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ICTP 40th Anniversary

SMR.1566 - 1

Introductory School on

RECENT DEVELOPMENTS IN SUPERSYMMETRIC GAUGE THEORIES

14 - 25 June 2004

INSTANTON PHYSICS

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ICTP
June 2004

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Lectures on Instanton Physics

QFT: Path Integral

$\downarrow$

Correlation Functions

$\swarrow \qquad \searrow$

Spectrum $\qquad$ S-Matrix

Path Integral on $\mathbb{R}^{3,1}$ ← Minkowski Space

$$\mathcal{Z} \sim \int [d\Phi]\, e^{\frac{i}{g^2} S_M[\Phi]}$$

defined via Wick rotation to $\mathbb{R}^4$ ← Euclidean spacetime

$t = -i\tau$ ← Euclidean time

$$\int_{-\infty}^{+\infty} dt = -i \int_{+i\infty}^{-i\infty} d\tau \underset{\substack{\uparrow \\ \text{contour rotation}}}{=} -i \int_{-\infty}^{+\infty} d\tau$$

Euclidean Path Integral:

$$Z[J] \sim \int [d\Phi]\, e^{-\frac{1}{g^2} S[\Phi] + \int d^4x\, J\Phi}$$

$S[\Phi] \in \mathbb{R}$ and banded below

Euclidean action

- generates perturbation theory
- defined non-perturbatively via lattice regularisation

aim: calculate Z at weak coupling $g^2 \ll 1$

idea: Expand around minima of $S[\Phi]$

Semiclassical Approximation

Minima of $S[\Phi]$

I) <u>The vacuum</u> $\Phi \equiv$ constant

generates ordinary perturbation theory

II) <u>Instantons</u> $\overset{\text{defn}}{\equiv}$ non-trivial minima of finite action

finite action $\Rightarrow$ localized region of action density in spacetime

$\mathbb{R}^4$

cf soliton

Yang-Mills Instanton

action:

$$S_{cl}/g^2 = \frac{8\pi^2}{g^2} - i\theta$$

gauge coupling

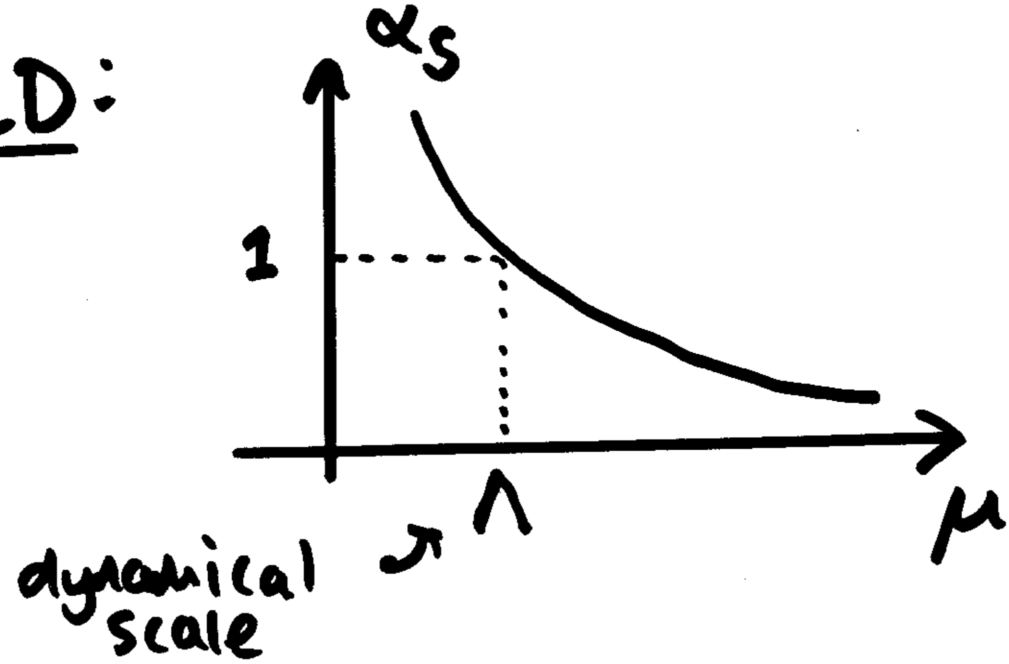
vacuum angle

contribution: $\sim e^{-\frac{8\pi^2}{g^2} + i\theta}$

- non-perturbative in g^2
- characteristic θ dependence

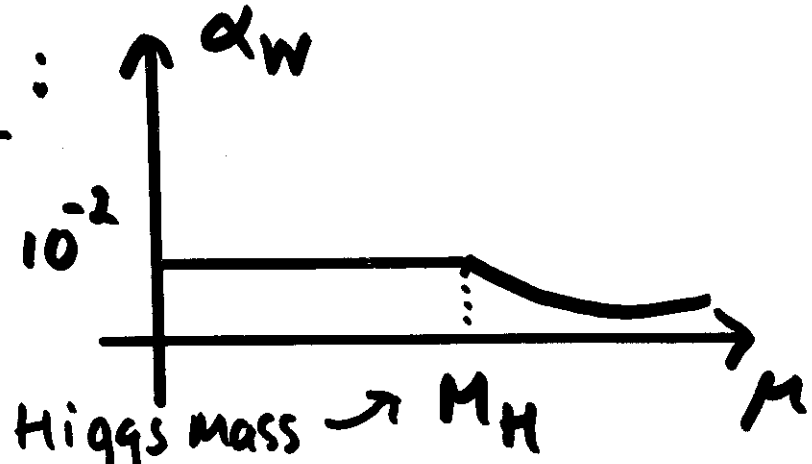
Instantons in the Standard Model

QCD:



coupling large in IR
$\Rightarrow$ semiclassical analysis invalid
qualitative effects: U(1) problem

Electro-Weak:



Interesting effects B+L violation
... but unobservably small

$$e^{-8\pi^2/g_W^2} \sim e^{-10^3}$$

Instantons & SUSY

BPS property

non-renormalization
holomorphy $\Rightarrow$

instantons sometimes give the whole answer!

