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SPRING SCHOOL ON SUPERSTRING THEORY AND RELATED TOPICS

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Integrability in Super Yang-Mills

Joseph MINAHAN
Uppsala University
Department of Theoretical Physics
P.O. Box 803
S-75108 Uppsala
SWEDEN

Please note: These are preliminary notes intended for internal distribution only.

Outline of Lectures

(1) Review of $N=4$ SYM

(2) ~~Computation~~ Spin chains

(3) Integrability, the R-matrix and Yang-Baxter Eq.

(4) Bethe ansatz

(5) Some elementary calculations

(6) The Bethe ansatz for "long" operators.

(7) Semiclassical string computations

(8) Classical Integrability.

(9) Higher Loops in Perturbation Theory.

List of References

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Introductory Remarks

Why is it interesting.
Integrability often means we can compute quantities.

AdS/CFT conjecture, is it true?

Ideally compute quantities on both sides of the conjecture, show that they are the same.

What we will do is show that this is true for a ~~tar~~ class of operators, UP TO A CERTAIN POINT.

Does this mean that AdS/CFT fails?
Not really. Ultimately we are relating strong coupling of one theory to the weak coupling of another. We need to be careful.

Nevertheless, we will find surprising agreement between string theory on $AdS_5 \times S_5$ and perturbative Yang-Mills.

Even if one does not believe in AdS/CFT, integrability still allows us to compute the anomalous dimensions of many nontrivial operators.

Things to remember $N=4$ SYM is not QCD
This is a conformal theory, so the physical information is found in correlation functions between gauge invariant operators.
A conformal theory has no mass scale, hence no running of the coupling, no confinement etc.
Still we can still hope to find a true gauge string duality.

For our purposes we will consider single trace ops. only. $\mathcal{N}=4$ SYM $SU(N)$ $N\to\infty$

Gauge multiplet:

(adjoint rep) $\Phi^i{}_{A\bar{B}}$ 6 scalar fields $i=1\ldots6$

$\Psi_{\alpha a, A\bar{B}}$ 8 fermi fields $a=1\ldots4$, $\alpha=1,2$, $\dot{\alpha}=1,2$

$\bar{\Psi}_{\dot{\alpha}\bar{a}, A\bar{B}}$

$A_{M, A\bar{B}}$ 2 gauge fields

Gauge invariant operators:

Traces of Φ_i, $\Psi_{\alpha a}$, $\chi_{\dot{\alpha}\bar{a}}$, $F_{\mu\nu}$ and covariant derivatives.

What we are most interested in are the dimensions of these operators.

AdS/CFT: String E = Operator Δ

To find dimension, compute 2-pt. function.

$$\mathcal{O} = Tr(\ldots(x))$$

$$\langle \mathcal{O}(x)\,\bar{\mathcal{O}}(y)\rangle = \frac{C}{|x-y|^{2\Delta}}$$

Bare dimension: $\Delta^{(0)} = N_S + \frac{3}{2}N_f + 2N_F + N_{CD}$

If coupling $g^2=0$, then $\Delta=\Delta^{(0)}$

Since this is a conformal theory (no mass scale) Δ is a function of g^2 only.

The Super conformal algebra:

D P_μ, K_μ, $M_{\mu\nu}$, R^{ij} $Q_{\alpha a}$ $\tilde{Q}_{\dot\alpha \bar a}$ $S_{\alpha \bar a}$, $\tilde{S}_{\dot\alpha a}$

↑ ↑ ↑ ↑ ↑ (super charges 16) (special conf. supercharges 16)

Dilatation, translation, special conf., Lorentz, R-symmetry

$$[D, P_\mu] = -iP_\mu \qquad [D, K_\mu] = +iK_\mu \qquad [D, M_{\mu\nu}] = 0$$

$$[M_{\mu\nu}, P_\lambda] = -i(\eta_{\mu\lambda} P_\nu - \eta_{\nu\lambda} P_\mu) \qquad [M_{\mu\nu}, K_\lambda] = -i(\eta_{\mu\lambda} K_\nu - \eta_{\nu\lambda} K_\mu)$$

$$\{Q_{\alpha a}, \tilde{Q}_{\dot\alpha \bar b}\} = \gamma^\mu_{\alpha\dot\alpha} \delta^{a\bar b} P_\mu \qquad [P_\mu, K_\nu] = 2i(M_{\mu\nu} - \eta_{\mu\nu} D)$$

$$\{S_{\alpha\bar a}, \tilde{S}_{\dot\alpha b}\} = \gamma^\mu_{\alpha\dot\alpha} \delta_{\bar a b} K_\mu$$

$$[D, Q_{\alpha a}] = -\tfrac{i}{2} Q_{\alpha a} \qquad [D, S_{\alpha\bar a}] = \tfrac{i}{2} S_{\alpha\bar a}$$

$$[K_\mu, Q_{\alpha a}] = \gamma_{\mu\alpha\dot\alpha} \tilde{S}_{a\dot\beta} \varepsilon^{\dot\alpha\dot\beta}$$

$$[P_\mu, S_{\alpha\bar a}] = \gamma_{\mu\alpha\dot\alpha} \tilde{Q}_{\bar a\dot\beta} \varepsilon^{\dot\alpha\dot\beta}$$

$$\{Q^a_\alpha, S^{\bar b}_\beta\} = -i\varepsilon_{\alpha\beta} \sigma^{ij}_{a\bar b} R^{ij} + \sigma^{\mu\nu}_{\alpha\beta} \delta^{a\bar b} M_{\mu\nu} - \varepsilon_{\alpha\beta} \delta^{a\bar b} D$$

$\mathcal{O}(x)$ under a rescaling $x \to \lambda x$

$$\mathcal{O}(x) \to \lambda^{-\Delta} \mathcal{O}(x\lambda)$$

$$[D, \mathcal{O}(x)] = i\left(-\Delta + x\frac{\partial}{\partial x}\right)\mathcal{O}(x)$$

$$[D, [K_\mu, \mathcal{O}(0)]] = [[D, K_\mu], \mathcal{O}(0)] + [K_\mu, [D_\mu, \mathcal{O}(0)]]$$

$$= -i(\Delta - 1)[K_\mu, \mathcal{O}(0)]$$

Unitary field theories have a lower bound on Δ.

Primary operator: $[K_\mu, \mathcal{O}(0)] = 0$

Generate all operators in the representation with $Q_{\alpha\bar a}$ and P_μ

Chiral Primaries: Primaries are also annihilated by some of the Q's (They are always annihilated by the S fields)

$$[Q_\alpha^a, \mathcal{O}(0)] = 0 \quad \text{for some } \alpha \text{ and } a$$

$$[S_\alpha^{\bar{b}}, \mathcal{O}(0)] = 0 \quad \text{for all } \alpha \ \bar{b}$$

$$\left[\{Q_\alpha^a, S_\beta^{\bar{b}}\}, \mathcal{O}(0)\right]$$

$$= \left\{Q_\alpha^a, [S_\beta^{\bar{b}}, \mathcal{O}(0)]\right\} + \left\{S_\beta^{\bar{b}}, [Q_\alpha^a, \mathcal{O}(0)]\right\} = 0$$

$$\left[\left[-i\mathcal{E}_{\alpha\beta} R^{IJ} \sigma^{IJ}_{ab} - \mathcal{E}_{\alpha\beta}\delta^{ab} D + \sigma^{\mu\nu}_{\alpha\beta}\delta^{a\bar{b}} M_{\mu\nu}\right], \mathcal{O}(0)\right]$$

If $\mathcal{O}(0)$ is a scalar, then Δ is related to R-charge of $\mathcal{O}(0)$

Write scalar fields as

$$Z = \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2) \quad W = \frac{1}{\sqrt{2}}(\phi_3 + i\phi_4), \quad Y = \frac{1}{\sqrt{2}}(\phi_5 + i\phi_6)$$

R-charges of the fields:

SO(6) is rank 3

Cartan Subalgebra R_{12}, R_{34}, R_{56}

$$[R_{12}, Z] = Z \quad , \quad [R_{12}, \bar{Z}] = -\bar{Z}$$

$$[R_{34}, Z] = [R_{56}, Z] = 0$$

R_{12}

Z: $(1, 0, 0)$ & $[D, Z] = -iZ$

$\mathcal{O} = \mathrm{Tr}[Z^J]$

$(J, 0, 0)$ $\quad [D, \mathcal{O}] = -iJ\mathcal{O}$

$[\{Q, S\}, \mathcal{O}] = 0$ This is a chiral primary

More precisely,

We have that $\sigma^{12}_{a\bar{a}} = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$

$\sigma^{34}_{a\bar{a}} = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & 1 & \\ & & & -1 \end{pmatrix}$

$\sigma^{56}_{a\bar{a}} = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & 1 \end{pmatrix}$

Hence, we can choose

Q^1_α, Q^2_α, $\tilde{Q}^{\bar{3}}_{\dot{\alpha}}$ $\tilde{Q}^{\bar{4}}_{\dot{\alpha}}$ to annihilate $\mathrm{tr}\, Z^J$ + adjoints.

Q^1_α Q^3_α, $\tilde{Q}^{\bar{2}}_{\dot{\alpha}}$ $\tilde{Q}^{\bar{4}}_{\dot{\alpha}}$ to annihilate $\mathrm{tr}\, X^J$

Q^1_α Q^4_α, $\tilde{Q}^{\bar{2}}_{\dot{\alpha}}$ $\tilde{Q}^{\bar{3}}_{\dot{\alpha}}$ to annihilate $\mathrm{tr}\, Y^J$

We can also say more. Since $\mathrm{tr}\, Z^J$ is a chiral primary then $\mathcal{O}$ is any other operator gotten by acting on this with R^{ij}

$\mathrm{Tr}\, Z^J$ is a highest weight state of symmetric traceless rep of $SO(6)$ $\Rightarrow$ symmetric traceless are chiral primaries.

