [42, 43, 44].

4.2 The Anomaly as Spontaneous Symmetry Breaking in AdS_5

Let us look for a deeper understanding of the anomaly from the dual gravity point of view. On the gauge theory side, the R-symmetry is global, but in the gravity dual it as usual becomes a gauge symmetry, which must not be anomalous, or the theory would not make sense at all. Rather, we will find that the gauge symmetry is spontaneously broken: the 5-d vector field dual to the R-current of the gauge theory 'eats' the scalar dual to the difference of the theta angles and acquires a mass.[3] A closely related mechanism was observed in studies of RG flows from the dual gravity point of view

[3] The connection between anomalies in a D-brane field theory and spontaneous symmetry breaking in string theory was previously noted in [45] (and probably elsewhere in the literature).

[46, 48]. There R-current conservation was violated not through anomalies but by turning on relevant perturbations or expectation values for fields. In these cases it was shown [46, 48] that the 5-d vector field dual to the R-current acquires a mass through the Higgs mechanism. We will show that symmetry breaking through anomalies can also have the bulk Higgs mechanism as its dual.

In the absence of fractional branes there are no background three-form fluxes, so the $U(1)$ R-symmetry is a true symmetry of the field theory. Because the R-symmetry is realized geometrically by invariance under a rigid shift of the angle β, it becomes a local symmetry in the full gravity theory, and the associated gauge fields $A = A_\mu dx^\mu$ appear as fluctuations of the ten-dimensional metric and RR four-form potential [22, 19]. The natural metric ansatz is of the familiar Kaluza-Klein form:

$$\begin{aligned} ds^2 &= h(r)^{-1/2}\left(dx_n dx^n\right) + \\ & h(r)^{1/2} r^2 \left[\frac{dr^2}{r^2} + \frac{1}{9}\left(g^5 - 2A\right)^2 + \frac{1}{6}\sum_{r=1}^{4}\left(g^r\right)^2\right], \end{aligned} \tag{72}$$

where $h(r) = L^4/r^4$, and $L^4 = \frac{27}{16}(4\pi\alpha'^2 g_s N)$. It is convenient to define the one-form $\chi = g^5 - 2A$, which is invariant under the combined gauge transformations

$$\beta \to \beta + \lambda, \qquad A \to A + d\lambda. \tag{73}$$

The equations of motion for the field A_μ appear as the $\chi\mu$ components of Einstein's equations,

$$R_{MN} = \frac{g_s^2}{4\cdot 4!}\tilde{F}_{MPQRS}\tilde{F}_N{}^{PQRS}. \tag{74}$$

The five-form flux will also fluctuate when we activate the Kaluza-Klein gauge field; indeed, the unperturbed $\tilde{F}_5$ of (52) is not self-dual with respect to the gauged metric (72). An appropriate ansatz to linear order in A is

$$\begin{aligned} \tilde{F}_5 = dC_4 &= \frac{1}{g_s} d^4x \wedge dh^{-1} + \frac{\pi\alpha'^2 N}{4}\Big[\chi \wedge g^1 \wedge g^2 \wedge g^3 \wedge g^4 \\ & \qquad - dA \wedge g^5 \wedge dg^5 + \frac{3}{L}\star_5 dA \wedge dg^5\Big]. \end{aligned} \tag{75}$$

The five-dimensional Hodge dual $\star_5$ is defined with respect to the AdS$_5$ metric $ds_5{}^2 = h^{-1/2} dx_n dx^n + h^{1/2} dr^2$. It is straightforward to show that the supergravity field equation $d\tilde{F}_5 = 0$ implies that the field A satisfies the equation of motion for a massless vector field in AdS$_5$ space:

$$d \star_5 dA = 0\ . \tag{76}$$

Using the identity $dg^5 \wedge dg^5 = -2g^1 \wedge g^2 \wedge g^3 \wedge g^4$, we can check that the expression for C_4 is

$$
\begin{aligned}
C_4 &= \frac{1}{g_s} h^{-1} d^4x + \frac{\pi \alpha'^2 N}{2} \Big[\beta g^1 \wedge g^2 \wedge g^3 \wedge g^4 - \frac{1}{2} A \wedge dg^5 \wedge g^5 \\
&\qquad - \frac{3}{2r} h^{-1/4} \star_5 dA \wedge g^5 \Big]. \qquad (77)
\end{aligned}
$$

Another way to see that A is a massless vector in AdS_5 is to consider the Ricci scalar for the metric (72)

$$
R = R(A=0) - \frac{h^{1/2} r^2}{9} F_{\mu\nu} F^{\mu\nu} \qquad (78)
$$

so that on reduction from ten dimensions the five-dimensional supergravity action will contain the action for a massless vector field.