<u>Example</u> $\mathcal{N}=2$ SUSY Yang-Mills

Seiberg + Witten

$$\underbrace{\mathcal{F}}_{\text{holomorphic prepotential}}(\Psi) = \frac{i}{2\pi}\Psi^2 \log\left(\Psi^2/\Lambda^2\right) \leftarrow \text{1-loop}$$

$$-\, i/\pi \sum_{k=1}^{\infty} \underbrace{\mathcal{F}_k}_{k\text{-instantons}} \left(\frac{\Lambda}{\Psi}\right)^{4k} \Psi^2$$

Exact analysis of SW EM duality

vs

Explicit semi-classical calculation of coefficients $\mathcal{F}_k$

Instantons in String Theory

instanton $\equiv$ finite action configuration in Euclidean space-time

IIB string on $\mathbb{R}^{10}$:

$\mathbb{R}^{10}$

D-instanton
$\equiv$ D(-1) brane

$$S_{cl} = \frac{2\pi}{g_s} - i\,2\pi\, C_{(0)}$$

string coupling

RR scalar

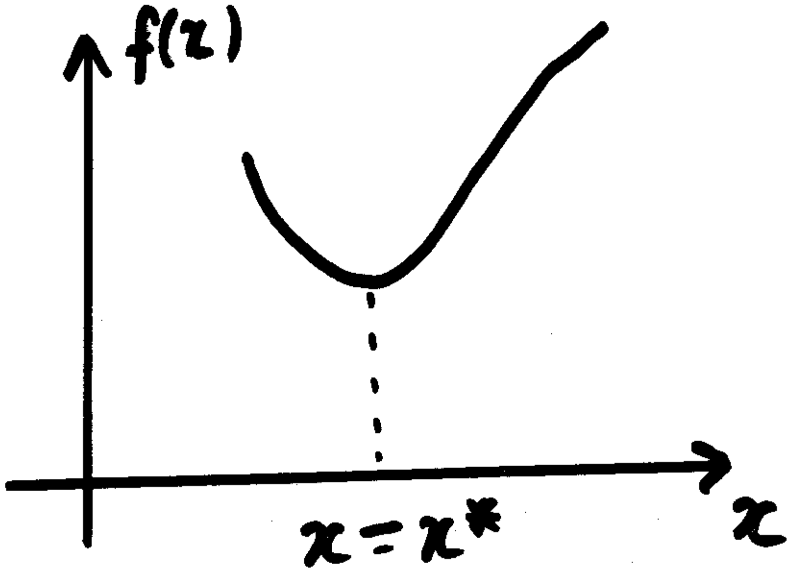
Gauge Theory / String Theory duality

D-instantons $\longleftrightarrow$ Yang-Mills instantons

- Tests of AdS/CFT

0-dimensional example

$$J = \int_{-\infty}^{+\infty} dx \; e^{-1/g^2 f(x)}$$



f attains minimum value at $x = x^*$

$\Rightarrow f'(x^*) = 0$

$g^2 \to 0 \Rightarrow J$ dominated by neighborhood of $x = x^*$

Saddle-point approximation

Expand around $x = x^*$ in integrand

$$x = x^* + \delta x \qquad \Rightarrow dx = d(\delta x)$$

$$f(x) = f(x^*) + \cancel{f'}(x^*)\,\overset{\nearrow 0}{\delta x}$$

$$+ \tfrac{1}{2} f''(x^*)\,\delta x^2 + O(\delta x^3)$$

$$\mathcal{I} = \int_{-\infty}^{+\infty} dx\; e^{-\frac{1}{g^2} f(x)}$$

$$= e^{-f(x^*)/g^2} \int_{-\infty}^{+\infty} d(\delta x)\; \underbrace{e^{-\frac{f''(x^*)}{2g^2}\delta x^2 + \dots}}_{\text{Gaussian}}$$

$$\Rightarrow$$

$$\mathcal{I} \simeq \sqrt{2\pi g^2}\; e^{-\frac{1}{g^2} f(x^*)}$$

$$J \overset{g^2\to 0}{\sim} \sqrt{2\pi g^2}\, e^{-\frac{1}{g^2} f(x^*)} \left[1 + O(g^2) \right]$$

- leading exponential behaviour governed by minimum of f
- prefactor comes from Gaussian integral over fluctuations
- first term in an <u>asymptotic series</u> in g^2

  $\Rightarrow$ zero radius of convergence

<u>Multi-component version</u>

$$f(\vec{x}) \qquad \vec{x} = (x_1, \ldots, x_n)$$

f has isolated minimum at $\vec{x} = \vec{x}^*$

$$J_{(n)} = \int d^n x \; e^{-1/g^2 f(\vec{x})}$$

expand around minimum:

$$\vec{x} = \vec{x}^* + \delta\vec{x} \quad \Rightarrow$$

$$f(\vec{x}) = f(\vec{x}^*) + \left.\frac{\partial f}{\partial \vec{x}}\right|_{\vec{x}=\vec{x}^*} \cdot \delta\vec{x}$$

$$+ \delta\vec{x}^T \cdot M \cdot \delta\vec{x} + O(\delta\vec{x}^3)$$

$$M_{ij} = \frac{1}{2} \left.\frac{\partial^2 f}{\partial x_i \partial x_j}\right|_{\vec{x}=\vec{x}^*}$$