$\therefore$ For example

$\mathrm{Tr}\, ZXY + \mathrm{Tr}\, ZYX$ is a chiral primary

(We used the cyclic property of the trace)

$\mathrm{Tr}\, ZXY - \mathrm{Tr}\, ZYX$ is not

$\mathrm{Tr}\, Z\bar{Z}$ is not chiral primary (not traceless)

$\mathrm{Tr}\, Z\bar{X}$ is chiral prim.

Chiral primaries: $\Delta = \Delta^{(0)}$ "Dimensions are protected"

The role of large N

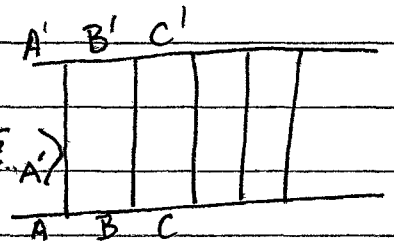
Suppose $g^2 = 0$

$$\theta_1 = \frac{1}{\sqrt{J}\, N^{J/2}} \operatorname{tr} Z^J \qquad \theta_2 = \frac{1}{\sqrt{K}\, N^{K/2}} \operatorname{tr} Z^K \frac{1}{N^{(J-K)/2}\sqrt{J-K}} \operatorname{tr} Z^{J-K}$$

$$\operatorname{tr} Z^J = Z_{A\bar{B}} Z_{B\bar{C}} Z_{C\bar{D}} \cdots Z_{..\bar{A}}$$

$$\langle \theta_1(x)\, \bar{\theta}_1(y) \rangle =$$

$$= \langle Z_{A\bar{B}} \cdots Z_{..\bar{A}} \; \bar{Z}_{\bar{A}'B'} \cdots \bar{Z}_{..A'} \rangle \times \frac{1}{J} \cdot \frac{1}{N^J}$$



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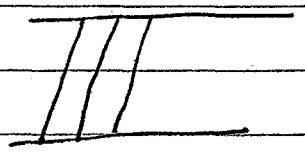

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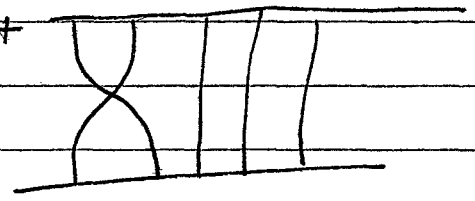

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1st term

factor $\delta_{A\bar{A}'} \delta_{\bar{B}B'} \delta_{\bar{B}\bar{B}'} \delta_{\bar{A}A'} = N^J$ } J such terms

2nd term also has N^J factor

3rd term: "nonplanar"

$$\delta_{A\bar{B}'} \delta_{C'\bar{B}} \delta_{B\bar{A}'} \delta_{\bar{C}B'} \delta_{C\bar{C}'} \cdots \delta_{\bar{A}A'}$$

$$= \delta_{A\bar{C}} \delta_{\bar{A}'C'} \delta_{C\bar{C}'} \cdots \delta_{\bar{A}A'}$$

$$= \delta_{A\bar{A}'} \cdots \delta_{\bar{A}A'} = N^{J-2}$$ suppressed.

Check that mixing with Double trace is also suppressed.

=> Large N limit, only consider planar graphs. => Only consider mixing between single trace operators.

Review from last time

Chiral primaries : $\mathcal{O}(x)$ has $\Delta = \Delta^{(0)}$

$\mathrm{tr}\, Z^J$ is an example

other ~~examples~~ primaries can be generated from $\mathrm{tr}\, Z^J$ by acting on this with SO(6) generators.

$\mathrm{tr}\, Z^J$ is in the symmetric traceless rep (under SO(6)) of SO(6) (Obviously symmetric, also traceless since there are no $\bar{Z}$ terms)

$\Rightarrow$ all symmetric traceless reps are chiral primaries.

$\mathrm{tr}(XYZ) \pm \mathrm{tr}(ZYX)$ $\pm$ is chiral primary (using cyclic symmetry), $-$ is not chiral primary (there is a contribution to the trace)

$\mathrm{tr}(Z\bar{X})$ is chiral primary

Most general combinations of scalar fields are not chiral primaries. To simplify life further, we will only consider

$\mathcal{O}(x) = \mathrm{tr}(\phi_{i_1} \cdots \phi_{i_L})$, $\Delta^{(0)} = L$

One-loop corrections

In general, corrections due to interactions, to $\Delta^{(0)}$

$$\Delta = \Delta^{(0)} + \gamma \qquad \text{"anomalous dimension"}$$

For small coupling $\gamma \sim g^2 N = \lambda$ "t' Hooft" coupling

$$\langle \mathcal{O}(x)\,\bar{\mathcal{O}}(y)\rangle = \frac{C}{|x-y|^{2\Delta}} = \frac{C}{|x-y|^{2\Delta^{(0)}+2\gamma}}$$

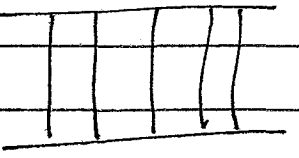
$$\approx \frac{C}{|x-y|^{2\Delta^{(0)}}}\left(1-\gamma \ln \Lambda^2 |x-y|^2\right)$$

Compute log divergences.

$$S = \frac{1}{2g^2}\int d^4x \left\{ -\frac{1}{2}\mathrm{Tr}\,F^2 + \mathrm{Tr}\,D_\mu \Phi_i\, D^\mu \Phi^i - \mathrm{Tr}[\Phi_i,\Phi_j]^2 \right.$$

$$\left. + \text{fermions} \right.$$

$$\mathcal{O}(x) \equiv \frac{1}{N^{L/2}}\mathrm{Tr}\left[\Phi_{i_1}\cdots\Phi_{i_L}(x)\right] \Rightarrow$$



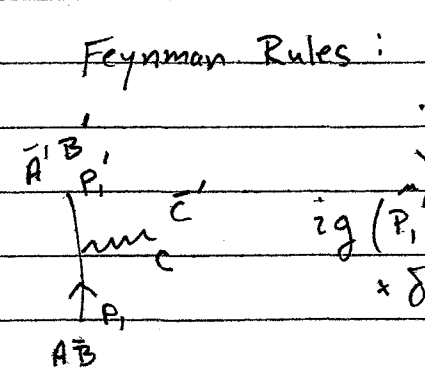
Planar contribution to correlation function

$$\mathcal{O}(x) = \int \prod_{k=1}^{L}\left(\frac{d^4p_k}{(2\pi)^4}\, e^{-i p_k\cdot x}\right)\mathrm{Tr}\left(\Phi_{i_1}(p_1)\cdots\Phi_{i_L}(p_L)\right)$$

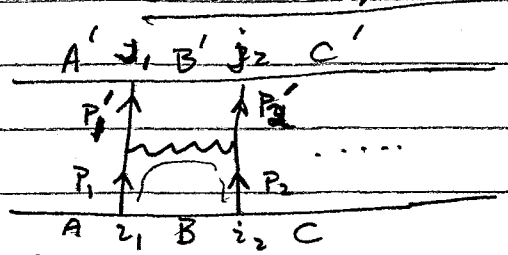
$$\langle \mathcal{O}(x)\bar{\mathcal{O}}(y)\rangle = \frac{1}{N^L}\cdot N^L \int \prod_{k=1}^{L}\left(e^{-ip_k\cdot(x-y)}\frac{d^4p_k}{(2\pi)^4}\frac{i}{p_k^2}\right)$$

$$\sim \frac{1}{|x-y|^{2L}} \qquad \text{by dimensional analysis}$$

Feynman Rules:



One-loop corrections



$$\int \frac{d^4p_1}{(2\pi)^4}\frac{d^4p_2}{(2\pi)^4}\cdots\int \frac{d^4p_1'}{(2\pi)^4}\frac{d^4p_2'}{(2\pi)^4}\; e^{i(p_1'+p_2')\cdot y - i(p_1+p_2)\cdot x}$$

$$\frac{i}{p_1^2}\cdot\frac{i}{p_2^2}\cdot\frac{i}{p_1'^2}\cdot\frac{i(2\pi)^4\delta^4(p_1'+p_2'-p_1-p_2)(-ig)^2}{p_2'^2}$$

$$\frac{(-i)(p_1'+p_1)\cdot(p_2'+p_2)}{(p_1'-p_1)^2}\;\delta_{A\bar{A}'}\,\delta_{B\bar{B}}\,\delta_{B'\bar{B}'}\,\delta_{C\bar{C}}^{\mu\nu}\;\delta_{i_1}^{j_1}\,\delta_{i_2}^{j_2}$$

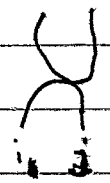
For large p_1' this is

$$\left[\int \frac{d^4p_1}{(2\pi)^4}\frac{d^4p_2}{(2\pi)^4}\frac{i}{p_1^2}\cdot\frac{i}{p_2^2}N^2\cdot\cdots\; e^{i(p_1+p_2)\cdot(y-x)}\right]$$

$$\times\int\frac{d^4p_1'}{(2\pi)^4}\cdot\frac{(-i)}{[(p_1')^2]^2}\,g^2N\,\delta_{i_1}^{j_1}\delta_{i_2}^{j_2}\left[\quad\right]\frac{\lambda}{8\pi^2}\ln\Lambda\,\delta_{i_1}^{j_1}\delta_{i_2}^{j_2}$$