The story changes when we add wrapped D5-branes. As described in Section 2, the 5-branes introduce M units of RR flux through the three-cycle of $T^{1,1}$. Now, the new wrinkle is that the RR three-form flux of (39) is not gauge-invariant with respect to shifts of β (73). To restore the gauge invariance, we introduce a new field $\theta \sim \int_{S^2} C_2$:

$$
F_3 = dC_2 = \frac{M\alpha'}{2} \left(g^5 + 2\partial_\mu \theta dx^\mu \right) \wedge \omega_2 \qquad (79)
$$

so that F_3 is invariant under the gauge transformation $\beta \to \beta + \lambda$, $\theta \to \theta - \lambda$. Let us also define $W_\mu = A_\mu + \partial_\mu \theta$. In terms of the gauge invariant forms χ and $W = W_\mu dx^\mu$,

$$
F_3 = \frac{M\alpha'}{2} (\chi + 2W) \wedge \omega_2. \qquad (80)
$$

From (80) we can immediately see how the anomaly will appear in the gravity dual. Assuming that the NS-NS three form is still given by (40), we find that up to terms of order $g_s M^2/N$ the three-form equation implies

$$
d \star_5 W = 0 \Rightarrow \frac{L^2}{r^2} \partial_i W^i + \frac{1}{r^5} \partial_r r^5 W_r = 0 \qquad (81)
$$

which is just what one would expect for a massive vector field in five dimensions. To a four dimensional observer, however, a massive vector field would satisfy $\partial_i W^i = 0$. Thus in the field theory one cannot interpret the $U(1)$ symmetry breaking as being spontaneous, and the additional W_r term in (81) appears in four dimensions to be an anomaly.

Another way to see that the vector field becomes massive is to compute its equation of motion. To do this calculation precisely, we should derive the $\chi\mu$ components of Einstein's equations, and also find the appropriate expressions for the five-form and metric up to quadratic order in $g_s M$ and linear order in fluctuations. This approach is somewhat nontrivial. A more heuristic approach is to consider the type IIB supergravity action to quadratic order in W, ignoring the 5-form field strength contributions:

$$\begin{aligned} S &= -\frac{1}{2\kappa_{10}^2}\int d^{10}x\sqrt{-G_{10}}\left[R_{10} - \frac{g_s{}^2}{12}|F_3|^2\right] + \ldots \qquad (82)\\ &\sim -\frac{1}{2\kappa_{10}^2}\int d^{10}x\sqrt{-G_{10}}\left[-\frac{h^{1/2}r^2}{9}F_{\mu\nu}F^{\mu\nu} - \right.\\ &\qquad \left.\left(\frac{g_s M\alpha'}{2}\right)^2\frac{36}{hr^4}W_\mu W^\mu\right] + \ldots \qquad (83)\end{aligned}$$

This is clearly the action for a massive four-dimensional vector field, which has as its equation of motion

$$\partial_\mu(hr^7 F^{\mu\nu}) = \tilde{m}^2 hr^7 W^\nu \qquad (84)$$

which in differential form notation is $d(h^{7/4}r^7 \star_5 dW) = -\tilde{m}^2 h^{7/4} r^7 \star_5 W$. From the action (83), we see that the mass-squared is given by

$$\tilde{m}^2 = (g_s M\alpha')^2\frac{81}{2h^{3/2}r^6}. \qquad (85)$$

This result, however, ignores the subtlety of the type IIB action in presence of the self-dual 5-form field. A more precise calculation [49], which takes the mixing into account, gives instead the following equation for the transverse vector modes:

$$\left(\frac{1}{hr^7}\partial_r hr^7\partial_r + h\partial_i\partial_i - \frac{(9M\alpha')^4}{64h^2r^{10}}\right)W_i = 0, \qquad (86)$$

This shows that the 10-d mass actually appears at a higher order in perturbation theory compared to the result (85) that ignores the mixing with the 5-form.

Let us compare this result to earlier work. In [46, 47, 48] it was shown that the 5-d vector field associated with a $U(1)_R$ symmetry acquires a mass in the presence of a symmetry-breaking relevant perturbation, and that this mass is related in a simple way to the warp factor of the geometry.[4] It is conventional to write the 5-d gauged supergravity metric in the form

$$\tilde{G}_{\mu\nu}dx^\mu dx^\nu = e^{2T(q)}\eta_{ij}dx^i dx^j + dq^2. \qquad (87)$$

[4]We are grateful to O. DeWolfe and K. Skenderis for pointing out the relevance of this work to the present calculation.