<u>Exercise</u>

$$J_{(n)} \sim \det^{-1/2}(M) \, e^{-1/g^2 f(\vec{x}^*)}$$

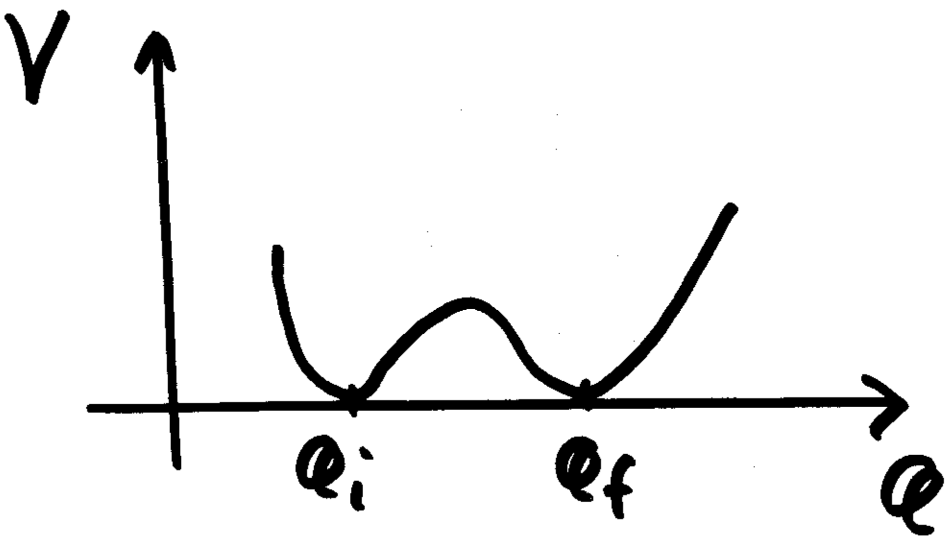
$$\times \left[1 + O(g^2)\right]$$

0+1-dimensional example $\equiv$ Quantum Mechanics

particle moving in one-dimension:

$$\mathcal{L} = \tfrac{1}{2}\dot{\varphi}^2 - V(\varphi)$$

Double-Well potential:



$$V(\varphi_i) = V'(\varphi_i) = 0$$

$$V(\varphi_f) = V'(\varphi_f) = 0$$

Explicit Example:

$$V(\varphi) = \left(\varphi^2 g^2 - \frac{\mu^2}{g}\right)^2$$

$$\varphi_i = -\mu/g \qquad \varphi_f = +\mu/g$$

<u>Problem</u>: Calculate amplitude for <u>tunneling</u> from initial state $|i\rangle$ to final state $|f\rangle$

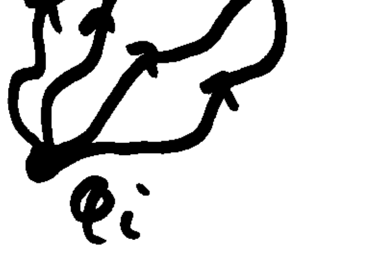
$$|i\rangle = \delta(q - q_i) \qquad t = -\infty$$

$$|f\rangle = \delta(q - q_f) \qquad t = +\infty$$

$$A \propto \lim_{T\to\infty} \langle i | e^{i\hat{H}T/\hbar} | f \rangle$$

Feynman - Kac prescription:

$$A \propto \int_{q(-\infty)=q_i}^{q(+\infty)=q_f} [dq] \; e^{i/\hbar \, S[q]}$$



$$S[q] = \int_{-\infty}^{+\infty} dt \, \mathcal{L}[q]$$

perform <u>Wick Rotation</u> to Euclidean time: $t = -i\tau$

Euclidean time

$$\int_{-\infty}^{+\infty} dt = -i \int_{+i\infty}^{-i\infty} d\tau = -i \int_{-\infty}^{+\infty} d\tau$$

contour rotation

<u>Result</u>

$$A \sim \int_{\varphi(-\infty)=\varphi_i}^{\varphi(+\infty)=\varphi_f} [d\varphi] \; e^{-\frac{1}{\hbar} S_E[\varphi]}$$

where,

$$S_E[\varphi] = \int_{-\infty}^{+\infty} d\tau \, \mathcal{L}_E[\varphi]$$

$$\mathcal{L}_E[\varphi] = \tfrac{1}{2}\left(\frac{\partial \varphi}{\partial \tau}\right)^2 + V(\varphi) \geq 0$$

Semiclassical Approximation

$\hbar \to 0 \Rightarrow A$ dominated by minima of S_E

stationary points of action:

- real time $\delta S / \delta \varphi = 0$

$\Rightarrow$ classical Equation of Motion (EOM)

$$\partial^2 \varphi / \partial t^2 = -V'(\varphi)$$

no solutions obeying boundary conditions

$$\varphi(-\infty) = \varphi_i$$
$$\varphi(+\infty) = \varphi_f$$

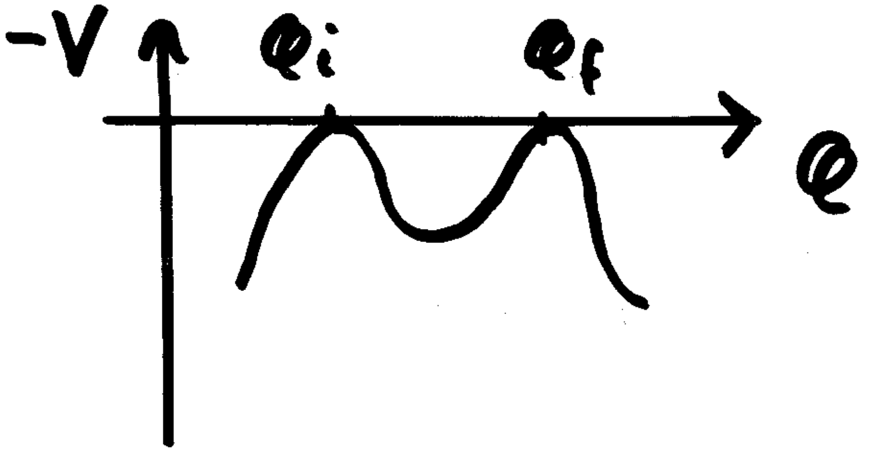
$\Rightarrow$ tunneling forbidden classically

