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SUPERSYMMETRIC GAUGE THEORIES AND DUALITY (Part II)

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2004 Trieste Lectures on Susy Gauge Theories and Duality

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4d $N = 1$ SQCD and related theories will be discussed, continuing from lectures last week. These are VERY ROUGH notes, that have not been proofread, and contain no references. For a better and more polished review, see my 1995 review lectures, with N. Seiberg.

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1. Introduction

4d (asymptotically free) gauge theories flow from weak coupling in the UV, to strong coupling in the IR. Pure $\mathcal{N} = 0$ glue[1] eventually flows to a theory with a mass gap and confinement of color flux (coming from non-dynamical sources that can be introduced) into thin tubes, leading to a potential $V(R) \sim \sigma R$ for sources separated by distance R (σ is the string tension).

$\mathcal{N} = 0$ $SU(N_c)$ QCD with N_f massless quark flavors exhibits (at least) two known phases: for N_f small, the theory confines and the $SU(N_f) \times SU(N_f)$ chiral symmetry spontaneously breaks to the diagonal $SU(N_f)_D$, via $\langle \overline{\psi}\psi \rangle = \Lambda^3$, leading to $N_f^2 - 1$ massless Goldstone boson pions[2]. In the medium IR limit, the effective field theory is the chiral lagrangian for the interacting pions; and in the extreme IR limit, the interactions in the chiral lagrangian are irrelevant, and all that's left are $N_f^2 - 1$ free pions.

On the other hand, if N_f is such that the theory is just barely asymptotically free ($N_f \sim 11N_c/2$), there is a different phase: an interacting 4d CFT. To see how this can happen, consider the beta function to two loops:

$$\beta(g) = \frac{dg}{d(log\mu)} = -b_1 g^3 + b_2 g^5. \tag{1.1}$$

Asymptotic freedom means that the first term (one loop) coefficient is negative, as explicitly indicated; but when the theory is just barely asymptotically free, the two loop coefficient turns out to be positive – i.e. both of the coefficients b_1 and b_2 in (1.1) are positive. Note that there can then be a zero of the beta function at $g_*^2 = b_1/b_2$. Should we believe that such a RG fixed point actually exists, without non-perturbative effects leading to a mass gap or something? Well, if the theory is really just barely asymptotically free, then b_1 is small, b_2 is large, and g_*^2 is small, so it's plausible that non-perturbative effects don't change the qualitiative picture that $\beta(g)$ has a zero. E.g. consider the large N_c limit. Then if the theory is just barely asymptotically free $b_1 \sim O(1)$ (rather than $\mathcal{O}(N_c)$ as would be the case for generic N_f) and $b_2 \sim O(N_c^2)$ (corresponding to the 't Hooft expansion in powers of $g^2 N_c$), and the zero of the beta function is at parametrically small 't Hooft coupling $\lambda_* \equiv g_*^2 N \sim 1/N_c$. This is called the "Banks-Zaks" RG fixed point **scenario**.

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[1] $\mathcal{N}$ denotes the number of 4d supersymmetries, e.g. $\mathcal{N} = 1$ means 4 supercharges, Q_α and $\overline{Q}_{\dot{\alpha}}$. "Pure glue" means only gauge fields, without dynamical matter fields.

[2] with quark masses, these pions wouldn't be massless, but would still be light for the light flavors

This scenario is believed to be indeed realized, provided that N_f is larger than some minimal value, with added confidence in its existence coming from supersymmetric analogs. In this phase, the coupling starts in the extreme UV at zero, and as one flows to the IR it's attracted to the RG fixed point coupling $g(\mu) \to g_*$. From then on, since $\beta(g_*) = 0$, the coupling stops running, and we have a scale invariant renormalization group (RG) fixed point. Because the theory is scale invariant there, the potential between two charged sources separated by distance R is determined by dimensional analysis to be $V(R) = g_*^2/R$. Because the gauge theory is non-Abelian, there are interaction terms, as in the classical Yang-Mills action, coming from the gauge group commutator terms. So this is an *interacting*, four dimensional, RG fixed point, an interacting 4d conformal field theory (CFT)[3]. We'll see many such interacting RG fixed points in the susy context, where exact results allow them to be explored in more detail, but it's interesting to keep in mind that there can also be non-supersymmetric, interacting 4d CFTs.

Added insights in for non-susy QCD phases come from other strong coupling techniques, or lattice gauge theory. Here we will focus on susy gauge theories, where it's possible to obtain some exact results for the effective action. We'll see analogs of the above different phases, as well as some others, in this context.