The result of [46] is that $m^2 = -2T''$. To relate the 5-d metric (87) to the 10-d metric (97) we must normalize the 5-d metric so that the graviton has a canonical kinetic term. Doing this carefully we find

$$\tilde{G}_{\mu\nu} dx^\mu dx^\nu = \left(hr^4/L^4\right)^{5/6} \left(h^{-1/2} \eta_{ij} dx^i dx^j + h^{1/2} dr^2\right). \tag{88}$$

The factor $\left(hr^4/L^4\right)^{5/6}$ arises due to the radial dependence of the size of $T^{1,1}$ through the usual Kaluza–Klein reduction. The radial variables q and r are related, at leading order in $g_s M^2/N$, by

$$\log(r) \sim \frac{q}{L} - \frac{g_s M^2}{2\pi N} \left(\frac{q}{L}\right)^2 . \tag{89}$$

We can also show that $-2T = -2\log(r) +$(terms which do not affect the mass to leading order in $g_s M^2/N$), so now computing the mass-squared by the prescription of [46] we obtain

$$m^2 = \frac{4}{\alpha'(3\pi)^{3/2}} \frac{(g_s M)^2}{(g_s N)^{3/2}}. \tag{90}$$

where this mass applies to a vector field V with a canonical kinetic term for the metric (88). For these calculations it is convenient to work with the transverse 4-d vector modes V_i and to decouple the longitudinal modes such as V_r. The equation of motion of V is

$$(e^{-2T} \frac{\partial}{\partial q} e^{2T} \frac{\partial}{\partial q} + e^{-2T} \partial_i \partial_i - m^2) V_i = 0. \tag{91}$$

In fact, this equation follows from (86) after a rescaling [49]

$$V_i = (hr^4/L^4)^{2/3} W_i. \tag{92}$$

The nonvanishing vector mass is consistent with gauge invariance because the massless vector field A has eaten the scalar field θ, spontaneously breaking the gauge symmetry, as advertised. It is interesting that the anomaly appears as a bulk effect in AdS space, in contrast to some earlier examples [3, 50] where anomalies arose from boundary terms.

The appearance of a mass implies that the R-current operator should acquire an anomalous dimension. From (90) it follows that

$$(mL)^2 = \frac{2(g_s M)^2}{\pi (g_s N)} . \tag{93}$$

Using the AdS/CFT correspondence (perhaps naively, as the KT metric is not asymptotically AdS) we find that the dimension of the current J^μ dual to the vector field W^μ is

$$\Delta = 2 + \sqrt{1 + (mL)^2} \ . \tag{94}$$

Therefore, the anomalous dimension of the current is

$$\Delta - 3 \approx (mL)^2/2 = \frac{(g_s M)^2}{\pi(g_s N)} \ . \tag{95}$$

We can obtain a rough understanding of this result by considering the relevant weak coupling calculation in the gauge theory. The leading correction to the current-current two-point function comes from the three-loop Feynman diagram composed of two triangle diagrams glued together, and the resulting anomalous dimension γ_J is quadratic in M and N. γ_J must vanish when $M = 0$, and it must be invariant under the map $M \to -M$, $N \to N + M$, which simply interchanges the two gauge groups. Thus, the lowest order piece of the anomalous dimension will be of order $(g_s M)^2$. Our supergravity calculation predicts that this anomalous dimension is corrected at large $g_s N$ by an extra factor of $1/(g_s N)$. Of course, it would be interesting to understand this result better from the gauge theory point of view.

5 Deformation of the Conifold

It was shown in [11] that, to remove the naked singularity found in [10] the conifold (13) should be replaced by the deformed conifold

$$\sum_{i=1}^{4} z_i^2 = \varepsilon^2 \ , \tag{96}$$

in which the singularity of the conifold is removed through the blowing-up of the $\mathbf{S}^3$ of $T^{1,1}$. The 10-d metric of [11] takes the following form:

$$ds_{10}^2 = h^{-1/2}(\tau) dx_n dx_n + h^{1/2}(\tau) ds_6^2 \ , \tag{97}$$

where ds_6^2 is the metric of the deformed conifold (98). This is the same type of "D-brane" ansatz as (47), but with the conifold replaced by the deformed conifold as the transverse space.

The metric of the deformed conifold was discussed in some detail in [18, 30, 51]. It is diagonal in the basis (36):

$$ds_6^2 = \frac{1}{2}\varepsilon^{4/3} K(\tau) \Big[\frac{1}{3K^3(\tau)} (d\tau^2 + (g^5)^2) + \cosh^2\left(\frac{\tau}{2}\right) [(g^3)^2 + (g^4)^2]$$

$$+\sinh^2\left(\frac{\tau}{2}\right)[(g^1)^2+(g^2)^2]\Big] , \quad (98)$$

where

$$K(\tau)=\frac{(\sinh(2\tau)-2\tau)^{1/3}}{2^{1/3}\sinh\tau} . \quad (99)$$

For large τ we may introduce another radial coordinate r via

$$r^2=\frac{3}{2^{5/3}}\varepsilon^{4/3}e^{2\tau/3} , \quad (100)$$

and in terms of this radial coordinate $ds_6^2 \to dr^2 + r^2 ds_{T^{1,1}}^2$.

