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ICTP 40th Anniversary

SMR.1566 - 13

Introductory School on

RECENT DEVELOPMENTS IN SUPERSYMMETRIC GAUGE THEORIES

14 - 25 June 2004

S-DUALITY IN N=2 SUPERSYMMETRIC GAUGE THEORIES (Part II)

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Lecture 4

We want to determine the quantum geometry of mod. space

$$\tau = \frac{\partial a_D}{\partial a} = \frac{i}{2\pi}(2N_c - N_f)\ln\frac{a}{\Lambda}$$

$$\mathcal{F} = \frac{i}{4\pi}(N_c - N_f/2)\, a^2 \ln\frac{a^2}{\Lambda^2}$$

$$K = \mathrm{Im}\, A_D \bar{A}\ , \qquad Z = a n_e + a_D n_m$$

CMS. $\begin{pmatrix} a_D(u) \\ a(u) \end{pmatrix}$ $u \to \infty$ we find monodromy

$$\pi_1(\mathcal{M}) \longrightarrow G$$

$$\left.\begin{array}{l} \ln u \to \ln u + 2\pi i \\ \ln a \to \ln a + i\pi \end{array}\right\} \quad \begin{array}{l} a_D = \frac{i}{\pi}\left(a \ln \frac{u}{\Lambda^2} + a\right) \\ a \end{array}$$

$$\begin{pmatrix} a_D \\ a \end{pmatrix} \to \begin{pmatrix} -1 & 2 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} a_D \\ a \end{pmatrix}$$

$$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},\quad S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},\quad P = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$M_\infty = P T^{-2}$$

$\mathrm{Im}\,\tau$ is harmonic, we need more singularities to avoid $\mathrm{Im}\,\tau < 0$, and also we need to maintain positivity.

$$ds^2 = \mathrm{Im}\frac{da_D}{du}\frac{d\bar{a}}{d\bar{u}}\, du\, d\bar{u}$$

$$= -\frac{i}{2}\epsilon_{\alpha\beta}\frac{da^\alpha}{du}\frac{d\bar{a}_D^\beta}{d\bar{u}}\, du\, d\bar{u}$$

$M = ISL(2,\mathbb{R})$. If $N_f = 0$ $SL(2,\mathbb{R})$

Z invariant $M \in SL_2(\mathbb{Z}) \equiv \Gamma$

Hence the trans. functions are in Γ.

$$\mathrm{Im} \int \tau\,(F + i\tilde{F})^2 + \int A_D \wedge dF$$

$$\Downarrow$$

$$\tau \to -1/\tau .$$

$$\mathrm{Im} \int A_D(q)\,\bar{A}(q) = \mathrm{Im} \int A_D \bar{A}$$

$$\begin{pmatrix} A_D(q) \\ A(q) \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} A_D \\ A \end{pmatrix} \Rightarrow \tau(q) = \frac{\alpha\tau+\beta}{\gamma\tau+\delta}$$

$$\begin{pmatrix} q_m & q_e \\ r & s \end{pmatrix} \in SL_2(\mathbb{Z}) \qquad q_m, q_e \text{ are coprime}$$

Imagine $a_D(q) \simeq 0$ at some u_0. Then ~~the~~

$$\tau_D \simeq -\frac{i}{\pi} \log a_D(q)$$

$$a(q) = -\int \tau_D(q)\, da_D(q) = \frac{i}{\pi} a_D(q) \log a_D(q) + a_0 .$$

$$\left.\begin{aligned} a_D(q) &\to a_D(q) \\ a(q) &\to a(q) - 2a_D(q) \end{aligned}\right\} \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}$$

monopole. Then conj. back

$$M(q_m, q_e) = \begin{pmatrix} 1 + 2q_m q_e & 2q_e^2 \\ -2q_m^2 & 1 - 2q_m q_e \end{pmatrix}$$

$$M(m,n) = M_\infty M^{-1}(m',n') \qquad \Big|\; (q_m, q_e)\cdot M = (q_m\, q_e)$$

$$m = \pm 1, \quad m' = \pm 1, \quad n = \pm(n' + m')$$

Conjugate with M_∞.