2. The classical theory, and classical moduli spaces of vacua

Consider a susy gauge theory with gauge group G and matter chiral superfields Q_i in representations r_i of G. We have the option of including or not including a tree-level superpotential $W_{tree}(Q)$.

After integrating out the auxiliary fields, the scalar components of the chiral superfields have a potential of the form

$$V_{scalar} = V_D + V_F; \qquad V_D = \sum_A D_A^2, \quad V_F = (K^{-1})^{i\bar{j}} W_i \overline{W}_{\bar{j}}. \tag{2.1}$$

Here A labels the adjoint of G, $A = 1 \ldots |G|$, and $D_A = \sum_i Q_i^\dagger T_{r_i}^A Q_i$ (with $T_{r_i}^A$ the G generators in representation r_i) upon integrating out the auxiliary fields. A supersymmetric vacuum has $V = 0$, which requires

$$\text{susy vac: } \sum_i Q_i^\dagger T_{r_i}^A Q_i \text{ for all } A, \quad \text{and } \partial_i W_{tree} = 0 \text{ for all i.} \tag{2.2}$$

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[3] Scale invariance often leads to full conformal invariance. The exception is if the trace of the stress tensor is not zero, but is a total derivative.

Consider first the case of $W_{tree} = 0$. Then typically there are **vacuum valleys**, a.k.a. D-flat directions, a.k.a. a classical moduli space of supersymmetric vacua:

$$\mathcal{M}_{cl} = \{\langle Q_i \rangle \mid V_D = 0\}/\text{gauge transformations}. \tag{2.3}$$

The classical moduli space space is naturally complex – indeed, setting $V_D = 0$ can be understood as the complexification of modding out by gauge transformations. It is a theorem that

$$\mathcal{M}_{cl} = \{\langle \text{complex gauge invariant polynomials} \rangle\}/\text{classical relations}. \tag{2.4}$$

Let's give an example. Consider $G = U(1)$, with matter fields Q_i of charge $+1$ and $\widetilde{Q}_{\tilde{i}}$ of charge -1, with i and $\tilde{i} = 1 \ldots N_f$. Then

$$V_D = \sum_{i=1}^{N_f} Q_i^\dagger Q_i - \sum_{\tilde{i}=1}^{N_f} Q_{\tilde{i}}^\dagger Q_{\tilde{i}}.$$

Setting $V_D = 0$ and modding out by gauge transformations $Q_i \to e^{i\alpha Q_i}$ and $\widetilde{Q}_{\tilde{i}} \to e^{-i\alpha}\widetilde{Q}_{\tilde{i}}$ imposes one complex condition on the $2N_f$ complex scalars Q_i and $\widetilde{Q}_{\tilde{i}}$, so $dim(calM_{cl}) = 2N_f - 1$ (we'll always count complex dimensions, i.e. half the real dimension). We can understand this counting physically: when the Q and $\widetilde{Q}$ vevs are non-zero, the gauge group is broken by the Higgs mechanism. The Higgs counting is that we started off with $2N_f$ light matter fields, but one complex field got eaten by the (super version of the) Higgs mechanism, leaving $2N_f - 1$ light moduli fields. The gauge invariants are $M_{i\tilde{j}} = Q_i\widetilde{Q}_{\tilde{j}}$. For $N_f = 1$, there is a single gauge invariant M, and the classical moduli space is the space of unrestricted vevs $\langle M \rangle$. For $N_f > 1$, there are N_f^2 gauge invariants, and the $2N_f - 1$ dimensional classical moduli space comes from imposing the classical relations

$$M_{i_1\tilde{j}_1} M_{i_2\tilde{j}_2} \epsilon^{i_1 i_2 \ldots i_{N_f}} \epsilon^{\tilde{j}_1\tilde{j}_2\ldots\tilde{j}_{N_f}} = 0, \tag{2.5}$$

stating that the matrix M has rank 1.