We also have

$$= -2\,\frac{1}{8\pi^2}\ln\Lambda\;\delta_{i_1}^{j_1}$$



Finally contribution to $\mathcal{L}$ of 4-scalar term is

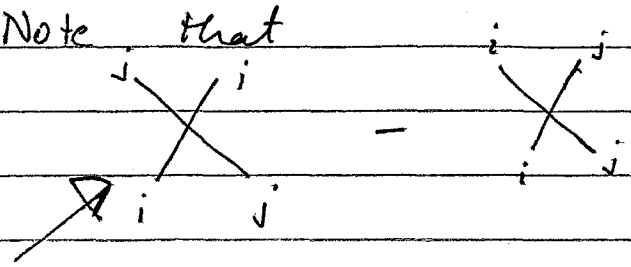
$$-\frac{g^2}{2}\sum_{i,j}\left(\mathrm{tr}\,\phi_i\phi_j\phi_i\phi_j - \mathrm{tr}\,\phi_i^2\phi_j^2\right)$$

$$4\cdot\frac{ig^2N}{2}\int\frac{d^4p_1}{(2\pi)^4}\frac{d^4p_2}{(2\pi)^4}\frac{d^4p_1'}{(2\pi)^4}\frac{d^4p_2'}{(2\pi)^4}\,\frac{i}{p_1^2}\cdot\frac{i}{p_2^2}\,\frac{i}{p_1'^2}\,\frac{i}{p_2'^2}\,(2\pi)^4\delta^4(p_1'+p_2'-p_1-p_2)$$

$$\left[\int\frac{d^4p}{(2\pi)^4}e^{-ip(x-y)_1}\frac{i}{p^2}\right]^{L-2}\qquad e^{i(p_1'+p_2')y - i(p_1+p_2)x}\qquad \delta_{i_1j_2}\delta_{i_2j_1}$$

$$=\left[\quad\right]^L g^2N\,\frac{\ln\Lambda}{(4\pi^2)}\;\delta_{i_1j_2}\delta_{i_2j_1}$$

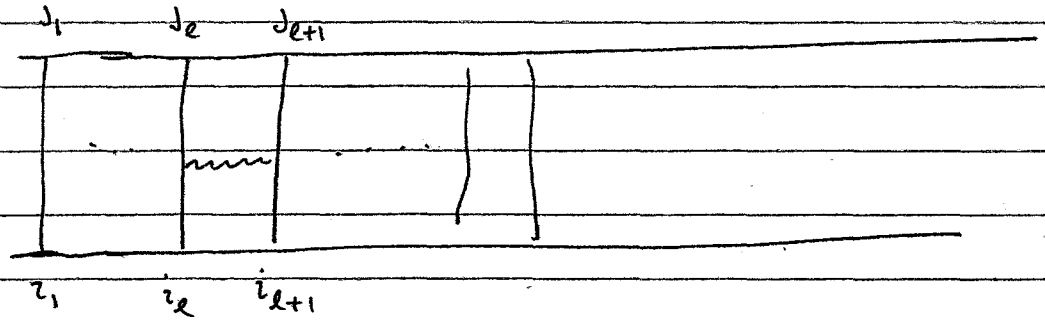
Note that

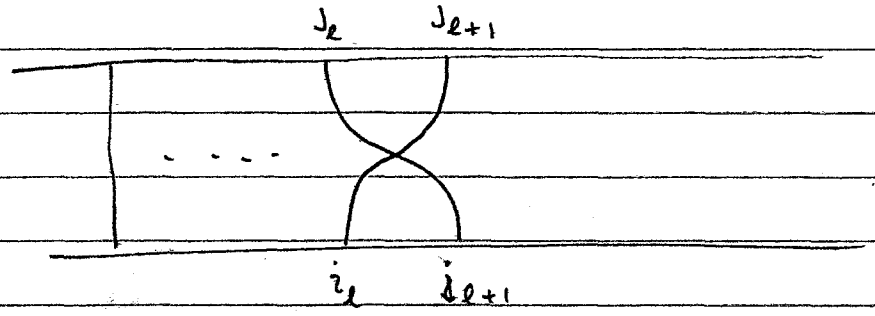


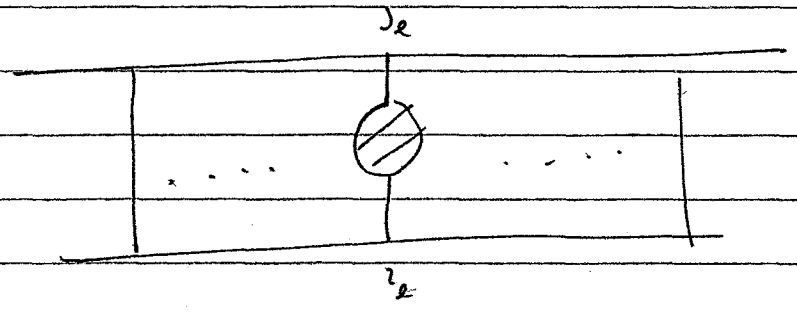
This leads to

$$-\frac{2\lambda}{16\pi^2}\delta_{i_\ell}^{\;j_{\ell+1}}\delta_{i_{\ell+1}}^{\;j_{\ell+1}} \Rightarrow \text{hence the other term leads to}$$

$$+\frac{\lambda}{16\pi^2}\left(\delta_{i_\ell}^{\;j_\ell}\delta_{i_{\ell+1}}^{\;j_{\ell+1}} \pm \delta_{i_\ell\, i_{\ell+1}}\delta^{j_\ell\, j_{\ell+1}}\right)$$







We end up with the following mixing matrix

$$\Gamma = \frac{\lambda}{16\pi^2} \sum_{\ell=1}^{L} (2 - 2P_{\ell,\ell+1} + K_{\ell,\ell+1})$$

$$P^{j_\ell j_{\ell+1}}_{i_\ell i_{\ell+1}} = \delta^{j_{\ell+1}}_{i_\ell} \delta^{j_\ell}_{i_{\ell+1}} \qquad K^{j_\ell j_{\ell+1}}_{i_\ell i_{\ell+1}} = \delta^{j_\ell j_{\ell+1}}_{i_\ell i_{\ell+1}}$$

Check Chiral primaries, θ

$K_{\ell,\ell+1}\theta = 0$ primaries are traceless

$P_{\ell,\ell+1}\theta = \theta$ (primaries are symmetric

$\Gamma\theta = 0$ Good!

Suppose we consider scalar fields made up of two types of operators

$$Z = \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2) \qquad W = \frac{1}{\sqrt{2}}(\phi_3 + i\phi_4)$$

Then $K_{\ell,\ell+1}\theta = 0$ (there are no $\bar{Z}$ or $\bar{W}$ so no traces)

$$\Gamma = \frac{\lambda}{8\pi^2} \sum_{\ell=1}^{L} (1 - P_{\ell,\ell+1})$$

Z = "up spin" $|\uparrow\rangle$

W = "down spin" $|\downarrow\rangle$

$$\Rightarrow P_{\ell,\ell+1} = \frac{1}{2}(1 + \vec{\sigma}_\ell \cdot \vec{\sigma}_{\ell+1})$$

$$= \frac{1}{2}\left(1 + \begin{pmatrix}1&0\\0&-1\end{pmatrix}_\ell \otimes \begin{pmatrix}1&0\\0&-1\end{pmatrix}_{\ell+1} + 2\begin{pmatrix}0&1\\0&0\end{pmatrix}_\ell \otimes \begin{pmatrix}0&0\\1&0\end{pmatrix}_{\ell+1} + 2\begin{pmatrix}0&0\\1&0\end{pmatrix}_\ell \otimes \begin{pmatrix}0&1\\0&0\end{pmatrix}_{\ell+1}\right)$$

$P_{\ell,\ell+1}\,|\cdots\uparrow\uparrow\cdots\rangle = |\cdots\uparrow\uparrow\cdots\rangle$

$P_{\ell,\ell+1}\,|\cdots\uparrow\downarrow\cdots\rangle = |\cdots\downarrow\uparrow\cdots\rangle$

$$\Gamma = \frac{\lambda}{16\pi^2} \sum_{\ell=1}^{\infty} (1 - \vec{\sigma}_\ell \cdot \vec{\sigma}_{\ell+1})$$

"Ferromagnetic spin chain"

Heisenberg spin chain (or magnet)

(Note that $\sum_\ell \vec{\sigma}_\ell$ commutes with Γ)

$XXX_{\frac{1}{2}}$ model. "XXX" operators.

More general case is an SO(6) spin-chain. where each site in the chain is an SO(6) vector (corresponding to the type of scalar in the operator ~~product~~). The cyclic property of the trace makes this a closed chain with L sites where if we shift by L sites we come back to the original position. Nearest neighbor interactions.

Can we find the eigenvalues of Γ for XXX operators $\frac{1}{2}$, or more general SO(6) operators?

Yes, in principle (and sometimes in practice)

Use the fact that theory is integrable.

Quantum level: Show that there are L commuting charges, one of which is the Hamiltonian. Simultaneously can be diagonalized. Since H generates time translations, these charges are constant in time.

$$\langle \psi(t) | Q | \psi(t) \rangle = \langle Q \rangle$$
$$= \langle \psi(0) | e^{iHt} Q e^{-iHt} | \psi(0) \rangle$$
$$= \langle \psi(0) | Q | \psi(0) \rangle \quad \text{if } [H, Q] = 0$$

Enough constants of the motion $\Rightarrow$ solve problem.