$$(q_m, q_e) \begin{pmatrix} -1 & 2 \\ 0 & -1 \end{pmatrix} = (-q_m, -q_e + 2q_m)$$

$$(1, 0), \qquad (1, -1)$$

$$u = \frac{1}{2}a^2$$

$$u = \infty \quad \begin{cases} a \simeq \sqrt{2u} \\ a_D \simeq \dfrac{i\sqrt{2u}}{\pi} \ln u \end{cases}$$

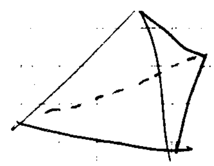
$$u = 1 \quad \begin{cases} a_D \simeq c_0(u-1) \\ a \simeq a_0 + \dfrac{i}{\pi} a_D \ln a_D \end{cases}$$

$u = -1$ $\{$ same with $a_D \to a - a_D$ $\}$

What about $m=0$ gluons?

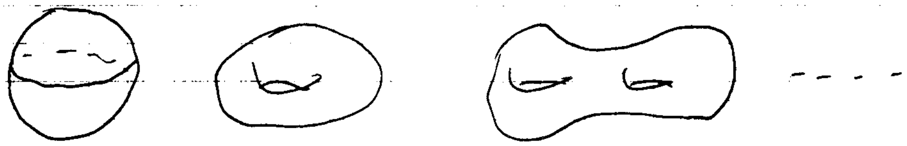
Mathematics.

cpt. Riemann surface. $\chi = 2-2g$ $g =$ genus or # handles.



$V = 4$ $F = 4$ $E = 6$

$\chi = 4 - 6 + 4 = 2$

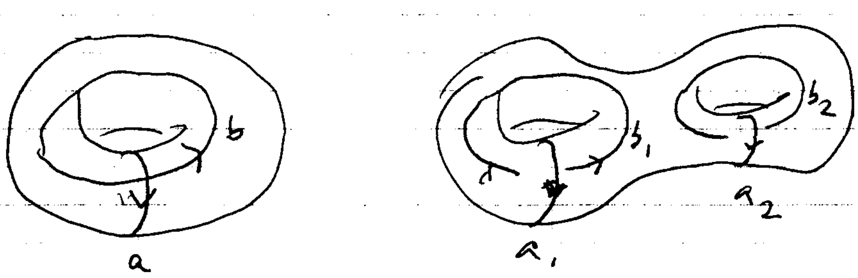


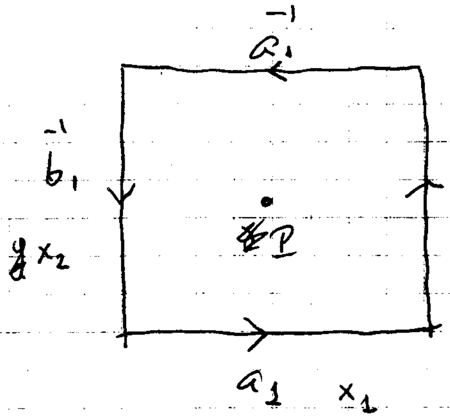
$g=0$ $g=1$ $g=2$ ...

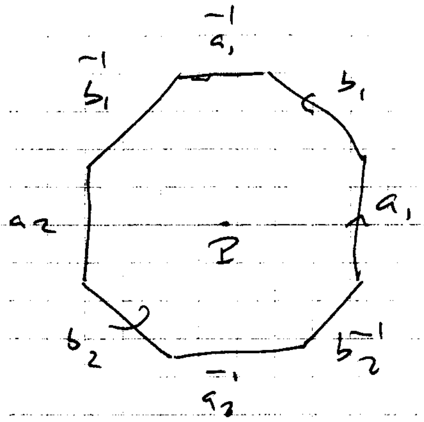
covering

S^2 $\mathbb{C}$ $\mathbb{H}$ uniformization Thms.

$g \geq 1$ not simply connected. $H^1 \neq 0$, $\pi^1 \neq 0$.







$$\prod_i a_i b_i a_i^{-1} b_i^{-1} = 1$$

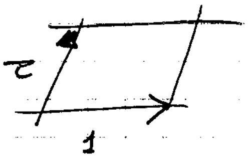
Think of them as cycles or as loops $S^1 \to \Sigma$

H^1 $\qquad\qquad\qquad\qquad$ π_1

Here you can break $\qquad\qquad$ Here not.