Let's give another example. Consider $G = SU(N_c)$, with N_f fundamental flavors, Q_f in the fundamental and $\widetilde{Q}_f$ in the anti-fundamental. (We include both to cancel gauge anomalies). For $N_f < N_c$, there is a classical moduli space of vacua where

$$\langle Q \rangle = \langle \widetilde{Q} \rangle = \begin{pmatrix} a_1 & 0 & \ldots & 0 \\ 0 & a_2 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & a_{N_f} & 0 \end{pmatrix},$$

and generically on this space the gauge group is Higgsed as $SU(N_c) \to SU(N_c - N_f)$. The counting of light fields, left uneaten by the Higgs mechanism, is thus:

$$dim(\mathcal{M}_{cl}) = 2N_c N_f - \left[(N_c^2 - 1) - ((N_c - N_f)^2 - 1)\right] = N_f^2.$$

The classical moduli space is parametrized by the N_f^2 unrestricted expectation values of the gauge invariant mesons $M_{i\tilde{j}} = Q_{ic}\widetilde{Q}^c_{\tilde{j}}$, with c a color index $c = 1 \dots N_c$ (which we often suppress).

For $N_f > N_c$, the gauge group is generically completely broken, and the Higgs counting then gives for the moduli space dimension

$$dim(\mathcal{M}_{cl}) = 2N_c N_f - (N_c^2 - 1).$$

For example, for $N_f = N_c$, we get a $N_f^2 + 1$ dimensional moduli space. The description in terms of the gauge invariants is more complicated, because we can form baryonic objects, and because there are non-trivial classical relations. E.g. for $N_f = N_c$, we form $B = \det Q$, $\widetilde{B} = \det \tilde{Q}$, and the classical moduli space is given by

$$\mathcal{M}_{cl} = \{\langle M_{i\tilde{j}}\rangle,\ \langle B\rangle,\ \langle\widetilde{B}\rangle \mid \det M - B\widetilde{B} = 0, \} \tag{2.6}$$

which is indeed $N_f^2 + 1$ complex dimensional.

The classical moduli spaces are singular spaces. E.g. the spaces (2.5) and (2.6) are topologically singular near the origin. Even for $U(1)$ with $N_f = 1$, where the moduli space is the space of unrestricted $M = Q\widetilde{Q}$ vevs, there is a singularity in the Kahler metric at the origin, because $K_{cl} = \sqrt{M^\dagger M}$. These singularities are at exactly the places where they should be: singularities reflect the fact that some additional light fields should be included in the effective action, so they occur classically where the gauge group is (partially or fully) unHiggsed.

3. The quantum theory

3.1. holomorphy and the gauge coupling

The classical gauge theory has holomorphic coupling $\tau = \theta/2\pi + 4\pi i/g^2$. In the quantum theory, the non-zero beta function say's that there's dimensional transmutation, with a dynamical scale Λ. Supersymmetry makes Λ also a holomorphic quantity, and implies that the holomorphic coupling running is exactly given by the 1-loop expression:

$$\frac{d}{d\log\mu}2\pi i\left(\frac{\theta}{2\pi}+\frac{4\pi i}{g^2(\mu)}\right)=-b_1. \tag{3.1}$$

The RHS is independent of $\theta\sim Re(\tau)$, and hence constant, with b_1 the 1-loop beta function:

$$b_1=3T(G)-\sum_f T(r_f), \tag{3.2}$$

with $T(G)$ and $T(r_f)$ the quadratic Casimirs in the adjoint ($T(G)$ is also called the dual Coxeter number) and the representation r_f of the matter fields: $Tr_{r_f}(T^AT^B)=T(r_f)\delta^{AB}$. For $SU(N_c)$, $T(G)=N_c$, and $T(fund)=\frac{1}{2}$. Integrating (3.1) leads to

$$e^{-8\pi^2/g^2(\mu)+i\theta}=\left(\frac{\Lambda}{\mu}\right)^{b_1}, \tag{3.3}$$

with Λ the holomorphic dynamical scale, whose phase is related to the theta angle. Anomalous global rotations have the effect of shifting the theta angle θ, which can be understood as saying that Λ is charged under anomalous symmetries, as we'll soon discuss further.

Another argument for the fact that the one-loop beta function is one-loop exact is the infamous multiplet of anomalies argument: supersymmetry related the beta function to an anomaly in a chiral $U(1)_R$ current, which is one-loop exact by the Adler Bardeen theorem.

The above argument is infamous because it led to much confusion – since explicit calculation revealed that the beta function is *not* one -loop exact, it has a non-zero two-loop and higher loop contributions. In fact, we'll want those higher loop contributions to argue for an analog of the RG fixed point, a'la Banks-Zaks, for susy gauge theories. The point is that the physical gauge coupling is not the holomporphic one: it differs from it by effects coming from the wavefunction renormalization of the fields. Indeed, the physical exact beta function can be expressed exactly in terms of the anomalous dimensions of the fields: this is the exact NSVZ beta function, which we might discuss further in a later lecture.