At $\tau=0$ the angular metric degenerates into

$$d\Omega_3^2=\frac{1}{2}\varepsilon^{4/3}(2/3)^{1/3}[\frac{1}{2}(g^5)^2+(g^3)^2+(g^4)^2] , \quad (101)$$

which is the metric of a round $\mathbf{S}^3$ [18, 30]. The additional two directions, corresponding to the $\mathbf{S}^2$ fibered over the $\mathbf{S}^3$, shrink as

$$\frac{1}{8}\varepsilon^{4/3}(2/3)^{1/3}\tau^2[(g^1)^2+(g^2)^2] . \quad (102)$$

The simplest ansatz for the 2-form fields is

$$\begin{aligned}F_3 &= \frac{M\alpha'}{2}\{g^5\wedge g^3\wedge g^4+d[F(\tau)(g^1\wedge g^3+g^2\wedge g^4)]\}\\ &= \frac{M\alpha'}{2}\{g^5\wedge g^3\wedge g^4(1-F)+g^5\wedge g^1\wedge g^2 F\\ &\qquad +F'd\tau\wedge(g^1\wedge g^3+g^2\wedge g^4)\} , \end{aligned} \quad (103)$$

with $F(0)=0$ and $F(\infty)=1/2$, and

$$B_2=\frac{g_s M\alpha'}{2}[f(\tau)g^1\wedge g^2+k(\tau)g^3\wedge g^4] , \quad (104)$$

$$\begin{aligned}H_3=dB_2 &= \frac{g_s M\alpha'}{2}\Big[d\tau\wedge(f'g^1\wedge g^2+k'g^3\wedge g^4)\\ &\qquad +\frac{1}{2}(k-f)g^5\wedge(g^1\wedge g^3+g^2\wedge g^4)\Big] . \end{aligned} \quad (105)$$

As before, the self-dual 5-form field strength may be decomposed as $\tilde F_5=\mathcal{F}_5+\star\mathcal{F}_5$. We have

$$\mathcal{F}_5=B_2\wedge F_3=\frac{g_s M^2(\alpha')^2}{4}\ell(\tau)g^1\wedge g^2\wedge g^3\wedge g^4\wedge g^5 , \quad (106)$$

where

$$\ell = f(1-F) + kF \ , \tag{107}$$

and

$$\star \mathcal{F}_5 = 4 g_s M^2 (\alpha')^2 \varepsilon^{-8/3} dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3 \wedge d\tau \frac{\ell(\tau)}{K^2 h^2 \sinh^2(\tau)} \ . \tag{108}$$

5.1 The First-Order Equations and Their Solution

In searching for BPS saturated supergravity backgrounds, the second order equations should be replaced by a system of first-order ones. Luckily, this is possible for our ansatz [11]:

$$\begin{aligned} f' &= (1-F)\tanh^2(\tau/2) \ , \\ k' &= F\coth^2(\tau/2) \ , \\ F' &= \frac{1}{2}(k-f) \ , \end{aligned} \tag{109}$$

and

$$h' = -\alpha \frac{f(1-F) + kF}{K^2(\tau)\sinh^2\tau} \ , \tag{110}$$

where

$$\alpha = 4 (g_s M \alpha')^2 \varepsilon^{-8/3} \ . \tag{111}$$

These equations follow from a superpotential for the effective radial problem [52].

Note that the first three of these equations, (109), form a closed system and need to be solved first. In fact, these equations imply the self-duality of the complex 3-form with respect to the metric of the deformed conifold: $\star_6 G_3 = i G_3$. The solution is

$$\begin{aligned} F(\tau) &= \frac{\sinh\tau - \tau}{2\sinh\tau} \ , \\ f(\tau) &= \frac{\tau\coth\tau - 1}{2\sinh\tau}(\cosh\tau - 1) \ , \\ k(\tau) &= \frac{\tau\coth\tau - 1}{2\sinh\tau}(\cosh\tau + 1) \ . \end{aligned} \tag{112}$$

Now that we have solved for the 3-forms on the deformed conifold, the warp factor may be determined by integrating (110). First we note that

$$\ell(\tau) = f(1-F) + kF = \frac{\tau\coth\tau - 1}{4\sinh^2\tau}(\sinh 2\tau - 2\tau) \ . \tag{113}$$

This behaves as τ^3 for small τ. For large τ we impose, as usual, the boundary condition that h vanishes. The resulting integral expression for h is

$$h(\tau) = \alpha \frac{2^{2/3}}{4} I(\tau) = (g_s M \alpha')^2 2^{2/3} \varepsilon^{-8/3} I(\tau) \ , \tag{114}$$

where

$$I(\tau) \equiv \int_\tau^\infty dx \frac{x \coth x - 1}{\sinh^2 x} (\sinh(2x) - 2x)^{1/3} \ . \tag{115}$$