• <u>Euclidean time</u> $\delta S_E / \delta \varphi = 0$

$\Rightarrow$ Euclidean EOM

$$\frac{\partial^2 \varphi}{\partial \tau^2} = + V'(\varphi)$$

$\equiv$ motion of particle in the inverse potential $-V(\varphi)$



can now easily find solution obeying

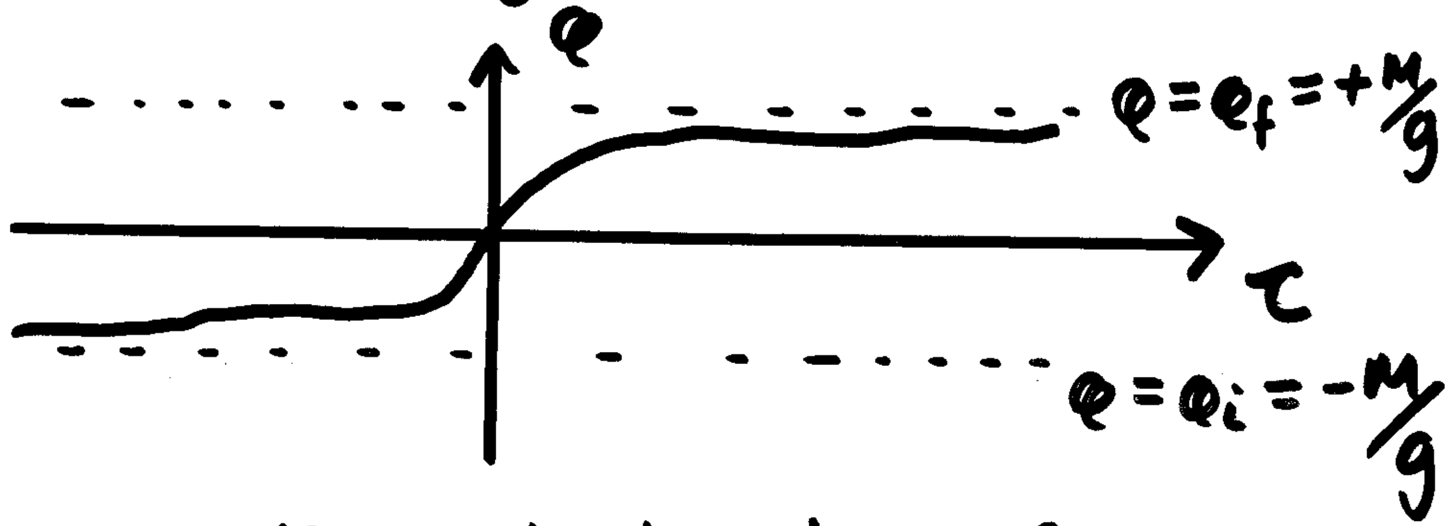
$$\varphi(\tau = -\infty) = \varphi_i$$
$$\varphi(\tau = +\infty) = \varphi_f$$

$\equiv$ Instanton

Explicit case $V(\varphi) = \left(g\varphi^2 - \frac{M^2}{g}\right)^2$

solution:

$$\varphi_{cl}(\tau) = \frac{M}{g}\tanh(M\tau)$$



$\equiv$ localized kink at $\tau = 0$

- Action and EOM invariant under Euclidean time translation

$$T : \tau \longrightarrow \tau - \tau_0$$

T acts non-trivially on kink to produce new solution

$$\varphi = \varphi_{cl}(\tau - \tau_0)$$ $\leftarrow$ kink at $\tau = \tau_0$

Semiclassical Approximation

1st pass

Expand around instanton

$$\varphi(\tau) = \underset{\text{instanton}}{\varphi_{cl}(\tau)} + \underset{\text{fluctuation}}{\delta\varphi(\tau)}$$

$$S_E[\varphi] = S_E[\varphi_{cl}] + \int d\tau \, \cancel{\frac{\delta S}{\delta \varphi}}\Bigg|_{\varphi \equiv \varphi_{cl}}^{\to 0} \cdot \delta\varphi$$

$$+ \frac{1}{2} \int d\tau \, \delta\varphi \, \Delta \, \delta\varphi \quad + O(\delta\varphi^3)$$

$$\Delta = \frac{\delta^2 S_E}{\delta \varphi^2}\Bigg|_{\varphi \equiv \varphi_{cl}}$$

$$= \underbrace{-\frac{\partial^2}{\partial \tau^2} + V''(\varphi_{cl})}_{\text{small fluctuation operator}}$$

$$\Rightarrow \quad \mathcal{A} \sim e^{-S_{cl}/\hbar} \times \int [d(\delta q)]\, e^{-\frac{1}{2\hbar}\int d\tau\, \delta q \Delta \delta q}$$

Gaussian

$$\sim \det{}^{-1/2}\Delta \; e^{-S_{cl}/\hbar}\left[1 + O(\hbar)\right]$$

$$S_{cl} = S_E[q_{cl}]$$

What is $\det\Delta$?

find spectrum of differential operator Δ

$$\underbrace{\Delta \psi_\lambda(\tau)}_{\text{Eigenfunction}} = \underbrace{\lambda}_{\text{Eigenvalue}} \psi_\lambda(\tau)$$

Δ self-adjoint $\Rightarrow$ λ real spectrum of Δ includes:

- discrete "bound-states"

  $\{\lambda_1, \ldots, \lambda_r\}$

- continuum "scattering states"

  $\{\lambda(e)\}$ $\quad e \in [0,1]$

$$\det(\Delta) \sim \prod_{i=1}^{r} \lambda_i \times e^{\int de \, \log(\lambda(e))}$$

- will not need to evaluate this explicitly. Non-zero eigenvalues <u>cancel</u> between bosons and fermions in SUSY theories

Zero-modes

Δ always has an eigenfunction with eigenvalue zero

$$\Delta \psi_0(\tau) = 0$$

zero mode

<u>Exercise</u>: show that

$$\psi_0(\tau) = \frac{\partial}{\partial \tau} q_{cl}(\tau)$$

$\psi_0(\tau)$ corresponds to an infinitesimal translation of kink:

$$q_{cl}(\tau) \to q_{cl}(\tau + \delta)$$

corresponding zero eigenvalue

$$\Rightarrow \det{}^{-1/2}(\Delta) = \infty \quad !!$$

Semiclassical Approximation

2nd pass

$$A \sim \int_{\varphi(-\infty)=\varphi_i}^{\varphi(+\infty)=\varphi_f} [d\varphi] \; e^{-\frac{1}{\hbar} S_E[\varphi]}$$

1) Fadeev-Popov trick: insert unity as

$$1 = \int d\tau_0 \, \frac{\partial f}{\partial \tau_0} \, \delta(f)$$

zero mode

with $f = \int_{-\infty}^{+\infty} d\tau \, \big(\varphi(\tau) - \varphi_{cl}(\tau-\tau_0)\big) \, \psi_0(\tau-\tau_0)$