There is another source of confusion, coming from e.g. the Seiberg-Witten curve for $\mathcal{N}=2$ theories: there one studies e.g. the Coulomb branch, where the gauge group is broken to the Cartan subalgebra: $G\to U(1)^r$, and obtains the exact gauge couplings for the $U(1)^r$ photons. The solution has the appearance of a beta function that's one-loop exact in perturbation theory, but with additional non-perturbative corrections. This beta function is not to be confused with the two discussed above – it is a third kind of beta function, distinct from the above two, for the effective $U(1)^r$ couplings on the Coulomb branch.

3.2. anomalies, instantons

Classical $U(1)$ flavor symmetries can be violated by the ABJ anomaly: $\partial_\mu J^\mu \sim (\text{coeff})F\widetilde{F}$, coming entirely from the triangle diagram with the flavor current J^μ at one vertex, and G gauge fields at the other two vertices. The non-zero contributions come from the massless fermions running in the triangle loop, and this gives the coefficient as $\sum_f q(r_f)T(r_f)$, with $q(r_f)$ the charge of the fermion flavor in G representation r_f and $T(r_f)$ is the quadratic Casimir. Integrating both sides of the anomaly equation gives $\Delta q = \int d^4x \partial^\mu J_\mu \sim \int d^4x F\widetilde{F}$: the anomalous charge is conserved in perturbation theory, but has very specific violation in instanton background. This instanton charge violation is equivalent to the fact that instantons have fermion zero modes, as is also determined by the AS index theorem.

The instanton and its fermion zero modes lead to a 't Hooft vertex interaction. We can picture this vertex schematically as a blob, with legs corresponding to the fermion zero modes. The (anti) instanton 't Hooft vertex leads to a holomorphic interaction that's weighted by

$$e^{-S_{inst}} = e^{-8\pi^2/g_2+i\theta} \sim \Lambda^{b_1}. \tag{3.4}$$

The G instanton has $2T(G)$ gaugino zero modes and $2T(r_f)$ fermion zero modes for each flavor Q_f, in representation r_f of G.

3.3. pure $\mathcal{N}=1$ SYM

Pure SYM means just the G vector multiplet, with no matter chiral superfields. In this case, there is no moduli space of vacua and no massless fields. The only basic gauge invariant that can be formed is the glueball superfield

$$S = -\frac{1}{32\pi^2}\text{Tr}W_\alpha W^\alpha, \tag{3.5}$$

formed from the G adjoint gauge field strength chiral superfields W_α. The glueball field is expected to be massive in the IR. The lore, which is supported by lattice simulations and also the susy exact results we're discussing, is that this theory gets a mass gap, with confinement[4] and chiral symmetry breaking.

A G instanton leads to (accounting for the fermion zero modes)

$$\langle S(x_1)\dots S(x_h)\rangle = \Lambda^{3h}, \tag{3.6}$$

with $h = T(G)$ the dual Coxeter number. This shows that the classical $U(1)_R$ symmetry is anomalous, but that a Z_{2h} subgroup is anomaly free. $S(x_i)$ means the glueball superfield operator is placed at spacetime location x_i, but correlation functions involving only chiral superfields (or only anti-chiral superfields) are always independent of the operator locations; this follows upon effecting translations as a commutator with momentum, and then replacing the momentum with the supercharge anticommutator.

Now many different arguments lead to the conclusion that the Z_{2h} global symmetry is spontaneously broken to Z_2, by gaugino condensation:

$$\langle S\rangle = \Lambda^3 e^{2\pi ik/h}, \qquad k = 1\dots h, \tag{3.7}$$

with k labeling the h vacua associated with the $Z_{2h} \to Z_2$ SSB (there are no goldstone bosons, since it's a discrete symmetry that's spontaneously breaking). These h vacua are believed to have a mass gap and hence unbroken supersymmetry (since there's no massless goldstino fermion, as happens when global supersymmetry spontaneously breaks). One argument for this is the position independence of (3.6): take the locations of the S operator insertions widely separated, and then invoke "cluster decomposition". Another argument is that the $h = T(G)$ supersymmetric vacua agrees with the computation of the Witten index $\mathrm{Tr}(-1)^F$ (for groups other than $SU(N)$ and $Sp(N)$, there is an an interesting group theoretic subtlety in the Witten index computation, as explained by Witten in the late 90's). Another argument, as we'll see, is to add matter chiral superfields, and then given them mass and integrate them out, which again leads to $h = T(G)$ vacua.