We have not succeeded in evaluating this integral in terms of elementary or well-known special functions, but it is not hard to see that

$$I(\tau \to 0) \to a_0 + O(\tau^2) \ ; \tag{116}$$

$$I(\tau \to \infty) \to 3 \cdot 2^{-1/3} \left(\tau - \frac{1}{4}\right) e^{-4\tau/3} \ , \tag{117}$$

where $a_0 \approx 0.71805$. This $I(\tau)$ is nonsingular at the tip of the deformed conifold and, from (100), matches the form of the large-τ solution (55). The small τ behavior follows from the convergence of the integral (114), while at large τ the integrand becomes $\sim x e^{-4x/3}$.

Thus, for small τ the ten-dimensional geometry is approximately $\mathbb{R}^{3,1}$ times the deformed conifold:

$$\begin{aligned} ds_{10}^2 \ \to \ & \frac{\varepsilon^{4/3}}{2^{1/3} a_0^{1/2} g_s M \alpha'} dx_n dx_n + a_0^{1/2} 6^{-1/3} (g_s M \alpha') \Big\{ \frac{1}{2} d\tau^2 + \frac{1}{2} (g^5)^2 \\ & + (g^3)^2 + (g^4)^2 + \frac{1}{4} \tau^2 [(g^1)^2 + (g^2)^2] \Big\} \ . \end{aligned} \tag{118}$$

This metric will be useful in section 6 where we investigate various infrared phenomenon of the gauge theory.

Very importantly, for large $g_s M$ the curvatures found in our solution are small everywhere. This is true even far in the IR, since the radius-squared of the $\mathbf{S}^3$ at $\tau = 0$ is of order $g_s M$ in string units. This is the 't Hooft coupling of the gauge theory found far in the IR. As long as this is large, the curvatures are small and the SUGRA approximation is reliable.

5.2 $SO(4)$ invariant expressions for the 3-forms

In [53, 54] it was shown that the warped background of the previous section preserves $\mathcal{N} = 1$ SUSY if and only if G_3 is a $(2,1)$ form on the CY space. Perhaps the easiest way to see the supersymmetry of the deformed conifold solution is through a T-duality. Performing a T-duality along one of the longitudinal directions, and lifting the result to M-theory maps our background to a Becker-Becker solution supported by a G_4 which is a $(2,2)$ form on $T^2 \times$ CY. G-flux of this type indeed produces a supersymmetric background [55].

While writing G_3 in terms of the angular 1-forms g^i is convenient for some purposes, the $(2,1)$ nature of the form is not manifest. That G_3 is indeed $(2,1)$ was demonstrated in [56] with the help of a holomorphic basis. Below we write the G_3 found in [11] in terms of the obvious 1-forms on the deformed conifold: dz^i and $d\bar{z}^i$, $i = 1,2,3,4$:

$$
\begin{aligned}
G_3 = \frac{M\alpha'}{2\varepsilon^6 \sinh^4 \tau} \Big\{ & \frac{\sinh(2\tau) - 2\tau}{\sinh \tau} (\epsilon_{ijkl}\, z_i \bar{z}_j\, dz_k \wedge d\bar{z}_l) \wedge (\bar{z}_m\, dz_m) \\
& + 2(1 - \tau \coth \tau)(\epsilon_{ijkl}\, z_i \bar{z}_j\, dz_k \wedge dz_l) \wedge (z_m\, d\bar{z}_m) \Big\} .
\end{aligned}
\tag{119}
$$

We also note that the NS-NS 2-form potential is an $SO(4)$ invariant $(1,1)$ form:

$$
B_2 = \frac{i g_s M \alpha'}{2\varepsilon^4} \frac{\tau \coth \tau - 1}{\sinh^2 \tau} \epsilon_{ijkl}\, z_i \bar{z}_j\, dz_k \wedge d\bar{z}_l \ . \tag{120}
$$

The derivation of these formulae is given in [31]. Our expressions for the gauge fields are manifestly $SO(4)$ invariant, and so is the metric.

6 Infrared Physics

We have now seen that the deformation of the conifold allows the solution to be non-singular. In the following sections we point out some interesting features of the SUGRA background we have found and show how they realize the expected phenomena in the dual field theory. In particular, we will now demonstrate that there is confinement; that the theory has glueballs and baryons whose mass scale emerges through a dimensional transmutation; that there is a gluino condensate that breaks the $\mathbb{Z}_{2M}$ chiral symmetry down to $\mathbb{Z}_2$ and that there are domain walls separating inequivalent vacua. Other stringy approaches to infrared phenomena in $\mathcal{N} = 1$ SYM theory have recently appeared in [57, 58, 59].