Differential forms $\quad dx_1, \; dx_2$

$$T^2 = \mathbb{C} / \{ \mathbb{Z}\tau + \mathbb{Z} \}$$



The set of tori is given by $\boxed{\operatorname{Im} \tau > 0.}$
and $SL(2,\mathbb{Z})$ acts as usual

$\Gamma / \Gamma(2) = SL(2, \mathbb{Z}_2)$

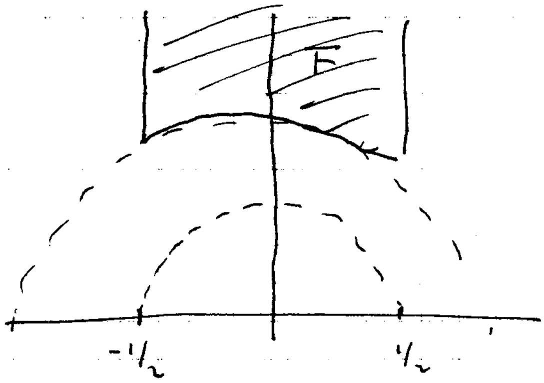
$\tau \to \tau + 1$

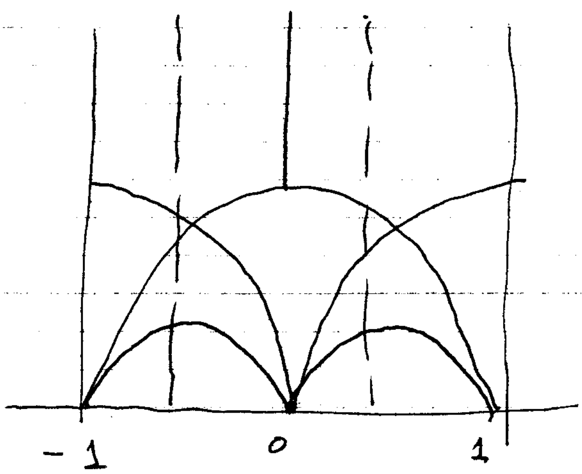
$\tau \to -1/\tau$

$\tau \to 1/\tau+1$

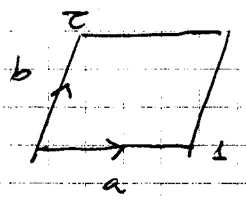
$\tau \to \tau/\tau+1$

$\tau \to (\tau+1)/\tau$





$$dz = dx_1 + i\,dx_2$$



$$\oint_{\gamma_i} dx_i = 1$$

$$\oint_a dz = 1 \qquad \oint_b dz = \tau$$

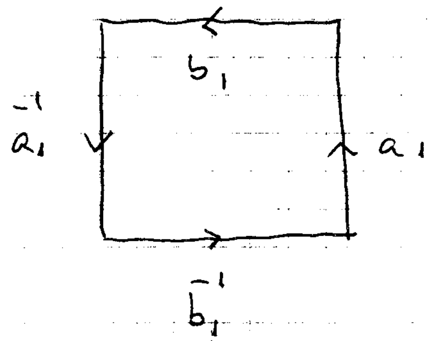
In general z complex coordinate

$$\omega_i(z) = f_i(z)\,dz \quad \text{globally defined}$$

Normalize.

$$\oint_{a_i} \omega_j = \delta_{ij} \qquad \oint_{b_i} \omega_j = \tau_{ij} \qquad dz \wedge dz = 0$$

1). $\tau_{ij} = \tau_{ji}$ 2) $\mathrm{Im}\,\tau_{ij} > 0$.



$$0 = \int \omega_i \wedge \omega_j$$

$$= \int_\Sigma d\varphi_i \wedge \omega_j$$

$$= \int_{\partial \Sigma_c} \varphi_i\,\omega_j = \int_{a_1} + \int_{b_1} + \int_{a_1^{-1}} + \int_{b_1^{-1}}$$

$$\int_{a_1} \varphi_i\,\omega_j + \int_{a_1^{-1}} \varphi_i\,\omega_j = \int_{a_1} \varphi_i\,\omega_j - \int_{a_1 + (b_1)} \varphi_i\,\omega_j$$

$$= \int_{a_1} \varphi_i\,\omega_j - \int_{a_1} \varphi_i\,(p + b_1)\,\omega_j =$$