Another argument for (3.7) was given long ago by VY. Consider writing an effective action for the glueball chiral superfield S. Since S is massive, it's not clear that this is a good idea (we're integrating out degrees of freedom which might be lighter than the included field S, which would be problematic), but not worrying about this leads to

[4] Because the gauginos are in the adjoint of the group, this is true confinement: sources charged under the center of the group cannot be screened by the dynamical matter, and the flux tube connecting such sources can't break, leading to a potential $V(R) \sim \sigma R$ for arbitrarily larger R.

an interesting and often useful expression. The classical lagrangian density has a term $W(S) = \tau_{cl} S$, with $\tau = \theta/2\pi + 4\pi i/g^2$, which is invariant under the classical $U(1)_R$ symmetry under which S has charge 2. In the quantum theory, this symmetry is anomalous, but the idea of VY is to use this to say that effecting a $U(1)_R$ global phase rotation $S \to e^{2i\alpha} S$ has the effect of shifting the theta angle, $\theta \to \theta - 2h\alpha$, which corresponds to a shift of the superpotential $W_{eff} \to e^{2i\alpha}(W_{eff} - hS)$. The effective superpotential with this property is

$$W_{VY}(S) = S\left(\log\left(\frac{\Lambda^{3h}}{S^h}\right) + h\right). \tag{3.8}$$

This superpotential has h supersymmetric vacua, with $\langle S \rangle$ expectation values given by (3.7).

Some exercises

1. Consider the following Kalher potentials and superpotentials, and determine the supersymmetric vacua. Determine whether or not there is a moduli space of vacua and, if there is, it's structure. If there are isolated vacua, determine how many susy vacua there are. If not explicitly given, take $K = K_{can}$, the classical Kahler potential having $K_{i\bar{j}} = \delta_{i\bar{j}}$. One of the examples breaks supersymmetry. (A): $K = \overline{Q}Q$, $W = \frac{1}{2}mQ^2$; (B): $K = \overline{Q}Q$, $W = \lambda Q$; (C): $K = \sqrt{\overline{X}X}$, $W = \lambda X$; (D) $W = gXYZ$; (E): $W = (X-1)Y^2$; (F): $W = (X-1)Y^2 + mX$.
2. Work out the quadratic Casimir $T(r_j)$ for r_j the spin j representation of $SU(2)$. Hint: you can compute it for the generator of rotations around the z axis, where you know that the eigenvalues are j, $j-1$, ..., $-j$. As a check, confirm that the fundamental $j = \frac{1}{2}$ has quadratic Casimir $T(fund) = \frac{1}{2}$ and the adjoint $j = 1$ has $T(adj) = 2$. Now work out the quadratic Casimirs of the adjoint and two index antisymmetric representations of $SU(N_c)$, using the fact that they can be computed by decomposing the representations into those of an $SU(2)$ subgroup, with $N_c \to 2+$ singlets. Hint: draw these as $N_c \times N_c$ matrices, and picture the $SU(2)$ as the upper 2×2 block, now find the charged $SU(2)$ representations (the singlets don't contribute to the Casimirs).
3. Consider an $SU(2)$ gauge theory, with a matter field Q in the $j = 3/2$ representation. Using your result from the above problem, verify that this matter content is asymptotically free. Draw the $SU(2)$ instanton, indicating how many gaugino and ψ_Q fermion zero modes there are. There is a unique anomaly free $U(1)_R$ symmetry; find the charge of Q under this anomaly free $U(1)_R$.

4. quantum effective actions: holomorphy and symmetries

Consider the quantum effective action for the classically massless moduli fields. Let's call the gauge invariant polynomials of the chiral superfields X_r; these give the massless moduli, like the mesons and baryons above, and are themselves chiral superfields (products of chiral superfields give chiral superfields). The IR effective action Lagrangian density (up to two-derivative terms, with higher derivative terms suppressed in the IR limit) takes the form

$$\mathcal{L} = \int d^4\theta K_{eff}(X_r, \overline{X}_r) + \int d^2\theta W_{eff}(X_r) + \sum_{ij} \int d^2\theta \tau_{ij}(X_r) W^i_\alpha W^{j\alpha} + h.c., \quad (4.1)$$

where W^i_α are for any $U(1)$ factors which might be left unHiggsed, as on the Coulomb branch of $\mathcal{N}=2$ theories, where the gauge group is broken to its Cartan subalgebra, as $G \to U(1)^r$. We start with a **non-Abelian** gauge group G, but by the time we get to the IR, often the non-Abelian gauge fields are often effectively massive (unless there is a non-Abelian Coulomb phase there), but any unbroken Abelian subgroup of G will remain massless (and free in the extreme UV), so such photons are to be included in the low-energy effective action, as above.