6.1 Dimensional Transmutation and Confinement

The resolution of the naked singularity via the deformation of the conifold is a supergravity realization of the dimensional transmutation. While the singular conifold has no dimensionful parameter, we saw that turning on the R-R 3-form flux produces the logarithmic warping of the KT solution. The scale necessary to define the logarithm transmutes into the the parameter ε that determines the deformation of the conifold. From (100) we see that $\varepsilon^{2/3}$ has dimensions of length and that

$$\tau = 3\ln(r/\varepsilon^{2/3}) + \text{const} . \tag{121}$$

Thus, the scale r_s entering the UV solution (55) should be identified with $\varepsilon^{2/3}$. On the other hand, the form of the IR metric (118) makes it clear that the dynamically generated 4-d mass scale, which sets the tension of the confining flux tubes, is

$$\frac{\varepsilon^{2/3}}{\alpha'\sqrt{g_s M}} . \tag{122}$$

The reason the theory is confining is that in the metric for small τ (118) the function multiplying $dx_n dx_n$ approaches a constant. This should be contrasted with the AdS_5 metric where this function vanishes at the horizon, or with the singular metric of [10] where it blows up. Consider a Wilson contour positioned at fixed τ, and calculate the expectation value of the Wilson loop using the prescription [60, 61]. The minimal area surface bounded by the contour bends towards smaller τ. If the contour has a very large area A, then most of the minimal surface will drift down into the region near $\tau = 0$. From the fact that the coefficient of $dx_n dx_n$ is finite at $\tau = 0$, we find that a fundamental string with this surface will have a finite tension, and so the resulting Wilson loop satisfies the area law. A simple estimate shows that the string tension scales as

$$T_s = \frac{1}{2^{4/3} a_0^{1/2} \pi} \frac{\varepsilon^{4/3}}{(\alpha')^2 g_s M} . \tag{123}$$

We will return to these confining strings in the next section.

The masses of glueball and Kaluza-Klein (KK) states scale as

$$m_{glueball} \sim m_{KK} \sim \frac{\varepsilon^{2/3}}{g_s M \alpha'} . \tag{124}$$

Comparing with the string tension, we see that

$$T_s \sim g_s M (m_{glueball})^2 . \tag{125}$$

Due to the deformation, the full SUGRA background has a finite 3-cycle. We may interpret various branes wrapped over this 3-cycle in terms of the gauge theory. Note that the 3-cycle has the minimal volume near $\tau = 0$, hence all the wrapped branes will be localized there. A wrapped D3-brane plays the role of a baryon vertex which ties together M fundamental strings. Note that for $M = 0$ the D3-brane wrapped on the $\mathbf{S}^3$ gave a dibaryon [8]; the connection between these two objects becomes clearer when one notes that for $M > 0$ the dibaryon has M uncontracted indices, and therefore joins M external charges. Studying a probe D3-brane in the background of our solution show that the mass of the baryon scales as

$$M_b \sim M \frac{\varepsilon^{2/3}}{\alpha'} . \tag{126}$$

6.2 Tensions of the q-Strings

The existence of the blown up 3-cycle with M units of RR 3-form flux through it is responsible for another interesting infrared phenomenon, the appearance of composite confining strings. To explain what they are, let us recall that the basic string corresponds to the Wilson loop in the fundamental representation. The classic criterion for confinement is that this Wilson loop obey the area law

$$- \ln\langle W_1(C)\rangle = T_1 A(C) \tag{127}$$

in the limit of large area. An interesting generalization is to consider Wilson loops in antisymmetric tensor representations with q indices where q ranges from 1 to $M-1$. $q = 1$ corresponds to the fundamental representation as denoted above, and there is a symmetry under $q \to M - q$ which corresponds to replacing quarks by anti-quarks. These Wilson loops can be thought of as confining strings which connect q probe quarks on one end to q corresponding probe anti-quarks on the other. For $q = M$ the probe quarks combine into a colorless state (a baryon); hence the corresponding Wilson loop does not have an area law.

It is interesting to ask how the tension of this class of confining strings depends on q. If it is a convex function,

$$T_{q+q'} < T_q + T_{q'} , \tag{128}$$

then the q-string will not decay into strings with smaller q. This is precisely the situation found by Douglas and Shenker (DS) [62] in softly broken $\mathcal{N} = 2$ gauge theory, and later by Hanany, Strassler and Zaffaroni (HSZ) [63] in the MQCD approach to confining $\mathcal{N} = 1$ supersymmetric gauge theory [64, 65]:

$$T_q = \Lambda^2 \sin\frac{\pi q}{M} , \qquad q = 1, 2, \ldots, M-1 \tag{129}$$

where Λ is the overall IR scale.