The R-matrix and integrability

Yang Baxter eg.

$$R_{12}(u)\,R_{13}(u+v)\,R_{23}(v) = R_{23}(v)\,R_{13}(u+v)\,R_{12}(u)$$

u and v are "spectral parameters"

This acts on $V_1 \otimes V_2 \otimes V_3$ each V_i is n dimensional vector space. (later $n=6$)

Consider $T_0(u) = R_{01}(u) R_{02}(u) \cdots R_{0L}(u)$ acting on $V_0 \otimes V_1 \otimes \cdots V_L$

"transfer or monodromy matrix"

V_0 is "auxillary space"

YBE $\Rightarrow$ $Tr_0\, T(u) = t(u)$ $\quad [t(u), t(v)] = 0$

~~$Tr_0 (R_{01}(u) \cdots R_{0L}(u))\, Tr_0 (R_{01}(v) \cdots R_{0L}(v))$~~

Consider 2 auxillary spaces, a, b.

$R_{ab}(u-v)\, T_a(u)\, T_b(v)$

$= R_{ab}(u-v)\left(R_{a1}(u) \cdots R_{aL}(u)\right)\left(R_{b1}(v) \cdots R_{bL}(v)\right)$

$= R_{ab}(u-v)\left(R_{a1}(u) R_{b1}(v)\; R_{a2}(u) R_{b2}(v) \cdots R_{aL}(u) R_{bL}(v)\right)$

$= R_{b1}(v) R_{a1}(u) R_{ab}(u-v)\; R_{a2}(u) R_{b2}(v) - - - - - -$

$= T_b(v)\, T_a(u)\, R_{ab}(u-v)$

$T_a(u)\, T_b(v) = R^{-1}_{ab}(u-v)\, T_b(v)\, T_a(u) R_{ab}(u-v)$

$(Tr_a T(u))(Tr_b T(v)) = Tr_b T(v)\, Tr_a T(u)$

$R^{-1}_{ab} R_{ab}\; V_a \otimes V_b = V_b \otimes V_a$

$t(u) = Tr\, T(u) \Rightarrow [t(u), t(v)] = 0$

$\therefore$ $t(u)$ is the generator for commuting charges.

Let $R_{12} = u - P_{12}$

Show that this satisfies YBE

$$R_{12}(u)\, R_{13}(u+v)\, R_{23}(v) = R_{23}(v)\, R_{13}(u+v)\, R_{12}(u)$$

Need to check terms with 2 or 3 P's

$P_{12} P_{13} P_{23} \overset{?}{=} P_{23} P_{13} P_{12}$ use $P_{12}P_{13} = P_{13}P_{23}$

$$\underbrace{P_{12} P_{13} P_{23}}_{P_{13} = P_{23} P_{13} P_{12}}\, V_1 \otimes V_2 \otimes V_3 = P_{12} P_{13}\, V_1 \otimes V_3 \otimes V_2 = P_{12}\, V_2 \otimes V_3 \otimes V_1 = V_3 \otimes V_2 \otimes V_1$$

$$u P_{13} P_{23} + (u+v) P_{12} P_{23} + v P_{12} P_{13} \overset{?}{=} v P_{13} P_{12} + (u+v)(P_{23} P_{12}) + u P_{23} P_{13}$$

$$= (u+v)\{P_{13} P_{23} + P_{12} P_{23}\} \qquad v P_{12} P_{23} + (u+v) P_{13} P_{23} + u P_{12} P_{23}$$

$t(u)$ is going to be a polynomial of order $L \Rightarrow L$ different commuting variables.

Check: $\frac{1}{n-2}\left[u(2u+2-n) - (2(u+2-n) P_{12} + 2u K_{12}\right] = R_{12}$ satisfies YBE

$$K_{12} K_{13} K_{23}\, V_1 \otimes V_2 \otimes V_3$$
$$= K_{12} K_{13}\, V_2 \cdot V_3\, V_1 \otimes \hat{e}_i \otimes \hat{e}_i$$
$$= K_{12}\, V_{1,i}\, V_2 \cdot V_3\, \hat{e}_j \otimes \hat{e}_i \otimes \hat{e}_j$$
$$= \delta_{ij}\, \alpha_{1,jk}\, V_2 \cdot V_3\, \hat{e}_k \otimes \hat{e}_k \otimes V_1$$

$$K_{23} K_{13} K_{12} = V_1 \cdot V_2\, V_3 \otimes \hat{e}_k \otimes \hat{e}_k$$

$$\{(2u+2-n) K_{13} K_{23} + (2(u+v)+2-n) K_{12} K_{23} + (2v+2-n) K_{12} K_{13}\}$$
$$V_1 \otimes V_2 \otimes V_3$$

$$= \{(2u+2-n)\, V_2 \cdot V_3\, \hat{e}_k \otimes V_1 \otimes \hat{e}_k + (2(u+v)+2-n)\, V_2 \cdot V_3\, \hat{e}_k \otimes \hat{e}_k \otimes V_1$$
$$+ (2v+2-n)\, V_1 \cdot V_3\, V_2 \otimes \hat{e}_k \otimes \hat{e}_k$$

~~$(2u+2-n)$~~ etc.

All terms have $V_a \cdot V_b\, \hat{e}_k \otimes \hat{e}_k \otimes V_c$ term.
$\hat{e}_k \otimes V_c \otimes \hat{e}_k$
$V_c \otimes \hat{e}_k \otimes \hat{e}_k$

a, b, c are all different.

All terms cancel.

Taylor expansion of u gives charges

$$T(0)\ V_0 \otimes V_1 \cdots V_L$$

$$= (-P_0)(-P_{02}) \cdots (-P_{0L})\ V_0 \otimes V_1 \otimes \cdots V_L$$

$$= (-1)^L\ V_1 \otimes V_2 \otimes \cdots V_L \otimes V_0$$

$$\left(t_0 = Tr\,(T(0))\right) V_1 \otimes \cdots V_L = V_2 \otimes \cdots V_L \otimes V_1\ (-1)^L$$

$$= \sum_i \hat{e}_i \cdot \vec{V}_1 \left[V_2 \otimes \cdots \otimes \hat{e}_i\right] (-1)^L = (-1)^L\ e^{iP}\ V_1 \cdots V_L$$

$$= [V_2 \otimes \cdots V_L \otimes V_1]$$

Shift operator.

$$T'(0)\ V_0 \otimes \cdots V_L$$

$$= (-1)^{L-1} \left\{ P_{01} \cdots P_{0L-1} + P_{01} \cdots P_{0L-2}\ P_{0L} + \cdots P_{02} \cdots P_{0L} \right\} V_0 \otimes V_1 \cdots V_L$$

$$= (-1)^{L-1} \{ V_1 \otimes \cdots V_0 \otimes V_L + V_1 \otimes \cdots V_L \otimes V_{L-1} \otimes V_0 + \cdots V_2 \otimes V_1 \otimes V_3 \cdots V_L \otimes V_0$$

$$\left(t_1 = Tr\ T'(0)\right) V_1 \otimes \cdots V_L$$

$$= (-1)^L \left[V_2 \otimes \cdots V_1 \otimes V_L + V_2 \otimes \cdots V_L \otimes V_{L-1} \otimes V_1 \cdots \right]$$

$$= (-1)^L e^{iP} \left\{ -\sum_{l=1}^{L} P_{l,l+1} \right\} V_1 \otimes \cdots V_L$$

$\therefore\ -\sum_{l=1}^{L} P_{l,l+1}$ is a conserved charge.

(and so is $\sum_{l=1}^{L} (1 - P_{l,l+1})$ is also conserved.

Check

$$t_1 = \frac{2}{n-2}\, t_0 \left(\sum_{l=1}^{L} K_{l,l+1} - 1 - \frac{n-2}{2} P_{l,l+1} \right)$$

$n = 6$ leaves correct ratios of coefficients.

We now wish to show two properties

Let $\Lambda(u)$ be the eigenvalues of $t(u)$ (specialize to $n=2$)

1) $\Lambda(u)^* = (-1)^L \Lambda(1-u)$ if u real

2) $\Lambda(u)\Lambda(u+1) = (u^2-1)^L + O(u^L)$

These are highly restrictive.

$$t(u)^{T} = Tr_0\left[R_{a1}^{T_1} R_{a2}^{T_2} \cdots R_{aL}^{T_L} \right]$$

where T_l tells us to transpose l indices only

$$(R_{al})^{T_e}_{(1-u)} = -\delta_{i_0}^{j_0}\delta_{i_l}^{j_l}\cdot u + \underbrace{\left(\delta_{i_0}^{j_0}\delta_{i_l}^{j_l} - \delta_{i_0 i_l}\delta^{j_0 j_l}\right)}_{S_{0l}}$$

Check that

$$S_{0l}\,|\uparrow\uparrow\rangle = -|\downarrow\downarrow\rangle$$
$$S_{0l}\,|\downarrow\downarrow\rangle = -|\uparrow\uparrow\rangle$$
$$S_{0l}\,|\uparrow\downarrow\rangle = |\uparrow\downarrow\rangle$$
$$S_{0l}\,|\downarrow\uparrow\rangle = |\downarrow\uparrow\rangle$$

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}_0 S_{0l} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}_0 = -P_{0l} \qquad \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}^2 = -1$$

$$\Rightarrow (R_{0l})^{T_e}(1-u) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}_0 \left[u - P_{0l}\right] \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}_0$$

$$\Rightarrow t(1-u)^{T} = (-1)^L t(u)$$

Consider the next

$$R_{ab}(-1)\,R_{al}(u)\,R_{bl}(u+1) = R_{bl}(u+1)\,R_{al}(u)\,R_{ab}(-1)$$

$$\Rightarrow \tfrac{1}{2}(1+P_{ab})\,R_{al}(u)\,R_{bl}(u+1) = R_{bl}(u+1)\,R_{al}(u)\,\tfrac{1}{2}(1+P_{ab})$$

$$\Rightarrow \tfrac{1}{2}(1+P_{ab})\,R_{al}(u)\,R_{bl}(u+1)\cdot\tfrac{1}{2}(1-P_{ab}) = 0$$

$\chi_{ab\ell}(u) = R_{a\ell}(u)\, R_{b\ell}(u+1)$ is upper triangular

$$\begin{pmatrix} \frac{1}{2}(1-P_{ab})\chi_{ab\ell}\frac{1}{2}(1-P_{ab}) & \frac{1}{2}(1-P_{ab})\chi_{ab\ell}\frac{1}{2}(1+P_{ab}) \\ 0 & \frac{1}{2}(1+P_{ab})\chi_{ab\ell}(u)\frac{1}{2}(1+P_{ab}) \end{pmatrix}$$

$\Rightarrow \prod_{\ell=1}^{L} \chi_{ab\ell}$ is upper triangular.