$$= -\int_a \big(\varphi_i(p+b_1) - \varphi_i(p)\big)\,\omega_j$$

$$= -\oint_b \omega_i \oint_a \omega_j$$

$$\int_{\Sigma} \omega_1 \wedge \omega_2 = \sum_i \left(\oint_{a_i} \omega_1 \oint_{b_i} \omega_2 - \oint_{b_i} \omega_1 \oint_{a_i} \omega_2 \right)$$

$$\int_{\Sigma} \omega_i \wedge \omega_j = \sum_k \oint_{a_k} \omega_i \oint_{b_k} \omega_j - \oint_{a_k} \omega_j \oint_{b_k} \omega_i$$

$$= \sum_k \left(\delta_{ik} \tau_{kj} - \delta_{jk} \tau_{ki} \right) = \tau_{ij} - \tau_{ji} = 0$$

Next do the same with:

$$\frac{i}{2} \int \omega \wedge \bar{\omega} \qquad \omega \wedge \bar{\omega} = f \bar{f} \underbrace{dz \wedge d\bar{z}}_{\substack{(dx+idy)\wedge(dx-idy) \\ -2i\,dx \wedge dy}}$$

$$\frac{i}{2} \int_{\Sigma} \omega \wedge \bar{\omega} \qquad \omega = \omega_i \xi_i$$

$$= \frac{i}{2} \sum_{i,j} \xi_i \bar{\xi}_j \underbrace{\int_{\Sigma} \omega_i \wedge \bar{\omega}_j}_{\mathrm{Im}\, \tau_{ij}} \geq 0$$

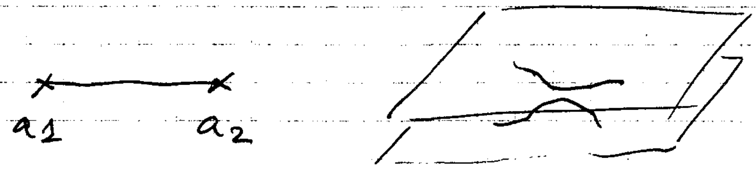
QED

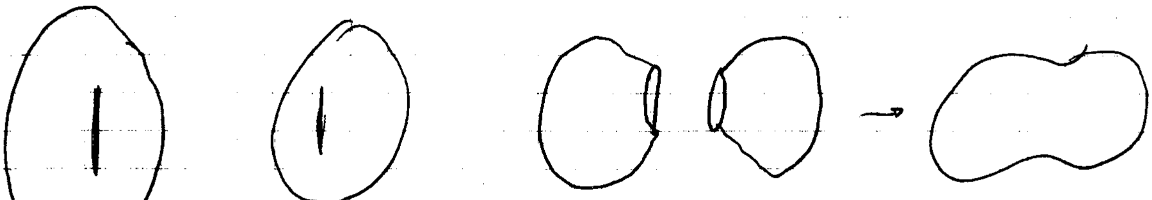
$$\boxed{\begin{array}{l} \tau_{ij} = \tau_{ji} \\ \mathrm{Im}\, \tau > 0 \end{array}}$$

Every RS is a branch covering over the sphere.

$$M \to \mathbb{P}^1 = S^2 = \mathbb{C}\cup\{\infty\}$$

$$w = \sqrt{(z-a_1)(z-a_2)} \qquad w^2 = (z-a_1)(z-a_2)$$

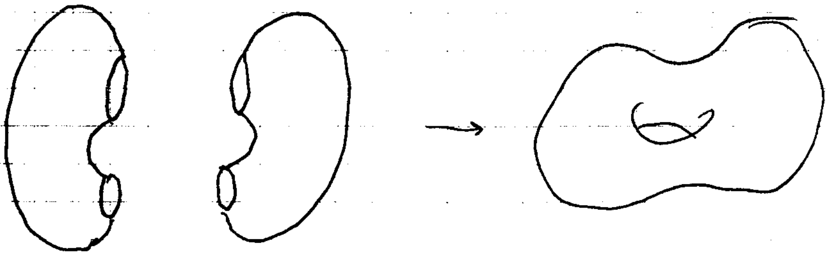




S^2

$$w = \sqrt{(z-a_1)(z-a_2)(z-a_3)}$$

4 branch points a_1, a_2, a_3, ∞, again two sheets.