Supersymmetry restricts the effective superpotential $W_{eff}(X_r)$ and the effective gauge coupling $\tau_{eff}(X_r) = \theta_{eff}/2\pi + 4\pi i/g^2_{eff}$ to be **holomorphic** functions of the chiral superfields. In these lectures, we'll focus on theories which do not have unbroken $U(1)$ factors, though one can also obtain exact results for the $\tau_{ij}(X_r)$ in $\mathcal{N}=1$ theories (as well as $\mathcal{N}=2$ theories) that do have $U(1)$ factors. A key development was Seiberg's argument that the effective superpotential (or the effective gauge coupling for that matter) is holomorphic in all of the holomorphic coupling constants, in addition to the chiral superfields. The idea is that the holomorphic coupling constants could have been expectation values of some background chiral superfields, in which case the superpotential would certainly have to be holomorphic in them; since the dynamics doesn't know whether or not the couplings are chiral superfield vevs, it must arrange itself to keep the coupling dependence holomorphic.

Holomorphy in the fields and couplings, combined with various known limits is **extremely** powerful – often sufficiently powerful to **exactly** determine W_{eff} (and τ_{eff} when there are $U(1)$ factors).

For the most part, we'll have nothing to say about the Kahler potential $K_{eff}(X_r, \overline{X}_r)$, which is unconstrained by the power of holomorphy[5]. And for the most part, the Kahler potential doesn't matter, as long as the Kahler metric doesn't have any singularities. And this is the one thing that we'll sometimes be able to say about the Kahler potential: whether or not it's smooth (without singularities) in terms of a particular set of field variables. A tool that we'll use for this, to be discussed shortly, is *'t Hooft anomaly matching.*

4.1. All symmetries are useful, even (especially!) the broken ones!

All symmetries are useful, even broken ones. Broken symmetries can be thought of symmetries under which some couplings are charged. Anomalous symmetries can be thought of as symmetries under which Λ^{b_1} is charged. Since the coupling is thought of as a vev of some background chiral superfield, the broken or anomalous symmetry can be thought of as being *spontaneously broken*, which means that the symmetry is still useful: it leads to selection rules. You've seen this same logic in a quantum mechanics class, when studying the Stark effect: the background electric field breaks rotational symmetry spontaneously, but we can still use the rotational symmetry, just keeping in mind that the E_{bkgd} has $l = 1$ and we get selection rules.

Let's consider some $U(1)$ symmetries, which are generally anomalous. Let $U(1)_R = U(1)_\lambda - U(1)_\psi$, under which all gauginos have charge $+1$ (as is always the case for a $U(1)_R$ symmetry) and all matter fermion quarks have charge -1, which charge zero for the squarks. Let $U(1)_{Q_f}$ be the (non-R) $U(1)$ under which only the single matter chiral superfield Q_f has charge 1. The relevant charges are

$$\begin{array}{ccc} & U(1)_R & U(1)_{Q_f} \\ X_r & 0 & * \\ \Lambda^{b_1} & 2T(G) - \sum_f 2T(r_f) & T(r_f) \\ W(X_r, \Lambda^{b_1}) & 2 & 0. \end{array} \tag{4.2}$$

We can immediately come to a very interesting conclusion. Suppose that $W_{tree} = 0$, so there is a classical moduli space of vacua. These vacua are physically inequivalent, e.g. the masses of the massive W-bosons depends on the vacuum choice. So the classical energy degeneracy of the classical moduli space isn't protected by any symmetry, and we

[5] For $\mathcal{N} = 2$ theories the added supersymmetry allows the Kahler potential to be exactly obtained too. On the Coulomb branch, it's related by supersymmetry to τ^{ij}_{eff} (both are expressed in terms of a prepotential). On the Higgs branch, it can be shown to be unrenormalized.

should expect this classical degeneracy to be lifted by quantum effects, by a dynamically generated $W_{dyn}(X_r, \Lambda^{b_1})$. But for $W_{tree} = 0$ such a W_{dyn} can only be generated and lift the classical moduli space degeneracy if $T(G) > \sum_f T(r_f)$. This follows from (4.2), because the classical moduli space must be recovered if we take $g \to 0$, corresponding to $\Lambda \to 0$, meaning that the powers of Λ must be in the numerator of W_{dyn}. To give an example, for $SU(N_c)$ SQCD with N_f fundamental flavors, a non-zero W_{dyn} can lift the classical moduli space degeneracy only if $N_f < N_c$. For $N_f \geq N_c$ there is an exactly degenerate quantum moduli space of vacua.