2) Interchange order of integration

$$\int [d\varphi] \int d\tau_0 \to \int d\tau_0 \int [d\varphi]$$

now expand field as,

$$\varphi(\tau) = \varphi_{cl}(\tau - \tau_0) + \delta\varphi(\tau)$$

to get: $A \sim$

δ-function constraint

$$\int_{-\infty}^{+\infty} d\tau_0 \int [d(\delta\varphi)] \underset{\text{Jacobian}}{J}\, \delta\left(\int_{-\infty}^{+\infty} d\tau\, \delta\varphi(\tau)\, \psi_0(\tau - \tau_0)\right) \times e^{-\frac{1}{\hbar} S_E[\varphi]}$$

Thus we have:

- $\int$ over "moduli space" of instanton solutions τ_0
- fluctuations $\delta\varphi(\tau)$ constrained to be <u>orthogonal</u> to zero mode

Expanding action as:

$$S_E = S_{cl} + \int_{-\infty}^{+\infty} d\tau \, \delta\varphi \, \Delta \, \delta\varphi + \mathcal{O}(\delta\varphi^3)$$

we get (exercise):

$$A \overset{\hbar \to 0}{\sim} \int_{-\infty}^{+\infty} d\tau_0 \, J_{cl} \left[\det{}'(\Delta)\right]^{-1/2} e^{-S_{cl}/\hbar}$$

where,

- $\det{}'[\Delta] \overset{\text{defn}}{\equiv}$ product over non-zero eigenvalues
- $J_{cl} = \sqrt{S_{cl}}$

Yang-Mills Instantons

$G = SU(N)$ Yang-Mills theory on $\mathbb{R}^4$

$$S[A] = -\tfrac{1}{2}\int d^4x \, \mathrm{Tr}_N\left[\underset{\sim}{F}_{mn}\underset{\sim}{F}_{mn}\right]$$

$$\underset{\sim}{F}_{mn} = \partial_m \underset{\sim}{A}_n - \partial_n \underset{\sim}{A}_m + \left[\underset{\sim}{A}_m, \underset{\sim}{A}_n\right]$$

field strength

$$\underset{\sim}{A}_m = A_m^a T^a$$

anti-Hermitian generators

$$\mathrm{Tr}_N[T^a T^b] = -\tfrac{1}{2}\delta_{ab}$$

gauge field

$m, n = 1,2,3,4 \qquad x_4 = \tau$

Euclidean Time

dual field strength:

$$^*\underset{\sim}{F}_{mn} = \tfrac{1}{2}\,\varepsilon_{mnkl}\,\underset{\sim}{F}_{kl}$$

Gauge configurations on $\mathbb{R}^4$ are classified by <u>topological charge</u>

$$k = -\frac{1}{16\pi^2}\int d^4x \, \mathrm{Tr}_N\left[\underset{\sim}{F}_{mn} {}^*\underset{\sim}{F}_{mn}\right] \in \mathbb{Z}$$

<u>$G = SU(2)$</u>

configurations of finite action must go to pure gauge at ∞

Polar coordinates on $\mathbb{R}^4$:

r, θ, ϕ, ψ

$$\underset{\sim}{A}_\mu \xrightarrow{r\to\infty} \underset{\sim}{U}^\dagger \partial_\mu \underset{\sim}{U}$$

$\underset{\sim}{U}(\theta,\phi,\psi) \in SU(2)$ defines a map

$$\underset{\sim}{U} : S^3 \longrightarrow SU(2) \cong S^3$$

Winding number: $k \in \pi_3(S^3) \cong \mathbb{Z}$

Bogomol'nyi Bound

$$-\int d^4x \, \mathrm{Tr}_N \left[\left(\underset{\sim}{F}_{mn} \pm {}^*\underset{\sim}{F}_{mn} \right)^2 \right] \geqslant 0$$

$$\Rightarrow -2 \int d^4x \, \mathrm{Tr}_N \left[\underset{\sim}{F}_{mn} \underset{\sim}{F}_{mn} \right]$$

$$\geqslant \pm 2 \int d^4x \, \mathrm{Tr}_N \left[\underset{\sim}{F}_{mn} {}^*\underset{\sim}{F}_{mn} \right]$$

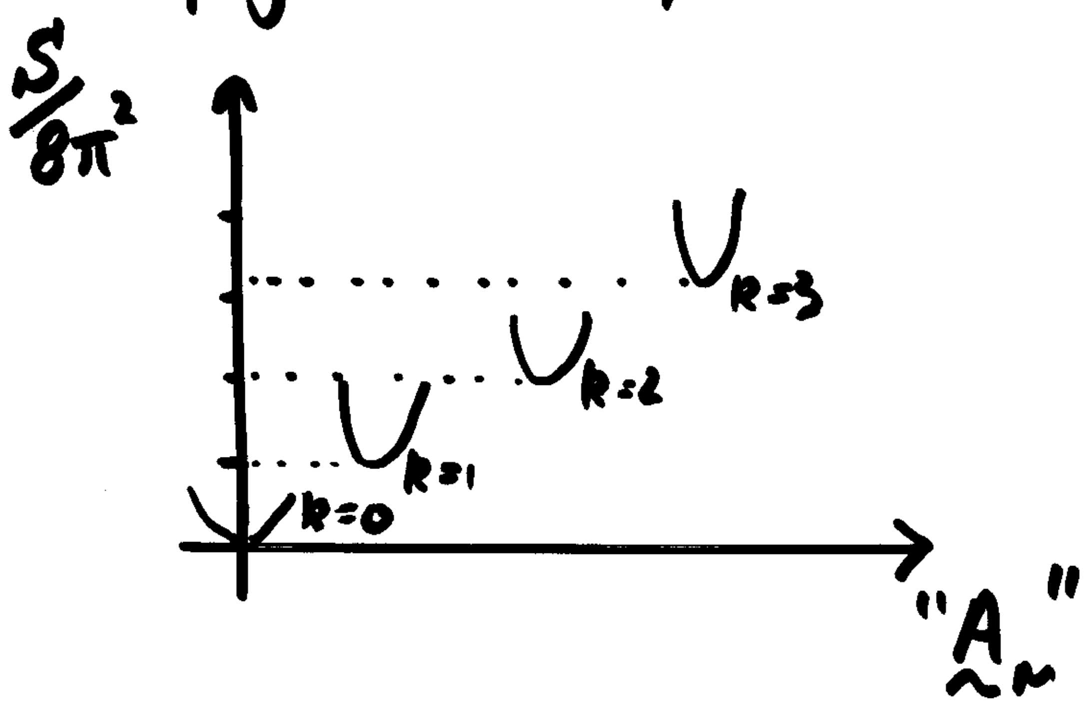
$$\Rightarrow \underset{\sim}{S}[A] \geqslant 8\pi^2 |k|$$

with equality iff

$$\underset{\sim}{F}_{mn} = \pm \underset{\sim}{F}_{mn}$$

(anti) self-dual YM eqn

for $k \begin{matrix} > 0 \\ < 0 \end{matrix}$ respectively configuration space:



$$Z = \int [d\underset{\sim}{A}]\, e^{-\frac{1}{g^2} S[\underset{\sim}{A}] + ik\theta}$$