To complete proof, you need to show that the antisymmetric part (on $V_1 \otimes \cdots V_L$) which is 1-dimensional is diagonal and equals $(u^2-1)^L I$

The symmetric part is proportional to u.

Straight forward if somewhat tedious.

About $\frac{1}{2}$ a page of algebra.

$$\Rightarrow \prod_{\ell=1}^{L} \chi_{ab\ell} = \begin{pmatrix} (u^2-1)^L & * \\ 0 & O(u^L) \end{pmatrix}$$

1) and 2) are highly restrictive.

$$T_a(u)\, T_b(u+1) = \prod_{\ell=1}^{L} \chi_{ab\ell}(u)$$

Try $A(u) =$

$$Tr_a T_a(u)\, Tr_b T_b(u+1) = (u^2-1)^L I + O(u^L)$$

Look for solutions.

$\langle\uparrow\uparrow|$ $\quad |\uparrow\uparrow\rangle$ $\quad \langle\uparrow\uparrow|$ $\quad |\uparrow\uparrow\rangle$

$+\langle\uparrow\downarrow|$ $\quad |\uparrow\downarrow\rangle$ $\quad$ tr $+\frac{1}{2}\langle\uparrow\downarrow| + \langle\downarrow\uparrow|$ $| \quad |\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle$

$+\langle\downarrow\uparrow|$ $\quad |\downarrow\uparrow\rangle$ $\quad \pm\frac{1}{2}\langle\uparrow\downarrow| - \langle\downarrow\uparrow|($ $\quad)|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle$

$+\langle\downarrow\downarrow|$ $\quad |\downarrow\downarrow\rangle$ $\quad \langle\downarrow\downarrow|($ $\quad)|\downarrow\downarrow\rangle$

Look for solutions.

Highly restrictive.

$\Lambda(u)$ is eigenvalue of $t(u)$

Two facts I won't prove here

Claim:

$$\Lambda(u) = (u-1)^L \prod_{j=1}^{n} \frac{u - iu_j + \frac{1}{2}}{u - iu_j - \frac{1}{2}} + u^L \prod_{j=1}^{n} \frac{u - iu_j - 3/2}{u - iu_j - \frac{1}{2}}$$

1) $\Lambda^*(1-u) = \Lambda(u)$ if u is real

Fact 1) implies that for $\forall k$ $u_k^* = u_j$ for some j

By construction $\Lambda(u)$ is a polynomial of order L in u

$\Rightarrow$ poles must cancel $u = iu_j + 1/2$

$$(iu_j - \tfrac{1}{2})^L \prod_{k=1}^{n} \left(i(u_j - u_k) + i \right) + (iu_j + \tfrac{1}{2})^L \prod_{k=1}^{n} \left(i(u_j - u_k) - 1 \right) = 0$$

Bethe equations

$$\Rightarrow \left(\frac{u_j + i/2}{u_j - i/2} \right)^L = \prod_{k \neq j} \frac{u_j - u_k + i}{u_j - u_k - i}$$

u_j are called Bethe roots.

On the ground state since $P_{ij} = 1$

$\Lambda(u) = (u-1)^L + u^L \Rightarrow$ define charges to be 0 on ground state (has no Bethe roots)

$t(0) = (-1)^L P_{01} P_{02} \cdots P_{0L} = e^{iP}$ shift operator

$t(u) = (u-1)^L e^{iP} e^{i \sum_{n=1}^{\infty} (-iu)^n Q_n \cdot \frac{1}{n}}$

Q_n are local charges.

Compare to $\Lambda(iu) \Rightarrow \frac{u - iu_j - \frac{i}{2}}{u - u_j + i/2} = e^{iP + \sum \frac{u^n}{n} Q_n} \equiv H(u)$

$$-i \log \left[\frac{u - u_j - i/2}{u - u_j + i/2} \right] = \text{some} \sum_{n=0}^{\infty} u^n Q_n \qquad Q_0 = P$$

$$Q_0 = -i \log \prod_{j=1}^{n} \frac{x_j + i/2}{x_j - i/2}$$

$$Q_1 = -i \frac{\partial}{\partial x} H(x) = +i \sum_{j=1}^{n} \left(\frac{1}{x_j + i/2} - \frac{1}{x_j - i/2} \right)$$

$$= \sum_{j=1}^{n} \frac{1}{x_j^2 + 1/4}$$

$$\gamma = \frac{\lambda}{8\pi^2} Q_1$$

More general solution for SO(6)

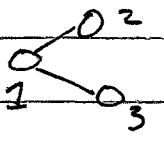
3 types of roots

$$\left(\frac{u_{1,j}+i/2}{u_{1,j}-i/2}\right)^L = \prod_{\substack{\ell=1\\ \ell\neq j}}^{n_1} \frac{u_{1,j}-u_{1,\ell}+i}{u_{1,j}-u_{1,\ell}-i} \prod_{\ell=1}^{n_2} \frac{u_{1,j}-u_{2,\ell}-i/2}{u_{1,j}-u_{2,\ell}+i/2} \prod_{\ell=1}^{n_3} \frac{u_{1,j}-u_{3,\ell}}{u_{1,j}-u_{3,\ell}}$$

$$1 = \prod_{\substack{\ell=1\\ \ell\neq j}}^{n_2} \frac{u_{2,j}-u_{2,\ell}+i}{u_{2,j}-u_{2,\ell}-i} \prod_{\ell=1}^{n_1} \frac{u_{2,j}-u_{1,i}-i/2}{u_{2,j}-u_{1,i}+i/2}$$

Same ($2 \to 3$)

Related to Dynkin diagrams.



$$\vec{\alpha}_1\cdot\vec{\alpha}_1 = \vec{\alpha}_2\cdot\vec{\alpha}_2 = \vec{\alpha}_3\cdot\vec{\alpha}_3 = 2$$
$$\vec{\alpha}_1\cdot\vec{\alpha}_2 = \vec{\alpha}_1\cdot\vec{\alpha}_3 = -1$$
$$\vec{\alpha}_2\cdot\vec{\alpha}_3 = 0$$

Vector rep. Highest weight satisfies $\vec{w}\cdot\vec{\alpha}_1 = 1$

$\vec{w}\cdot\vec{\alpha}_2 = \vec{w}\cdot\vec{\alpha}_3 = 0$

Total representation is

$$\vec{W} = L\vec{w} - n_1\vec{\alpha}_1 - n_2\vec{\alpha}_2 - n_3\vec{\alpha}_3$$

In terms of Cartan subalgebra ($SO(2)\times SO(2)\times SO(2)$ subgroup

$\vec{w} = (1,0,0)$ $\quad \vec{\alpha}_1 = (1,-1,0)$

$\vec{\alpha}_2 = (0,1,-1)$

$\vec{\alpha}_3 = (0,1,1)$

$$\vec{W} = (L-n_1,\ n_1-n_2-n_3,\ n_2-n_3) = (J_1, J_2, J_3)$$

Highest weights have $J_1 \geq J_2 \geq J_3 \geq 0$

$$\Rightarrow n_1 \geq 2n_2 \geq 2n_3$$

~~$L \geq n_1 + n_2 + n_3$~~ $\quad n_1 \leq \frac{(L+n_2+n_3)}{2}$

If we only have $u_{1,j}$ roots $\Rightarrow n_2 = n_3 = 0$

then this means we have a restricted space when we have $(1,0,0) \to (0,1,0)$

$Z \qquad W$ this restricted case are the bethe ~~functions~~ eqs. for XXX chain.

Charges from the roots

$$R_{12}(u) = \frac{1}{u-2}\left[u(2u-u-2) - (2u-u-2)P_{12} + u K_{12}\right]$$

Expand at $u=0$

$$t(0) = \mathrm{Tr}\, P_{01} P_{02} \cdots P_{0L} = e^{iP}$$ shift operator.

Expand about u gives local operators.

We also want the local operators to have 0 eigenvalue on the ground state.