This type of behaviour is also found in the supergravity duals of $\mathcal{N} = 1$ gauge theories [66]. Here the confining q-string is described by q coincident fundamental strings placed at $\tau = 0$ and oriented along the $\mathbb{R}^{3,1}$.[5] In the deformed conifold solution analyzed above both F_5 and B_2 vanish at $\tau = 0$, but it is important that there are M units of F_3 flux through the $\mathbf{S}^3$. In fact, this R-R flux blows up the q fundamental strings into a D3-brane wrapping an $\mathbf{S}^2$ inside the $\mathbf{S}^3$. Although the blow-up can be shown directly, for brevity we build on a closely related result of Bachas, Douglas and Schweigert [68]. In the S-dual of our type IIB gravity model, at $\tau = 0$ we find the $\mathbb{R}^{3,1} \times \mathbf{S}^3$ geometry with M units of NS-NS H_3 flux through the $\mathbf{S}^3$ and q coincident D1-branes along the $\mathbb{R}^{3,1}$. T-dualizing along the D1-brane direction we find q D0-branes on an $\mathbf{S}^3$ with M units of NS-NS flux. This geometry is very closely related to the setup of [68] whose authors showed that the q D0-branes blow up into an $\mathbf{S}^2$. We will find the same phenomenon, but our probe brane calculation is somewhat different from [68] because the radius of our $\mathbf{S}^3$ is different.

After applying S-duality to the KS solution, at $\tau = 0$ the metric is

$$\frac{\varepsilon^{4/3}}{2^{1/3} a_0^{1/2} g_s^2 M \alpha'} dx_n dx_n + bM\alpha'(d\psi^2 + \sin^2\psi d\Omega_2^2) \ , \tag{130}$$

where $b = 2a_0^{1/2} 6^{-1/3} \approx 0.93266$. We are now using the standard round metric on $\mathbf{S}^3$ so that ψ is the azimuthal angle ranging from 0 to π. The NS-NS 2-form field at $\tau = 0$ is

$$B_2 = M\alpha' \left(\psi - \frac{\sin(2\psi)}{2}\right) \sin\theta d\theta \wedge d\phi \ , \tag{131}$$

while the world volume field is

$$F = -\frac{q}{2} \sin\theta d\theta \wedge d\phi \ . \tag{132}$$

Following [68] closely we find that the tension of a D3-brane which wraps an $\mathbf{S}^2$ located at the azimuthal angle ψ is

$$\frac{\epsilon^{4/3}}{12^{1/3}\pi^2 g_s^2 \alpha'^2 b} \left[b^2 \sin^4\psi + \left(\psi - \frac{\sin(2\psi)}{2} - \frac{\pi q}{M}\right)^2 \right]^{1/2} \ . \tag{133}$$

Minimizing with respect to ψ we find

$$\psi - \frac{\pi q}{M} = \frac{1 - b^2}{2} \sin(2\psi) \ . \tag{134}$$

[5] Qualitatively similar confining flux-tubes were examined in [67] where the authors use the near horizon geometry of non-extremal D3-branes to model confinement.

The tension of the wrapped brane is given in terms of the solution of this equation by

$$T_q = \frac{\epsilon^{4/3}}{12^{1/3}\pi^2 g_s^2 \alpha'^2} \sin\psi \sqrt{1 + (b^2 - 1)\cos^2\psi} \ . \tag{135}$$

Note that under $q \to M - q$, we find $\psi \to \pi - \psi$, so that $T_{M-q} = T_q$. This is a crucial property needed for the connection with the q-strings of the gauge theory.

Although (134) is not exactly solvable, we note that $(1 - b^2)/2 \approx 0.06507$ is small numerically. If we ignore the RHS of this equation, then $\psi \approx \pi q/M$ and

$$T_q \sim \sin\frac{\pi q}{M} \ . \tag{136}$$

The deviations from this formulae are small: even when $\psi = \pi/4$ and correspondingly $q \approx M/4$, the tension in the KS case is approximately 96.7% of that in the $b = 1$ case.

It is interesting to compare (136) with the naive string tension (123) we obtained in the previous section. In the large M limit, we expect interactions among the strings to become negligible and the q-string tension to become just q times the ordinary string tension (123). Indeed, we find that $g_s T_q = q T_s$ in the large M limit. The extra g_s appears because we have been computing tensions in the dual background. When we S-dualize back to the original background with RR-flux and q F-strings, all the tensions are multiplied by g_s.

An analogous calculation for the MN background [57] proceeds almost identically. In this background only the F_3 flux is present; hence after the S-duality we find only $H_3 = dB_2$. The value of B_2 at the minimal radius is again given by (131). There is a subtle difference however from the calculation for the KS background in that now the parameter b entering the radius of the $\mathbf{S}^3$ is equal to 1. This simplifies the probe calculation and makes it identical to that of [68]. In particular, now we find

$$\frac{T_q}{T_{q'}} = \frac{\sin\frac{\pi q}{M}}{\sin\frac{\pi q'}{M}} \ , \tag{137}$$

without making any approximations.