Euclidean path integral

Semiclassical Approximation ($g^2 \ll 1$)

$$Z \sim \sum_{k=-\infty}^{+\infty} e^{-\frac{8\pi^2}{g^2}|k| + ik\theta}\left[1 + O(g^2)\right]$$

solutions of (A)SDYM eqn known as <u>(anti-) instantons</u>

- automatically satisfy full Yang-Mills equation

$$\underset{\text{(A)SDYM}}{\mathcal{D}_\mu \underset{\sim}{F}_{\mu\nu} = \pm \mathcal{D}_\mu {}^* \underset{\sim}{F}_{\mu\nu}} = 0 \quad \text{Bianchi identity}$$

covariant derivative

$$\mathcal{D}_\mu \underset{\sim}{X} = \partial_\mu \underset{\sim}{X} + [\underset{\sim}{A}_\mu, \underset{\sim}{X}]$$

<u>instanton contribution</u>

$$\propto e^{-8\pi^2/g^2 |k| + ik\Theta}$$

- non-perturbative in g^2
- characteristic Θ-dependence

Conventions Phys. Rept 371 p240

$$SO(4) \cong \underbrace{SU(2)_L \times SU(2)_R}_{\text{covering group}}$$

LH Weyl spinor: $\underset{\sim}{\lambda}_\alpha$ $\alpha = 1,2$

$(\underset{\sim}{2}, 0)$

RH Weyl spinor: $\underset{\sim}{\bar{\lambda}}_{\dot{\alpha}}$ $\dot{\alpha} = 1,2$

$(\underset{\sim}{0}, \underset{\sim}{2})$

Euclidean σ-matrices

$\sigma_{m\alpha\dot{\alpha}}$, $\bar{\sigma}_m^{\dot{\alpha}\alpha}$ $m = 1,2,3,4$

$\sigma = (i\vec{\tau}, \mathbb{1})$ $\bar{\sigma} = (-i\vec{\tau}, \mathbb{1})$

$\vec{\tau} = (\tau_1, \tau_2, \tau_3)$

Pauli matrices

SO(4) generators:

$$\sigma_{mn} = \tfrac{1}{4}(\sigma_m \bar{\sigma}_n - \sigma_n \bar{\sigma}_m)$$

$$\bar{\sigma}_{mn} = \tfrac{1}{4}(\bar{\sigma}_m \sigma_n - \bar{\sigma}_n \sigma_m)$$

are $\begin{matrix} \text{SD} \\ \text{ASD} \end{matrix}$ respectively:

$$\sigma_{mn} = \tfrac{1}{2} \varepsilon_{mnkl} \sigma_{kl}$$

$$\bar{\sigma}_{mn} = -\tfrac{1}{2} \varepsilon_{mnkl} \bar{\sigma}_{kl}$$

generate $\begin{matrix} SU(2)_L \\ SU(2)_R \end{matrix}$ respectively

$k=1 \quad G=SU(2)$

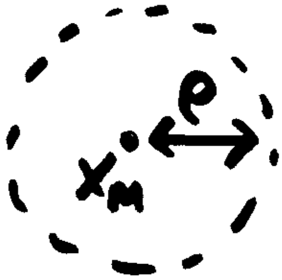
BPST instanton (regular gauge)

$$\underset{\sim}{A}_\mu^{cl} = \frac{2(x-X)_\mu \sigma_{\mu\nu}}{(x-X)^2+\rho^2}$$

$$\Rightarrow \underset{\sim}{F}_{\mu\nu} = \frac{4\rho^2 \sigma_{\mu\nu}}{((x-X)^2+\rho^2)^2}$$

self dual tensor

$\boxed{\mathbb{R}^4}$



localized region of action density

size $\sim \rho$

position $\sim x_\mu = X_\mu$

additional parameter: global gauge rotation:

$$\underset{\sim}{A}_\mu^{cl} \to U^\dagger \underset{\sim}{A}_\mu^{cl} U \qquad U \in SU(2)$$

Symmetry of action <u>broken</u> by instanton $\Rightarrow$ <u>collective coordinate</u>

Symmetry	Collective coordinate	#
translation	$x_\mu \in \mathbb{R}^4$	4
dilatation	$\rho \in \mathbb{R}^+$	1
global gauge	$U \in SU(2)$	3
		8 ← total #

<u>$G = SU(N)$</u> <u>$k=1$</u>

Embed BPST instanton in an $SU(2)$ subgroup of $SU(N)$

$$\underset{\substack{\uparrow \\ N\times N}}{\underset{\sim}{A}{}^{cl}_{\mu}} = \left(\begin{array}{c:c} \underset{\sim}{A}{}^{BPST}_{\mu} & 0_{2\times N-2} \\ \hdashline 0_{N-2\times 2} & 0_{N-2\times N-2} \end{array}\right)$$

again consider orbit under global gauge group

$$\underset{\sim}{A}_\mu^{cl} \longrightarrow \mathcal{U}^\dagger \underset{\sim}{A}_\mu^{cl}\, \mathcal{U} \ , \quad \mathcal{U} \in SU(N)$$

stabilizer: $SU(N-2) \times U(1)$

$\Rightarrow$ orbit is the coset $SU(N)/SU(N-2)\times U(1)$ of dimension $N^2-1-(N-2)^2 = 4N-5$

Symmetry	Collective coordinate	#
translation	$X_\mu \in \mathbb{R}^4$	4
dilatation	$\rho \in \mathbb{R}^+$	1
global gauge	$\hat{U} \in \dfrac{SU(N)}{SU(N-2)\times U(1)}$	$4N-5$
		$4N$ ← total #