On the ground states, we see that

$$R_{ij}(u) = \frac{(u-1)(2u-u-2)}{(u-2)}$$

$$\therefore \quad \Lambda(u) = \frac{(u-1)^L (2u-u-2)^L}{(u-2)^L} e^{iP} e^{i\sum_{n=1}^{\infty} (-2iu)^n Q_n \cdot \frac{1}{n}}$$

Comparing to our expression, we see that

$$H(iu) = e^{i\sum_{n=0}^{\infty} u^n Q_n \frac{1}{n}} \qquad Q_0 = P$$

$$\therefore \quad \sum_{n=0}^{\infty} \frac{1}{n} u^n Q_n = -i \log H(2u)$$

$$H(iu) = \prod_{j=1}^{n_1} \frac{u - u_{1,j} - i/2}{u - u_{1,j} + i/2}$$

$$P = -i \log \prod_{j=1}^{n_1} \frac{u_{1,j} + i/2}{u_{1,j} - i/2} = -i \sum \log \frac{u_{1,j}+i/2}{u_{1,j}-i/2}$$

$$Q_1 = -i \frac{\partial}{\partial u} \log H(iu)$$

$$= \sum \left(\frac{-i}{-u_{1,j} - i/2} + \frac{i}{-u_{1,j} + i/2} \right) = \sum_{j=1}^{n_1} \frac{1}{u_{1,j}^2 + 1/4}$$

Check: $\gamma = \frac{\lambda}{8\pi^2} Q_1$

~~Examples with 2 impurities~~

The trace condition:

The operators are

$$Tr[\phi_{i_1}\phi_{i_2}\cdots\phi_{i_L}] = Tr[\phi_{i_L}\phi_{i_1}\cdots\phi_{i_{L-1}}]$$

For a spin chain

$$e^{iPL}|\uparrow\downarrow\uparrow\cdots\rangle = |\uparrow\downarrow\uparrow\cdots\rangle$$

$e^{iPL}=1$ eigenvalues are $P=\frac{2\pi m}{L}$

For us $P=2\pi m$ $e^{iP}=1$

$$e^{iP} = \prod_{j=1}^{J}\frac{u_{1,j}+i/2}{u_{1,j}-i/2} = 1$$

Examples:

2 $u_{1,j}$ roots 0 $u_{2,j}$ or $u_{3,j}$ roots

· Call these u_j

Bethe eqs

Momentum condition $\frac{u_1+i/2}{u_1-i/2}\frac{u_2+i/2}{u_2-i/2}=1$

$$\Rightarrow u_1=-u_2$$

Reality condition $u_1^* = u_1$ or $u_1^*=u_2$

$\Rightarrow u_1$ is real

B. eqs.

$$\left(\frac{u+i/2}{u-i/2}\right)^L = \frac{u-(-u)+i}{u-(-u)+i} = \frac{u+i/2}{u-i/2}$$

$$\Rightarrow \left(\frac{u+i/2}{u-i/2}\right)^{L-1}=1$$

$$\Rightarrow \frac{u+i/2}{u-i/2} = e^{2\pi i m/(L-1)} \qquad u=\frac{i}{2}\frac{e^{\frac{2\pi m}{L-1}}+1}{e^{\frac{2\pi m}{L-1}}-1}$$

$$= \frac{1}{2}\cot\frac{\pi m}{L-1}$$

$$\gamma = \frac{\lambda}{8\pi^2} \cdot 2 \cdot \frac{4}{\cot^2 \frac{\pi m}{L-1} + 1} = \frac{\lambda}{\pi^2} \sin^2 \frac{\pi m}{L-1}$$

Compare with Beisert

Now choose two $u_{1,s}$ and a u_2

$u_{2,j}$ does not contribute to P

$u_{1,1} = -u_{1,2} = u$.

$u_{2,1}$ equation

2nd B.E. $\quad 1 = \dfrac{u_{2,j} - u - i/2}{u_{2,1} - u + i/2} \dfrac{u_{2,1} + u - i/2}{u_{2,2} + u + i/2} \Rightarrow u_{2,1} = 0$

1st B.E $\quad \left(\dfrac{u+i/2}{u-i/2}\right)^L = \dfrac{2u+i}{2u-i} \cdot \dfrac{u-i/2}{u+i/2} = 1$

we write $\quad \gamma = \dfrac{\lambda}{\pi^2} \sin^2 \dfrac{\pi m}{L}$

two $u_{1,s}$ u_2, u_3 $\Rightarrow u_3 = 0$

$$\left(\frac{u+i/2}{u-i/2}\right)^{L+1} = 1 \Rightarrow \gamma = \frac{\lambda}{\pi^2} \sin^2 \frac{\pi m}{L+1}$$

This is most of what I can solve exactly.

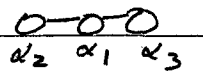
Homework, show that for any number u_1 of $u_{1,4}$ roots, we can have $u_{2,1} = 0$ or $u_{2,1} = 0$ and $u_{3,1} = 0$

and $\gamma_2(L) = \gamma(L+1) \qquad \gamma_{23} = \gamma(L+2)$

(The reason is that the operators are in the same super multiplets)

The representations:

Highest weight with $\vec{W} = (L-2, 2, 0)$ — 2 $x_{1,j}$ roots



Dynkin index

$$[\vec{W}\cdot\vec{\alpha}_2, \vec{W}\cdot\vec{\alpha}_1, \vec{W}\cdot\vec{\alpha}_3] = [2, L-4, 2]$$

Dynkin indices are always non-negative

$\vec{w} = (1,0,0)$

$$[(\vec{W}\cdot L - n_1\vec{\alpha}_1 - n_2\vec{\alpha}_2 - n_3\vec{\alpha}_3)\cdot\vec{\alpha}_2, (\quad)\cdot\vec{\alpha}_3, (\quad)\cdot\vec{\alpha}_3]$$

$$= [\tfrac{n_1}{2} - n_2,\ L - 2n_1 + n_2 + n_3,\ n_1 - 2n_3]$$

$n_2 \le n_1/2 \qquad n_3 \le n_1/2 \qquad n_1 \le \frac{1}{2}(L + n_1 + n_2)$

$2x_{1,j}$ roots $x_{2,j}$ root

$$[0, L-3, 2]$$

$2x_{1,j}$, $x_{2,j}$, $x_{3,j}$ $\qquad [0, L-2, 0]$

$[0, L, 0]$ is traceless symmetric rep with L indices

$[0, L-2, 0]$ is traceless symmetric rep with $L-2$ indices

Note that this is not chiral primary since $\Delta^{(0)} = L$, not $L-2$

Konishi: $\mathrm{Tr}[\phi_i \phi_i]$ $SO(6)$ singlet $[0, 2-2, 0]$

$$\gamma = \frac{\lambda}{\pi^2}\sin^2\frac{\pi}{3} \qquad (m=1 \text{ is only independent answer})$$

$$= \frac{3\lambda}{4\pi^2} \quad \checkmark$$

Let us concentrate on XXX operators

Z and W inside trace.

We restrict ourselves to the SU(2) subgroup of SO(6)

We only have $u_{1,j}$ roots. "u_j"

B.E. $$\left(\frac{u_j+i/2}{u_j-i/2}\right)^L = \prod_{k\neq j}^{J'} \frac{u_j-u_k+i}{u_j-u_k-i}$$

~~Tr 1+Z~~ $\mathrm{Tr}(Z^J W^{J'}) + \cdots$ $\quad J+J'=L$

Consider ~~BMN Limit~~ if u_j is real then

$e^{iP_j} \equiv \frac{u_j+i/2}{u_j-i/2}$ is a phase

P_j is "magnon" momentum $e^{i\sum P_j} = e^{iP} = 1$

$\sum P_j = 2\pi m$

If u_j is complex, then there is a u_j^*

$\frac{u_j+i/2}{u_j-i/2}\,\frac{u_j^*+i/2}{u_j^*-i/2}$ is a phase

P_j is complex but $P_j + P_j^*$ is real

$$\mathcal{E}_j = \frac{1}{u_j^2+1/4} \qquad u_j = \tfrac{1}{2}\cot\frac{P_j}{2}$$

$$= 4\sin^2\frac{P_j}{2} = 2(1-\cos P_j)$$

$$= 2(1-\cos(P_r + iP_I))$$

$$= 2(1-\cos P_r \cosh P_I + i \sin P_r \sinh P_I)$$

$$\mathcal{E}_j + \mathcal{E}_j^* = 4(1-\cos P_r \cosh P_I) < 4(1-\cos P_r)$$

∴ this describes a bound ~~state~~ pair of 2 magnons.

How should we think of Bethe equations.

Magnon has quantized momentum.

If there is only 1 then

$e^{iP_1 L} = 1 \qquad P_1 = \frac{2\pi m}{L}$ usual momentum quantization on a latticized circle.

Review:

XXX operators: $Q_{J_1,J_2} = \mathrm{tr}(\underbrace{Z\cdots Z}^{J_1} \underbrace{W\cdots W}^{J_2}) + \cdots$ Recharge: $(J_1, J_2, C$

L scalars $\quad J_1 + J_2 = L$

$H = \frac{1}{8\pi^2} \sum_{\ell=1}^{L} (1 - P_{\ell,\ell+1})$ = Heisenberg Ferromagnet.