In general $f: M \to \mathbb{P}^1$

$P \to f(P)$

choose local coord. z, ζ, $\zeta = z^n$

n = ramification at P

$b_f(P) \equiv n-1$. The # of ~~P bf~~ $P \exists\ b_f > 1$ is finite.

Use Euler

$$f: M \longrightarrow N \qquad (E, F, V)$$

make sure that $S = \{f(P),\ P\in M,\ b_f(P) > 0\}$ is in V.

Then $f^{-1}(E,F,V)$ gives a triangulation with nF, nE, but for the vertices we get $nV - B$ $\quad B = \sum_{P\in M} b_f(P)$

Hence: 8

$$2-2g = \cancel{n} \, nF - nE + nV - B =$$
$$= n(2-2\gamma) - B$$

For the sphere then $\gamma = 0$.

$$2-2g = 2n - B \Rightarrow \boxed{2+B-2n = 2g}$$

For double covering $n=$

$$\boxed{2-2g = n(2-2\gamma) - B}$$

$\gamma = 0$ $\qquad \boxed{1-g = n - B/2}$

$n = 2$ $\qquad -g-1 = -B/2 \qquad B = 2g+2$

$y^3 = z(z+1)(z-1)$. no branch at ∞.

$n = 3, \quad B = 2\cdot 3 = 6 \quad 1-g = 3 - 6/2 = 0$
$g = 1$

Σ is hyperelliptic if $\exists$ z of degree 2 i.e. two first order poles.

A hyperelliptic curve has a f'n of degree 2, i.e. is has two first order poles

$$z \quad z : M \to \mathbb{P}^1 \qquad (z) = \frac{Q_3 Q_4}{Q_1 Q_2}$$

now $\gamma = 0$

$$1 - g = 2 - B/2 \qquad \boxed{B = 2g+2}$$

let $P_1, \dots, P_{2g+2}$ be the branched points.

$z(P_j) \neq \infty$

$$w = \sqrt{\prod_{j=1}^{2g+2} (z - z(P_i))}$$

$z - z(P_i) = \zeta_i^2$, hence the divisor of w:

$$(w) = \frac{P_1 \cdots P_{2g+2}}{Q_1^{g+1} \, Q_2^{g+1}}$$

Abelian diff.

$$z^j dz / w \qquad j = 0, 1, \dots, g-1$$

$$(z) = \frac{Q_3 Q_4}{Q_1 Q_2} \qquad (dz) = \frac{P_1 \cdots P_{2g+2}}{Q_1^2 \, Q_2^2}$$

$$\left(\frac{z^j dz}{w}\right) = \frac{Q_1^{g+1} \, Q_2^{g+1}}{P_1 \cdots P_{2g+2}} \cdot \frac{P_1 \cdots P_{2g+2}}{Q_1^2 Q_2^2} \cdot \frac{Q_3^j \, Q_4^j}{Q_1^j \, Q_2^j}$$

$$= Q_1^{g-j-1} \, Q_2^{g-j-1} \, Q_3^j \, Q_4^j$$

$$\boxed{j = 0, 1, \dots, g-1 \quad \text{holomorphic}}$$

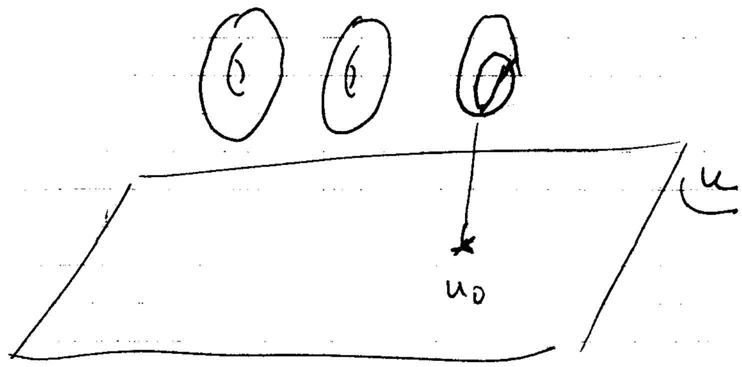
If $j = g$ however we have:

$$Q_1^{-1} Q_2^{-1} \quad Q_3^{j} Q_4^{j}$$

on the torus $x\,dx/w$ has a double pole at infinity.