General Analysis ($k>0$)

Expand gauge field around classical solution:

$$\underset{\sim}{A}_M = \underset{\sim}{A}_M^{cl} + \delta \underset{\sim}{A}_M$$

small fluctuation operator

$$S' = 8\pi^2 |k| + \int d^4x \, \delta \underset{\sim}{A}_m \, \Delta^{(B)}_{mn} \, \delta \underset{\sim}{A}_n$$

$$\Delta^{(B)}_{mn} = \left(\underset{\sim}{\mathcal{D}}^2_{cl} \, \delta_{mn} + 2 \underset{\sim}{F}^{cl}_{mn} \right)$$

Zero Modes

Solutions of $\Delta^{(B)}_{mn} \, \delta \underset{\sim}{A}_n = 0$

which are not pure gauge

\# of zero modes

$\equiv$ dimension of moduli space

space of gauge-inequivalent solutions

Counting Zero Modes

zero modes obey linearized SDYM equation

$$\underset{\sim}{D}^{cl}_m \delta \underset{\sim}{A}_n - \underset{\sim}{D}^{cl}_n \delta \underset{\sim}{A}_m = \varepsilon_{mnkl} \underset{\sim}{D}^{cl}_k \delta \underset{\sim}{A}_l \qquad - \text{I}$$

pure gauge modes eliminated by gauge fixing condition

$$\underset{\sim}{D}^{cl}_m \delta \underset{\sim}{A}_m = 0 \qquad - \text{II}$$

I and II can be rewritten as a Dirac equation

$$\underset{\sim}{\bar{\not{D}}}^{\dot{\alpha}\alpha}_{cl} \delta \underset{\sim}{A}_{\alpha\dot{\beta}} = 0 \qquad \begin{array}{l} \dot{\alpha} = 1,2 \\ \dot{\beta} = 1,2 \end{array}$$

$$- \text{III}$$

where

$$\underset{\sim}{\bar{\not{D}}}^{\dot{\alpha}\alpha}_{cl} = \underset{\sim}{D}^{cl}_m \bar{\sigma}_m^{\dot{\alpha}\alpha}, \quad \delta \underset{\sim}{A}_{\alpha\dot{\beta}} = \delta \underset{\sim}{A}_m \sigma_{m\,\alpha\dot{\beta}}$$

Atiyah-Singer Index Theorem

counts # of zero modes of Dirac operator

$$\bar{\not{D}}^{\dot{\alpha}\alpha}_{cl} = \bar{\sigma}^{\dot{\alpha}\alpha}_{\mu}\left(\partial_\mu + [\underset{\sim}{A}^{cl}_{\mu}, \ ]\right)$$

in instanton backround of topological charge $k>0$

$$\# = 2kN$$

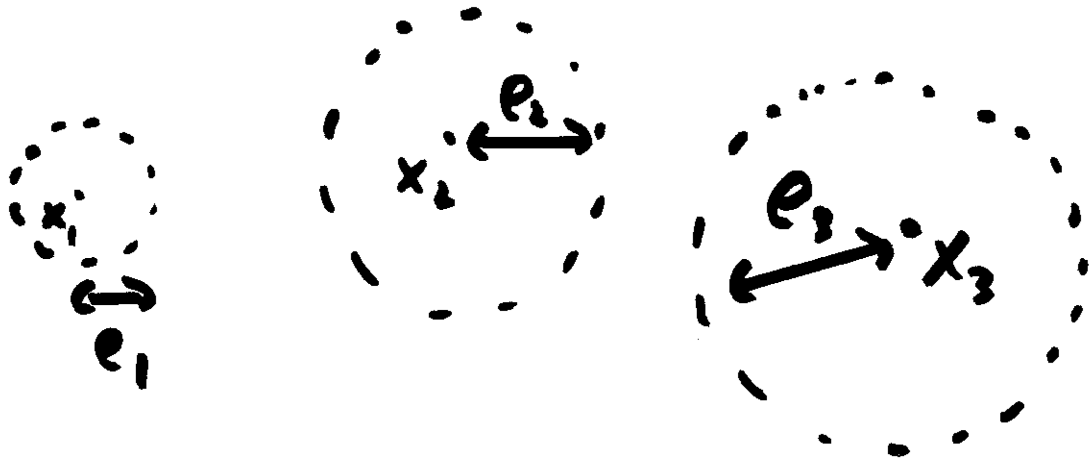
per spinor degree of freedom

$\delta \underset{\sim}{A}_{\alpha\dot{\beta}}$ for $\dot{\beta}=1,2$ provides two spinors $\Rightarrow$ Dimension of instanton moduli space is

$$\dim\left[\mathcal{M}_{k,N}\right] = 4kN$$

Multi-Instantons

linear superposition of single instanton solutions:



valid for $|x_i - x_j| \gg \rho_i, \rho_j$

"dilute gas"

parameters $= 4N \times k$

Exact multi-instanton solution given by ADHM construction

Non-linear PDE $\rightarrow$ Non-linear algebraic eqns

- no explicit parametrisation possible for $k \geq 3$

Appendix

(G&R p560 #7)

$$\int_0^\infty \log(a^2+b^2x^2)\,\frac{dx}{c^2+g^2x^2} = \frac{\pi}{cg}\log\left(\frac{ag+bc}{g}\right)$$

$$a,b,c,g>0$$

Instantons and Fermions

Example: $\mathcal{N}=1$ SUSY Yang-Mills

$G = SU(N)$

$\mathcal{N}=1$ vector multiplet $\supset$ $\underset{\sim}{A}_m$ ← gauge field

$\underset{\sim}{\lambda}_\alpha$ ← adjoint Weyl fermion

$$S = \int d^4x \, \mathrm{Tr}_N \Big[-\tfrac{1}{2} \underset{\sim}{F}_{mn} \underset{\sim}{F}_{mn} - 2 (\mathcal{D}_n \bar{\underset{\sim}{\lambda}}) \bar{\sigma}_n \underset{\sim}{\lambda} \Big]$$

