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SUPERSYMMETRIC GAUGE THEORIES AND DUALITY (Part III)

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6. Integrating in and out example: $SU(2)_1 \times SU(2)_2$ with matter $Q \in (\mathbf{2}, \overline{\mathbf{2}})$.

This theory has a classical moduli space of vacua, parameterized by the gauge invariant chiral superfield $M = Q^2$, when M gets an expectation value, the group is Higgsed as $SU(2)_1 \times SU(2)_2 \to SU(2)_D$, with $SU(2)_D$ the diagonally embedded subgroup. The exact dynamical superpotential is

$$W_{dyn} = \frac{\Lambda_1^5}{M} + \frac{\Lambda_2^5}{M} \pm 2\frac{\sqrt{\Lambda_1^5 \Lambda_2^5}}{M}, \tag{6.1}$$

where the terms have the following interpretation: the first and second come from instantons in the broken $SU(2)_1$ and $SU(2)_2$ factors, with Λ_1^5 and Λ_2^5 the expected instanton factors. The last term comes from gaugino condensation in the unbroken $SU(2)_D$, indeed the diagonal subgroup has $g_D^{-2} = g_1^{-2} + g_2^{-2}$ (as seen from the action, $\sim \sum_{i=1}^2 g_i^{-2} F_i^2$, upon setting $F_1^2 = F_2^2 = F_D^2$), which leads to

$$\Lambda_D^6 = \frac{\Lambda_1^5 \Lambda_2^5}{M^2},$$

upon matching the running couplings at the scale, set by M, of the Higgsing $SU(2)_1 \times SU(2)_2 \to SU(2)_D$.

Using symmetries and holomorphy, it can be argued (ILS) that (6.1) is exact. Here is another way to get this exact superpotential. Give M a mass term, by adding $W_{tree} = mM$.

At energy far below the scale m, the two $SU(2)$ gauge groups are then decoupled and we have

$$W_{low} = \epsilon_1 2\sqrt{m\Lambda_1^5} + \epsilon_2 2\sqrt{m\Lambda_2^5}, \tag{6.2}$$

coming from independent gaugino condensations in $SU(2)_1$ and $SU(2)_2$. Here $\epsilon_i = \pm 1$ (the four choices here correspond to four supersymmetric vacua), and we used the running coupling matching relation, $\Lambda_L^6 = m\Lambda^5$, as in the previous example. Knowing (6.2), we can re-derive (6.1) by inverse Legendre transform

$$W_{dyn}(M, \Lambda) = \left(W_{low}(m, \Lambda) - mM\right)_{\langle m \rangle}.$$

This "integrating in" of the field M is often useful, if we can be sure that the low-energy superpotential was correctly determined. Often, however, its usefulness is limited, because of the possibility of contributions to the low-energy superpotential that could not have been naively deduced.

Now something funny happens with (6.1) if we set $\Lambda_1 = \Lambda_2$: there is a branch where $W_{dyn} = 0$. There is an exactly degenerate moduli space of vacua in this case, where the instantons in the broken groups cancel off the gaugino condensation contribution in the low-energy $SU(2)_D$ (this is very weird if you think about it). We should then ask about what are the light fields everywhere on the moduli space – is it only the field M?

There are two ways to see that the answer must be no - there must be some additional massless fields at the origin of moduli space. One way to see this is by adding $W_{tree} = mM$: if there were no additional massless fields, this branch would lead to no supersymmetric vacua. The other branch of (6.1) would lead to two supersymmetric vacua, from the two signs of a square root. But there should be a total of four supersymmetric vacua for the low-energy $SU(2) \times SU(2)$ gaugino condensation - so we're missing two vacua. The other way to see that some additional massless fields are needed is via 't Hooft anomaly matching. Let's review the idea in this context.