$Z = |\uparrow\rangle \quad W = |\downarrow\rangle$

~~Generators of Charges:~~ $\sum_{n=0} u^n Q_n = -i \log \left(\frac{u - u - i/2}{\cdots} \right)$

Find eigenvalues by solving Bethe equations

$$\left(\frac{u_j + i/2}{u_j - i/2}\right)^L = \prod_{k \neq j}^{J_2} \frac{u_j - u_k + i}{u_j - u_k - i} \qquad u_j \text{ Bethe roots}$$

$$\sum u^n Q_n = -i \sum_j \log\left[\frac{u - u_j - i/2}{u - u_j + i/2}\right]$$

$$Q_0 = P = -i \sum_j \log \frac{u_j + i/2}{u_j - i/2} = \sum_j P_j$$

One magnon:

$$\left(\frac{u_1 + i/2}{u_1 - i/2}\right)^L = 1 \qquad e^{iP_1} = \frac{u_1 + i/2}{u_1 - i/2} \Rightarrow P_1 = -i \log \frac{u_1 + i/2}{u_1 - i/2}$$

e^{iP_1}

Spin-chain $P_1 = \frac{2\pi n_1}{L}$

$$|P_1\rangle = \frac{1}{\sqrt{L}} \left[|\downarrow\uparrow\uparrow\cdots\rangle + e^{\frac{2\pi i n_1}{L}} |\uparrow\downarrow\uparrow\cdots\rangle + e^{\frac{4\pi i n_1}{L}} |\uparrow\uparrow\downarrow\cdots\rangle + \cdots \right]$$

Can't have these states for our operator, since if we shift by one site we pick up a phase. $P_1 = 0$ is trivial:

$$\sum_{\ell=1}^{L} \sigma_\ell^+ |P_1\rangle = 0 \quad \text{if} \quad P_1 \neq 0 \bmod 2\pi$$

Hence the 1-magnon states have spin $\frac{L}{2} - 1$

2 Magnons: $P_1 = 0 \Rightarrow u_1 = \infty$ ($u_1 = \infty$ is equivalent to no roots at all)

Claim: If we have J_2 roots $P_j \neq 0 \bmod 2\pi$ total spin $\frac{L}{2} - J_2$

2-magnons: In this case state with J_2 down spins is highest weight in rep.

Q_{J_1,J_2} is highest weight

J_2 magnons

$$e^{i P_L L} = \prod_{k \neq j}^{J_2} \frac{u_j - u_k + i}{u_j - u_k - i} = \exp\left[i \sum_{k \neq j} \delta(p_j, p_k) \right]$$

↑ Phase shift for scattering magnon j and magnon k.

Translating a magnon around the circle, means that we have to take it through the other $J_2 - 1$ magnons. Hence the phase shifts have to be taken into account. (There is a repulsive potential between magnons. Can't have 2 of them on the same site.)

Going beyond BMN limit

Suppose we have a large number of Bethe roots $J' \sim L$, but each root still has $|u_j| \gg 1$

Can we still "solve" the BE's?

Can no longer ignore the R.H.S.

$$\frac{u_j + i/2}{u_j - i/2} = \frac{1 + \frac{i}{2u_j}}{1 - \frac{i}{2u_j}} \approx e^{\frac{i}{u_j}}$$

$$\frac{u_j - u_k + i}{u_j - u_k - i} \approx e^{\frac{2i}{u_j - u_k}} \quad \text{if } |u_j - u_k| \gg 1$$

Assume this is true for most roots.

B.E. $$e^{iL/u_j} \approx \prod_{k \neq j} e^{\frac{2i}{u_j - u_k}}$$

Take log:

$$\frac{L}{u_j} - 2\pi n_j = \sum_{k \neq j} \frac{2}{u_j - u_k}$$

This looks like the equations for a matrix model! where u_j are the eigenvalues of the matrix

$$V'(u) = \frac{L}{u} - 2\pi n_j$$

(this is different from a matrix model since in this case $V(u)$ is a meromorphic function).

$\sum_{k \neq j} \frac{2}{u_j - u_k}$ is the repulsive force between the eigenvalues coming from the Vandermonde det.

Ignore the repulsion, we find $u_j = \frac{L}{2\pi n_j}$ gives $V'(u) = 0$

For matrix models, we assume in the large N limit that the eigenvalues accumulate onto a number of cuts, centered around where $V'(u) = 0$. For us, the different cuts arise because of different choices for n_j.

Next we notice that

$V''(x) = -\frac{1}{x^2}$. Hence the central potential is unstable.

To see what happens, suppose we had a matrix model with

$$V(x) = \frac{g_1}{2L}x^2 \qquad V'(x) = \frac{g_1 x}{L} \qquad V''(x) = \frac{1}{L} > 0$$

$$\frac{x_j}{L} = 2 \sum_{k \neq j} \frac{1}{x_j - x_k}$$

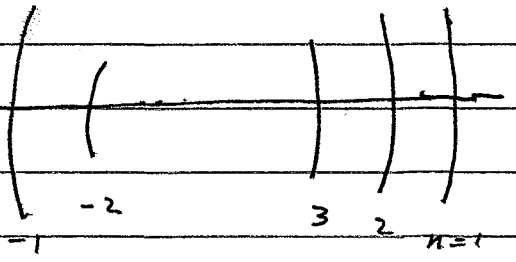
Suppose $V(x) = -\frac{1}{2L}x^2 \qquad V''(x) = -\frac{1}{L}$

$$V'(x) = -\frac{x}{L}$$

$$-\frac{x_j}{L} = 2 \sum_{k \neq j} \frac{1}{x_j - x_k} \qquad \text{Let } x_j = i v_j$$

$$-\frac{i v_j}{L} = \frac{2}{i} \sum_{k \neq j} \frac{1}{v_j - v_k} \Rightarrow \frac{v_j}{L} = \sum_{k \neq l} \frac{1}{v_j - v_k}$$

For our B.E. the effect is



with the roots assembling on many cuts.

Consider next ~~Resolvent~~ the scaling limit ~~G(x)~~. Let $x_j = L v_j$

$v_j - v_k \sim O(1)$ iff

accept for nearby roots on the same cut $\quad v_j - v_k \sim O(\frac{1}{L})$

Let $\rho(v) = \frac{1}{L} \sum_{j=1}^{J'} \delta(v - v_j)$

Then the Bethe Eqs. become

$$\frac{1}{v} - 2\pi n_j = 2 \int_{C_j} \frac{dv' \rho(v')}{v - v'} + \sum_{k \neq j} \int \frac{dv' \rho(v')}{v - v'} \qquad v \in C_j$$

C_j labels the various contours that the roots sit on. Each cut has an n_j associated with it.

Resolvent:

$$G(x) = \sum_k \int_{C_k} \frac{dx'\,\rho(x')}{x-x'}$$

$$G(x \pm i0) = \int_{C_j} \frac{dx'\,\rho(x')}{x-x'} \mp i\pi\rho(x) + \sum_{k\neq j} \int_{C_k} \frac{dx'\rho(x')}{x-x'} \qquad x \in C_j$$

$$\therefore\ G(x+i0) + G(x-i0) = 2\left\{\int_{C_j} \frac{dx'\,\rho(x')}{x-x'} + \sum_{k\neq j} \int_{C_k} \frac{dx'\rho(x')}{x-x'}\right\} = \frac{1}{x} - 2\pi k$$

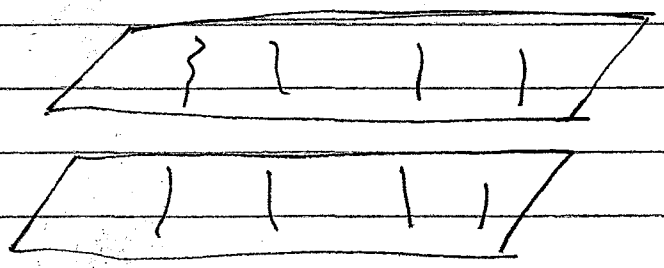
$$G(x+i0) - G(x-i0) = -2\pi i\,\rho(x) \qquad x \in C_j$$

We also will define $\rho(x) = G(x) - \frac{1}{2x}$

$$\rho(x+i0) + \rho(x-i0) = -2\pi n_j \qquad x \in C_j$$

In matrix models, locally near the end of the eigenvalue distribution $\rho(x) \sim \sqrt{x - x_0}$ where x_0 is a square-root branch cut.

Define $G(x)$ on a 2-sheeted surface with K square root branch cuts



$G(x)$ is not a function on this surface because of the n_j

However dG is a differential

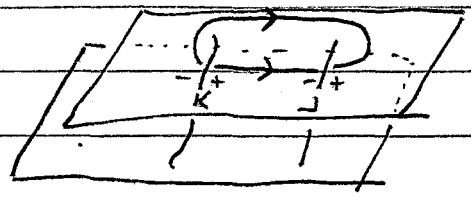
Physical sheet: $G(x), dG$ is non singular

unphysical sheet: $dG(x) \sim -\frac{dx}{x^2}$ at $x \to 0$

$y^2 = x^{2K} + r_1 x^{2K-1} - \cdots + r_{2K}$ hyperelliptic surface.

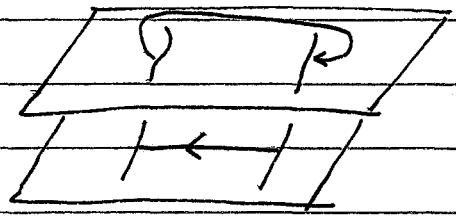
A cycles: $\int_A dG = 0$ is G is single valued on the physical sheet.

B-cycles:



$$\int dP = P(x_j + i0) - P(x_k - i0) + P(x_j - i0) - P(x_k + i0)$$

$$= -2\pi (n_j - n_k)$$



$$\int_{B_{jk}} dP = -2\pi (n_j - n_k)$$

$K-1$ independent B cycles.

$$dG = -\frac{1}{2x^2}\left(1 - \frac{\sqrt{\Gamma_{2k}}}{y} + \frac{\frac{1}{2}\Gamma_{2k-1}\, x}{\sqrt{\Gamma_{2k}}\, y}\right) + \sum_{\ell=0}^{K-2} \frac{\alpha_\ell x^\ell}{y}$$

Given the curve y, the α_ℓ are determined by choosing A-cycles to be $\emptyset$ $(K-1)$ parameters.