More generally, for $W_{tree} = 0$, if there is enough matter, so that $\sum_f T(r_f) \geq T(G)$, there is an exactly degenerate, quantum moduli space of vacua, unlifted by any dynamically generated superpotential. We'll explore what happens to the classical singularities near the origin in the quantum theory.

5. The basic $W_{dyn} \neq 0$ example: $SU(2)$ gauge theory with $N_f = 1$

For general $SU(N_c)$, $N_f = 1$ means one fundamental and one anti-fundamental (one of each is needed to avoid gauge anomalies, coming from the triangle diagram with a G gauge field at each vertex). For $SU(2)$, the $\mathbf{2} \sim \overline{\mathbf{2}}$, so $N_f = 1$ means two $SU(2)$ doublets, $Q_{f,c}$, with $f = 1, 2$ a flavor index and $c = 1, 2$ a color index; more generally $SU(2)$ with N_f flavors means Q_{fc} with $f = 1 \ldots 2N_f$ a flavor index and $c = 1, 2$ a color index[6] There is a single, independent, gauge invariant chiral superfield combination of the matter fields, $M = Q_{fc} Q_{gd} \epsilon^{fg} \epsilon^{cd}$. There is a classical moduli space, given by unconstrained vevs of M in the complex plane, with a singularity in the classical Kahler metric at the origin, since $K_{cl} \sim \sqrt{M^\dagger M}$. For $M \neq 0$, the associated expectation values of the fields Q_{fc} break $SU(2)$ entirely, and the Higgs counting is that the one light field M is the matter field left uneaten, after the 3 $SU(2)$ gauge bosons get a mass, $4 - 3 = 1$. We have the option of giving the matter fields a mass, via $W_{tree} = mM$. For now, let's take $m = 0$.

We should expect $W_{dyn} \neq 0$, lifting the classical degeneracy. W_{dyn} vanishes to all orders in perturbation theory, but non-perturbatively a $W_{dyn} \neq 0$ is indeed generated. Let's constrain its form using holomorphy and the classical $SU(2)_F \times U(1)_Q \times U(1)_R$

[6] Unlike $SU(N_c)$ with $N_c > 2$, the gauge anomaly triangle diagram, with a G gauge field at each of the three vertices, vanishes for $SU(2)$. But the number of $SU(2)$ fundamental fermions must be even, because of the Witten anomaly, coming from $\pi_4(SU(2)) = Z_2$.

flavor symmetries. The $SU(2)_F$ doesn't give any information, since M is already an $SU(2)_F$ singlet. The charges of the fields under $U(1)_Q$ and $U(1)_R$ are;

$$\begin{array}{ccc} & U(1)_Q & U(1)_{R_0} \\ M & 2 & 0 \\ \Lambda^5 & 2 & 2 \\ W & 0 & 2 \end{array} \tag{5.1}$$

The $U(1)_Q$ and $U(1)_{R_0}$ given above are anomalous, as seen from the non-zero charges of Λ^5, coming from the two ψ_{Q_f} fermion zero modes and the four gaugino fermion zero modes of the instanton 't Hooft vertex. There is an anomaly free $U(1)_{R_0}$ symmetry, which is a linear combination of the above $U(1)_Q$ and $U(1)_{R_0}$, under which $R(\psi_{Q_f}) = -2$, to cancel the $R = +4$ charge of the gauginos, so $R(Q_f) = -1$ and $R(M) = -2$ under the anomaly free $U(1)_R$. But for now we don't need to worry about this: all the above symmetries are useful, and the Λ charges account for all the needed quantum information.