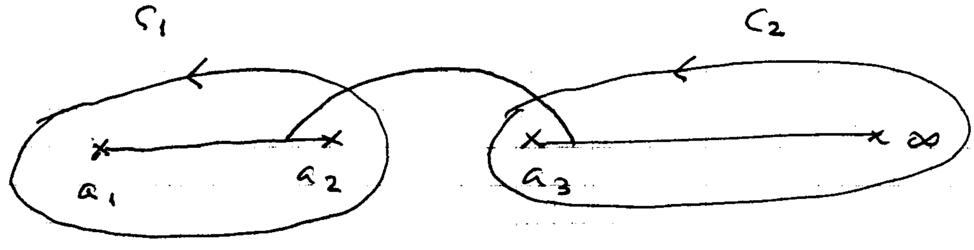
Now we can formulate the solution in detail.

We have seen that $\tau_{ij} \ni \operatorname{Im}\tau > 0$. In fact τ is a $g\times g$ matrix, thus we are looking for a torus. In fact we are looking for a 1-parameter family of tori, parametrized by u. We also know that at certain points τ get log. singularities. In RS. This can only happen when the surface degenerates; i.e. there are vanishing cycles.



we know how to construct the cycles. In the standard rep. a torus is just

$$w^2 = (z-a_1)(z-a_2)(z-a_3)$$



fix the curve by making sure there is a $\mathbb{Z}_2$ action:

$$y^2 = (x-1)(x+1)(x-u)$$

more carefully

$$y^2 = (x-\Lambda^2)(x+\Lambda^2)(x-u)$$

We want to find a differential λ

$$a(\gamma) \longrightarrow \oint_\gamma \lambda$$

we want λ to be meromorphic with zero residues so that the pairing is invariant under deformations of λ. Thus we need a λ either holo. or with a 2nd order pole. ~~these are~~ These are:

$$\frac{dx}{y} ; \quad \frac{x\,dx}{y} \qquad b_i = \oint_{\gamma_i} \lambda_1$$

$$\frac{b_1}{b_2} = \tau(u)$$

$$\lambda = a_1(u)\lambda_1 + a_2(u)\lambda_2$$

$$a_D = \oint_{\gamma_1} \lambda \qquad a = \oint_{\gamma_2} \lambda$$

$$\frac{d\lambda}{du} = f(u)\lambda_1 = f(u)\frac{dx}{y}$$

$$\frac{da_D}{du}$$

$$da_D/du = \oint_{\gamma_1} d\lambda/du \qquad da/du = \oint_{\gamma_2} d\lambda/du$$

we need $\quad \dfrac{d\lambda}{du} = f(u)\lambda_1$

so that $\quad \dfrac{da_D}{du} = f(u)b_1 \qquad \dfrac{da}{du} = f(u)b_2$

$$\tau = \frac{da_D/du}{da/du} = \frac{b_1}{b_2} = \tau_u .$$

In fact $f = -\sqrt{2}/4\pi$ is enough.

$$\lambda_1 = \frac{dx}{\sqrt{(x^2-1)(x-u)}}$$

$$\int du\, \lambda_1 = -\frac{dx}{\sqrt{x^2-1}} \cdot 2\sqrt{x-u}$$

$$\lambda = -\frac{\sqrt{2}}{4\pi} \cdot\cdot = \frac{\sqrt{2}}{2\pi} \frac{dx\sqrt{x-u}}{\sqrt{x^2-1}}$$

$$= \frac{\sqrt{2}}{2\pi} \frac{dx\,(x-u)}{y} = \frac{\sqrt{2}}{2\pi}(\lambda_2 - u\lambda_1)$$

$$\boxed{\lambda = \frac{\sqrt{2}}{2\pi}(\lambda_2 - u\lambda_1)}$$

— x —

Confinement.

$$\sqrt{2}\, a_D M\tilde{M} + mU(A_D).$$

Use standard Seiberg arguments.

$SU(N_c)$.

$$y^2 \equiv (x^2-u^2)^2 - \Lambda^4$$

$$= (x^2-u^2-\Lambda^2)(x^2-u^2+\Lambda^2)$$

$$= (x-\sqrt{u^2+\Lambda^2})(x+\sqrt{u^2+\Lambda^2})(x^2-\sqrt{u^2-\Lambda^2})$$
$$(x^2+\sqrt{u^2-\Lambda^2})$$

$u = \Lambda^2 \quad (x^2-2\Lambda^2)\,x^2$