We then need to determine y:

Recall the momentum condition:

$$\prod_j \frac{u_j + i/2}{u_j - i/2} = 1 \approx \exp\left[i \sum_j \frac{1}{u_j} \right]$$

$$\Rightarrow \sum_j \frac{1}{u_j} = \int_C \frac{dx' \rho(x')}{x'} = -G(0) = 2\pi m$$

$$\int_C \frac{dx' \rho(x')}{x'} = \sum_j \int_{C_j} dx' \rho(x') 2\pi n_j + 2 \sum_{j,k} \int_{C_j} \int_{C_k} \frac{dx' dx'' \rho(x') \rho(x'')}{x' - x''}$$

$$= \sum_j 2\pi n_j \int dx' \rho(x') \qquad \int_{C_k} = 0$$

$$= \sum_j 2\pi n_j S_j \qquad S_j = \int dx' \rho(x')$$

"filling fraction"

$$\sum_j S_j = J/L$$

Count parameters: K cuts has 2K parameters

K-1	independent	B-cycles
K	filling	fractions
1	momentum	condition

Solving for G solves our problem.

Remember that the generator of charges is

$$\sum_{n=0} u^n Q_n = -i \log \prod_{J=1}^{J'} \frac{u-u_j-i/2}{u-u_j+i/2}$$

$$\approx -i \sum_{J=1}^{J'} \log \exp\left[\frac{-i}{u-u_j}\right]$$

$$= -\sum_{J=1}^{J'} \frac{1}{u-u_j} \approx -\int \frac{dx' \rho(x')}{x-x'} = -G(x)$$

$$\sum_{n=0} L^n Q_n x^n$$

$\therefore$ $G(x)$ is the generator of all charges!

$$\Rightarrow \quad \gamma = 8 \frac{\lambda}{8\pi^2} \sum \frac{1}{u_j^2+1/4} \approx \frac{\lambda}{8\pi^2 L} \int \frac{dx' \rho(x')}{x'^2} = -\frac{\lambda}{8\pi^2 L} G'(0)$$

$$-G'(0) = \frac{8\pi^2 L}{\lambda} \gamma$$

Examples: In general it is very difficult to find $G(x)$. One case when it is easy is if there is only one cut. In this case $G(x)$ is a function.

$$\frac{1}{x} - 2\pi n = G(x+i0) + G(x-i0)$$

$$G(x) = \frac{1}{2x}\left(1 + \sqrt{ax^2+bx+1}\right) - \pi n$$

For large x

$$G(x) = \int \frac{dx'\,\rho(x')}{x-x'} \simeq \frac{1}{x}\int dx'\,\rho(x') = \frac{J'}{L}$$

$$a = (2\pi n)^2 \qquad \frac{1}{4}\frac{b}{2\pi n} = \frac{J'}{L} - \frac{1}{2}$$

$$G(0) \approx \frac{1}{2x}\left(1 - 1 - \frac{1}{2}bx\right) - \pi n$$

$$= -\frac{b}{4} - \pi n = -\pi n\left(\frac{2J'}{L}\right) = -2\pi m$$

$$\therefore\ n\left(\frac{2J'}{L}\right) = 2m$$

$$n \cdot \frac{J'}{L} = m$$

$J' \le \frac{L}{2}$

Suppose $J' = \frac{L}{2}$ $\quad n = 2m \quad J'/L = \frac{m}{n}$

BMN

$$\left(\alpha^{+}_{n}\right)^{\frac{L}{2}\frac{m}{n}}|0\rangle$$

$$b = 8\pi n\left(\frac{m}{n} - \frac{1}{2}\right)$$

$$= 4\pi(2m-n)$$

$$G'(0) = +\left(\frac{1}{8}b^2 - \frac{1}{2}a\right)\frac{1}{2}$$

$$= \frac{1}{16}\cdot\left((4\pi)^2(2m-n)^2 - 4\cdot(2\pi n)^2\right)$$

$$= -4\pi^2\cdot(m)(n-m)$$

$$\gamma = \frac{\lambda}{8\pi^2 L}4\pi^2 m(n-m) = \frac{\lambda}{2L}m(n-m)$$

Branch points at

$$\cancel{x = \pm\frac{4}{4\pi n}\left(\frac{1}{2} - \frac{m}{n}\right)}$$

$$\cancel{\pm\frac{1}{2\pi n}\left[\left(1 - \frac{2m}{n}\right)\right.}$$

$$\pm\frac{2}{n}\sqrt{m(n-m)}$$

$$\left(\alpha^{+}_{n}\right)^{\frac{m}{n}L}|0\rangle$$

Compare to naive BMN:

Roots at n # of roots is $\frac{m}{n}\cdot L$

Naive BMN is

$$\frac{1}{2}\frac{n^2}{L^2}\lambda\cdot\frac{m}{n}\cdot L = \frac{\lambda}{2L}nm$$

Look for a simple string solution that matches this one.

$$S = -\frac{\sqrt{\lambda}}{2\pi}\int d\tau d\sigma \left[\frac{1}{2}\partial_\alpha Z \partial_\alpha \bar{Z} + \frac{1}{2}\partial_\alpha W \partial_\alpha \bar{W} - \frac{1}{2}\partial_\alpha t\, \partial_\alpha t\right]$$

$Z = X_1 + iX_2 \quad W = X_3 + iX_4$

Where $Z\bar{Z} + W\bar{W} = 1$

Try the following solution

This is a circular string

$$t = \kappa\tau \qquad Z = \frac{e^{i(w_1\tau - m_1\sigma)}}{\cos\theta_0} \qquad W = \frac{e^{i(w_2\tau - m_2\sigma)}}{\sin\theta_0}$$

Conserved quantities: $Z = \rho_1 e^{i\phi_1}, \quad W = \rho_2 e^{i\phi_2}$

$$E = \frac{\sqrt{\lambda}}{2\pi}\int_0^{2\pi} d\sigma\, \partial_0 t = \sqrt{\lambda}\,\kappa$$

$$J_1 = \frac{\sqrt{\lambda}}{2\pi}\int_0^{2\pi} d\sigma \underbrace{\cos^2\theta_0}_{\lambda}\partial_0\phi_1 = \sqrt{\lambda}\, w_1 \cos^2\theta_0$$

$$J_2 = \sqrt{\lambda}\, w_2 \sin^2\theta_0$$

Equations of motion $\Rightarrow$ in order to enforce constraint on to sphere constraints, we put in a Lagrange multiplier $\Lambda(Z\bar{Z} + W\bar{W} - 1)$

$$\partial_\tau^2 Z - \partial_\sigma^2 Z - \Lambda Z = 0$$

$$\partial_\tau^2 W - \partial_\sigma^2 W - \Lambda W = 0$$

$$\Rightarrow w_1^2 - m_1^2 = w_2^2 - m_2^2$$

Constraints:

$$\partial_\tau Z \partial_\tau \bar{Z} + \partial_\sigma Z \partial_\sigma \bar{Z} + \partial_\tau W \partial_\tau \bar{W} + \partial_\sigma W \partial_\sigma \bar{W} - \partial_\tau t\, \partial_\tau t = 0$$

Re $\partial_\tau Z \partial_\sigma \bar{Z} + \partial_\sigma Z \partial_\tau \bar{Z} + (Z \to W) = 0$

$$\cos^2\theta_0\left(w_1^2 + m_1^2\right) + \underbrace{\left(w_2^2 + m_2^2\right)}_{\sin^2\theta_0} = \kappa^2$$

$$\cos^2\theta_0\,(w_1 \cdot m_1) + \underbrace{w_2 m_2}_{\sin^2\theta_0} = 0$$

For simplicity, let $m_1 = -m_2 = m \qquad w_1 = w_2 = w$

$$\cos^2\theta_0 = \sin^2\theta_0 = \frac{1}{2}$$

$$J_1 = J_2 = \sqrt{\lambda}\, w/2 = J$$

The constraint then gives

$$K^2 = W^2 + m^2$$

$$E^2 = (2J)^2 + m^2\lambda$$

for small λ we have that

$$E \approx 2J + \frac{m^2\lambda}{4J} + \cdots$$

More general solution

$$J_1 m_1 + J_2 m_2 = 0 \qquad J_1 + J_2 = L$$

$$\Rightarrow J_1 = \frac{L m_2}{m_2 - m_1}, \quad J_2 = \frac{L m_1}{m_1 - m_2}$$

$$E^2 = \frac{J_1}{\cos^2\theta_0} + m_1^2\lambda\cos^2\theta_0 + \frac{J_2}{\sin^2\theta_0} + m_2^2\lambda\sin^2\theta_0$$

$$\frac{J_1^2}{\cos^4\theta_0} - m_1^2 = \frac{J_2^2}{\sin^4\theta_0} - m_2^2$$

Let $y = \cos^2\theta_0$

$$\frac{J_1^2}{y^2} - m_1^2\lambda = \frac{J_2^2}{(1-y)^2} - m_2^2\lambda$$

$$E^2 = \frac{J_1}{y} + m_1^2\lambda y + \frac{J_2}{1-y} + m_2^2\lambda(1-y)$$

$$\frac{L^2 m_2^2}{(m_2-m_1)^2 y^2} - m_1^2\lambda = \frac{L^2 m_1^2}{(m_2-m_1)^2(1-y)^2} - m_2^2\lambda$$

In the limit of small λ

$$(1-y)^2 m_2 = y^2 m_1$$

$$y = \frac{m_2}{m_1 + m_2}$$