Our argument applied to the MN background leads very simply to the DS–HSZ formula for the ratios of q-string tensions (137). As we have shown earlier, this formula also holds approximately for the KS background. It is interesting to note that recent lattice simulations in non-supersymmetric pure glue gauge theory [69] appear to yield good agreement with (137).

6.3 Chiral Symmetry Breaking and Gluino Condensation

Our $SU(N+M)\times SU(N)$ field theory has an anomaly-free $\mathbb{Z}_{2M}$ R-symmetry. In section 3 we showed that the corresponding symmetry of the UV (large τ) limit of the metric is

$$\psi \to \psi + \frac{2\pi k}{M}\ , \qquad k = 1, 2, \ldots, M\ . \tag{138}$$

Recalling that ψ ranges from 0 to 4π, we see that the full solution, which depends on ψ through $\cos\psi$ and $\sin\psi$, has the $\mathbb{Z}_2$ symmetry generated by $\psi \to \psi + 2\pi$. As a result, there are M inequivalent vacua: there are exactly M different discrete orientations of the solution, corresponding to breaking of the $\mathbb{Z}_{2M}$ UV symmetry through the IR effects. The domain walls constructed out of the wrapped D5-branes separate these inequivalent vacua.

Let us consider domain walls made of k D5-branes wrapped over the finite-sized $\mathbf{S}^3$ at $\tau = 0$, with remaining directions parallel to $\mathbb{R}^{3,1}$. Such a domain wall is obviously a stable object in the KS background and crossing it takes us from one ground state of the theory to another. Indeed, the wrapped D5-brane produces a discontinuity in $\int_B F_3$, where B is the cycle dual to the $\mathbf{S}^3$. If to the left of the domain wall $\int_B F_3 = 0$, as in the basic solution derived in the preceding sections, then to the right of the domain wall

$$\int_B F_3 = 4\pi^2\alpha' k\ , \tag{139}$$

as follows from the quantization of the D5-brane charge. The B-cycle is bounded by a 2-sphere at $\tau = \infty$, hence $\int_B F_3 = \int_{\mathbf{S}^2} \Delta C_2$. Therefore from (43) it is clear that to the right of the wall

$$\Delta C_2 \to \pi\alpha' k\omega_2 \tag{140}$$

for large τ. This change in C_2 is produced by the $\mathbb{Z}_{2M}$ transformation (138) on the original field configuration (64).

It is expected that flux tubes can end on these domain walls [70]. Indeed, a fundamental string can end on the wrapped D5-brane. Also, baryons can dissolve in them. By studying a probe D5-brane in the metric, we find that the domain wall tension is

$$T_{wall} \sim \frac{1}{g_s}\frac{\varepsilon^2}{(\alpha')^3}\ . \tag{141}$$

In supersymmetric gluodynamics the breaking of chiral symmetry is associated with the gluino condensate $\langle\lambda\lambda\rangle$. A holographic calculation of the condensate was carried out by Loewy and Sonnenschein in [71] (see also [72] for previous work on gluino condensation in conifold theories.) They looked for the deviation of the complex 2-form field $C_2 - \frac{i}{g_s}B_2$ from its asymptotic large τ form that enters the KT solution:

$$\delta\left(C_2 - \frac{i}{g_s}B_2\right) \sim \frac{M\alpha'}{4}\tau e^{-\tau}[g_1\wedge g_3 + g_2\wedge g_4 - i(g_1\wedge g_2 - g_3\wedge g_4)]$$

$$\sim \frac{M\alpha'\varepsilon^2}{r^3}\ln(r/\varepsilon^{2/3})e^{i\psi}(d\theta_1 - i\sin\theta_1 d\phi_1)\wedge(d\theta_2 - i\sin\theta_2 d\phi_2)\ . \qquad (142)$$

In a space-time that approaches AdS_5 a perturbation that scales as r^{-3} corresponds to the expectation value of a dimension 3 operator. The presence of an extra $\ln(r/\varepsilon^{2/3})$ factor is presumably due to the fact that the asymptotic KT metric differs from AdS_5 by such logarithmic factors. From the angular dependence of the perturbation we see that the dual operator is $SU(2)\times SU(2)$ invariant and carries R-charge 1. These are precisely the properties of $\lambda\lambda$. Thus, the holographic calculation tells us that

$$\langle\lambda\lambda\rangle \sim M\frac{\varepsilon^2}{(\alpha')^3}\ . \qquad (143)$$

Thus, the parameter ε^2 which enters the deformed conifold equation has a dual interpretation as the gluino condensate.[6]

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