$\mathcal{N}=1$ SUSY:

$$\delta \underset{\sim}{A}_n = i \xi \sigma_n \bar{\underset{\sim}{\lambda}} + i \bar{\xi} \bar{\sigma}_n \underset{\sim}{\lambda}$$

$$\delta \underset{\sim}{\lambda}_\alpha = i \sigma_{mn} \xi_\beta \underset{\sim}{F}_{mn}$$

$$\delta \bar{\underset{\sim}{\lambda}}_{\dot\alpha} = i \bar{\sigma}_{mn} \bar{\xi} \underset{\sim}{F}_{mn}$$

see DHKM Section 4.1

path-integral quantization

$$Z \sim \int [d\underset{\sim}{A}][d\underset{\sim}{\lambda}][d\underset{\sim}{\bar{\lambda}}]\, e^{-\frac{1}{g^2} S + ik\Theta}$$

$\underset{\sim}{\lambda}$ and $\underset{\sim}{\bar{\lambda}}$ are Grassmann (anti-commuting) variables

Semiclassical Approximation ($g^2 \ll 1$)

Expand around bosonic instanton:

$$\underset{\sim}{A}_m = \underset{\sim}{A}_m^{cl} + \delta \underset{\sim}{A}_m$$

$$\underset{\sim}{\lambda}_\alpha = \delta \underset{\sim}{\lambda}_\alpha \qquad \underset{\sim}{\bar{\lambda}}_{\dot{\alpha}} = \delta \underset{\sim}{\bar{\lambda}}_{\dot{\alpha}}$$

$$S \simeq 8\pi^2 |k| + \int d^4x\, \delta \underset{\sim}{A}_m \Delta^{(B)}_{mn} \delta \underset{\sim}{A}_n + \int d^4x\, \delta \underset{\sim}{\bar{\lambda}}\, \Delta^{(F)} \delta \underset{\sim}{\lambda} + O(\delta^3)$$

fermion fluctuation operator,

$$\Delta^{(F)\,\dot{\alpha}\alpha} = \bar{\underset{\sim}{\not{D}}}{}^{\dot{\alpha}\alpha}_{cl} = \underset{\sim}{\mathcal{D}}{}^{cl}_{m}\, \bar{\sigma}_{m}^{\dot{\alpha}\alpha}$$

Dirac operator in the instanton backgrnd

Fermion Zero-Modes

satisfy

$$\bar{\underset{\sim}{\not{D}}}{}^{\dot{\alpha}\alpha}_{cl}\, \delta\underset{\sim}{\lambda}_{\alpha} = 0 \qquad - \text{(A)}$$

$$\underset{\sim}{\not{D}}{}^{\alpha\dot{\alpha}}_{cl}\, \delta\bar{\underset{\sim}{\lambda}}_{\dot{\alpha}} = 0 \qquad - \text{(B)}$$

Atiyah - Singer Index Theorem

$\Rightarrow$ (A) has $2kN$ normalisable solutions $(k>0)$

(B) has none

Symmetry and Zero Modes

As in bosonic sector, we can understand some fermion zero modes ~~as~~ via Goldstone argument

Broken Symmetry $\Rightarrow$ Zero Mode

SUSY variations:

SD projector

$$\delta \underset{\sim}{\lambda} = i\, \sigma_{mn}\, \xi\, \underset{\wedge}{F}_{mn} \qquad - (a)$$

ASD projector

$$\delta \underset{\sim}{\bar{\lambda}} = i\, \bar{\sigma}_{mn}\, \bar{\xi}\, \underset{\sim}{F}_{mn} \qquad - (B)$$

in a self-dual background (B) vanishes $\Rightarrow$

instanton invariant under RH supercharges $\bar{Q}_{\dot{\alpha}}$ $\quad \dot{\alpha} = 1, 2$

conversely,
instanton transforms under LH supercharges Q_α $\alpha = 1,2$

$\Rightarrow$ 2 linearly independent fermion zero modes

$$\underset{\sim}{\psi}^{(0)}{}_{\alpha\,(\beta)} \sim i\,(\sigma_{mn})_\alpha{}^\beta\, \underset{\sim}{F}{}^{cl}_{mn}$$

α: spinor index; (β): label

$N=1$ SUSY + classical conformal invariance

$\Rightarrow$ $N=1$ superconformal symmetry

instanton invariant under LH superconformal generators S_α

Action of RH superconformal charges $\bar{S}_{\dot\alpha}$ $\Rightarrow$ 2 more zero modes

Total $2 + 2 = 4$

References

The discussion of several topics given in the lectures are based on parts of [1]. In particular, the conventions used in the lectures are essentially identical to those of [1]. This reference also describes the ADHM formalism for multi-instantons which is beyond the scope of the course. The parts of [1] which are most relevant are (page numbers refer to the published version):

1 Section 2.1 (p239) contains the basic formulae for Yang-Mills instantons and also gives the conventions for spinors and γ-matrices.

2 Section 2.2 (p241) Zero modes and collective coordinates.

3 Section 3.1. (p264) Semiclassical approximation to the Yang-Mills path integral.

4 Introduction of Section 4 (p271). General discussion of instantons in supersymmetric gauge theory.

5 Section 4.1. Full conventions for $\mathcal{N} = 1, 2$ and 4 SUSY Yang-Mills, including Lagrangians and supersymmetry transformations and equations of motion.

Reference [1] also contains extensive references to the original instanton literature. An earlier paper which is particularly useful is Coleman [2]. In particular Coleman gives a classic discussion of tunneling in quantum mechanics. The analysis of the prepotential given in the lectures is based on Seiberg's original paper [3]. First principles instanton calculations for $\mathcal{N} = 2$ SUSY Yang-Mills are described in [4].

References

[1] N. Dorey, T. J. Hollowood, V. V. Khoze and M. P. Mattis, "The calculus of many instantons," Phys. Rept. **371** (2002) 231 [arXiv:hep-th/0206063].

[2] S. R. Coleman, "The Uses Of Instantons," in Aspects of Symmetry, Cambridge University Press, Cambridge (1985) p 265

[3] N. Seiberg, Phys. Lett. B **206** (1988) 75.

[4] N. Dorey, V. V. Khoze and M. P. Mattis, "Multi-Instanton Calculus in N=2 Supersymmetric Gauge Theory," Phys. Rev. D **54** (1996) 2921 [arXiv:hep-th/9603136].