The above charges uniquely determine the possible W_{dyn}, up to a single coefficient:

$$W_{dyn} = c\frac{\Lambda^5}{M}.$$

The fact that it's proportional to Λ^5 means that c comes entirely from a 1-instanton contribution. A non-perturbative superpotential means that the instanton should have precisely two fermion zero modes, to give the fermions in the Lagrangian term $W''_{dyn}\psi\psi$. Now supersymmetry ensures that the instanton has at least two fermion zero modes, which are superpartners to the 4 real bosonic translational zero modes. Naively, one might expect that supersymmetry ensures and additional two "superconformal" fermion zero modes, coming from the superpartners of the bosonic moduli ρ (the instanton size modulus) and three $SU(2)$ rotations. And above we claimed that the instanton has a total of 6: the two ψ_{Q_f} quark zero modes and the four gaugino zero modes.

The point is that the above counting of 6 zero modes is OK for determining charges, but it's actually incorrect once we account for the fact that the squark fields Q_{fc} have non-zero expectation values. These squark vevs break the scale and $SU(2)$ rotation symmetries, and it'd seem that there's then no instanton solution at all. Actually, one has to do a more detailed analysis of what's called constrained instantons, when there are Higgs vevs, and it can be shown (Affleck, Dine, Seiberg) that four of the naive zero modes pair up and get a mass, proportional to the Q_{fc} vevs. The upshot is that $c \neq 0$, and in an appropriate scheme we can take $c = 1$, so

$$W_{dyn} = \frac{\Lambda^5}{M}. \tag{5.2}$$

The qualitative form of W_{dyn} makes sense: it goes away when $\langle M \rangle \to \infty$, as expected by asymptotic freedom, since the gauge coupling only runs at energy scales above that set by the vev, $\mu > \sqrt{\langle M \rangle}$, so for large $\langle M \rangle$ the $SU(2)$ gauge coupling only runs a little from zero, and as $\langle M \rangle \to \infty$, the coupling is zero and we must recover the classical moduli space.

Now we add a mass term $W_{tree} = mM$. The full, exact superpotential is given by simply the sum

$$W_{full} = W_{dyn} + W_{tree} = \frac{\Lambda^5}{M} + mM. \tag{5.3}$$

Here we can show this by using symmetries, but this is a general result: the full superpotential is just the sum $W_{full} = W_{dyn} + W_{tree}$. A motivation for this is that sometimes this can be thought of as in the 1PI effective action, where the sources for the operators are added linearly in the action: the couplings in W_{tree} play the role of the operator sources here. If we integrate out the massive field M, we obtain

$$\frac{dW_{full}}{dM} = 0 \to \langle M \rangle = \pm\sqrt{\Lambda^5/m},$$

and plugging this back in gives the low-energy superpotential

$$W_{low}(m) = \pm 2\sqrt{\Lambda^5 m}. \tag{5.4}$$

This can be thought of, as usual for the effective action, as giving the operator expectation values

$$\frac{dW_{low}(m)}{dm} = \langle M \rangle = \pm\sqrt{\frac{\Lambda^5}{m}}.$$

And, as usual, for effective actions, $W_{dyn}(M)$ and $W_{low}(m)$ are related by a Legendre transform, which can be inverted: knowing $W_{low}(m)$ we can reconstruct $W_{dyn}(M)$ by the inverse Legendre transfrom.

Define the glueball chiral superfield $S \equiv -\frac{1}{32\pi^2}\mathrm{Tr}W_\alpha W^\alpha$, with W_α the super field-strengths of gauge group G. The Konishi anomaly relates the expectation values of S to W_{tree}, as $\langle Q \frac{\partial W_{tree}}{\partial Q} \rangle = \langle S \rangle$. This yields

$$\langle mM \rangle = \langle S \rangle \to \langle S \rangle = \pm\sqrt{m\Lambda^5}. \tag{5.5}$$

The results (5.4) and (5.5) can be connected with the low-energy pure-glue $SU(2)$ theory obtained below the energy scale m, where the massive matter fields can be integrated out. Matching the running coupling gives $m\Lambda^5 = \Lambda_L^6$, and (5.4) and (5.5) become

$$W = \pm 2\Lambda_L^3, \qquad \langle S \rangle = \pm\Lambda_L^3, \tag{5.6}$$

which is to be interpreted as coming from gaugino condensation in the low-energy $SU(2)$ theory with no matter fields. Indeed, the VY glueball superpotential (3.8) gives the minima for $\langle S \rangle$ and the value of $W_{low} = W_{VY}(\langle S \rangle)$ agreeing with (5.6).

6. Integrating in and out

6.1. $SU(2)_1 \times SU(2)_2$ *with matter* $Q \in (\mathbf{2}, \overline{\mathbf{2}})$.