The theory has an anomaly free $U(1)_R$ symmetry, under which $R(Q) = -1$ (draw the instanton 't Hooft vertices, with their fermion zero modes). The idea of 't Hooft anomaly matching is to imagine that we want to gauge this global symmetry. There is, however, an obstruction: triangle diagrams with three $U(1)_R$ gauge fields gives a non-zero contribution, which would make the theory with $U(1)_R$ gauged inconsistent (it's not a problem in our original theory, where $U(1)_R$ is a global symmetry). The value of this 't Hooft anomaly obstruction is easily computed from the triangle diagram:

$$\mathrm{Tr}U(1)_R^3 = \sum_{\text{massless fermions } f} R(f)^3 = 3(1)^3 + 3(1)^3 + 4(-2)^3 = -26,$$

where the contributions come from the $SU(2)_1$ and $SU(2)_2$ gauginos, and the matter fermions ψ_Q, having $R = -2$. There is also another obstruction, coming from the triangle diagram with a $U(1)_R$ gauge field at one vertex and gravitons at the other two; this obstruction is proportional to

$$\mathrm{Tr}U(1)_R = \sum_{\text{massless fermions } f} R(f) = 3(1) + 3(1) + 4(-2) = -2.$$

't Hooft's observation is that we can cancel the above obstructions with some otherwise decoupled spectator fields, carrying flavor (in this case $U(1)_R$) charge but neutral under the original gauge group G (in this case, $G = SU(2)_1 \times SU(2)_2$). Now, we can gauge the flavor symmetry and, whatever the G dynamics does, the gauged flavor symmetry must remain consistent. Subtracting the contribution from the spectators, which goes for the ride in terms of the G dynamics, this means that the G dynamics must preserve the 't Hooft anomaly obstructions, e.g. those computed above. 't Hooft anomalies are thus argued to be constant along all RG flows. Since anomalies can be computed from triangle diagrams, with only the massless fields needed running in the loop, this provides a very powerful constraint on the massless spectrum. E.g. in the above example, we can see that the massless spectrum at the origin of the moduli space (where $U(1)_R$ is unbroken) must contain more than just the field M, whose fermion component contributes to the 't Hooft anomalies (using $R(M) = -2$, so $R(\psi_M) = -3$) as

$$\mathrm{Tr}U(1)_R = -3, \qquad \mathrm{Tr}U(1)_R^3 = (-3)^3 = -27, \tag{6.3}$$

because these do not match the quantities computed above, using the microscopic spectrum.

We can fix both of the above two problems by conjecturing that there is a new massless field S at the origin of the moduli space. Away from the origin, it gets a mass via superpotential

$$W = -MS^2. \tag{6.4}$$

Since $R(W) = 2$ and $R(M) = -2$, we see $R(S) = 2$. Adding $W_{tree} = mM$ to (6.4), we find vacua $\langle M \rangle = 0$, $\langle S \rangle = \pm\sqrt{m}$, giving the two vacua that we were missing before. And adding the $R(\psi_S) = 1$ contributions to (6.3), we have agreement with the 't Hooft anomalies computed using the classical spectrum. The fact that $R(S) = 2$ suggests that the massless field S is a glueball, associated with $SU(2)_D$. This theory is a special case of $SO(N_c)$ with N_f fundamental flavors in the N_c dimensional vector representation, for $N_c = 4$ and $N_f = 1$. There is an analog of (6.4) for all $SO(N_c)$ when $N_f = N_c - 3$. If we add one more flavor, so $N_f = N_c - 2$, there is an Abelian Coulomb phase with massless monopoles at the origin, similar to the Seiberg Witten solution, and the field S in (6.4) and its generalizations can be understood as being some of those massless monopoles.

7. Integrating in the glueball field

Given a dynamical superpotential $W_{dyn}(X, \Lambda^{b_1}, g_p)$, it is sometimes useful to integrate in the glueball field, by treating S and $\log \Lambda^{b_1}$ as Legendre transform conjugate variables:

$$W_{eff} = S \log \Lambda^{b_1} + W_P(X_r, g_p, S), \tag{7.1}$$

with the property that integrating out S leads back to $W_{dyn}(X, \Lambda^{b_1}, g_p)$. We can get $W_P(X_r, g_p, S)$ by inverse Legendre transform

$$W_P(X_r, g_p, S) = \left(W_{dyn}(X, \Lambda^{b_1} \equiv Y, g_p) - S \log Y\right)_{\langle Y \rangle}. \tag{7.2}$$

For certain theories with mass gap, the work of DV et. al., as discussed in Narian's lectures, allows W_P to be independently directly computed, by perturbative computation. Upon integrating out S, this gives an independent means to compute *non-perturbative* superpotentials, associated with instantons etc., from *perturbative Feynman diagrams* !

8. Superpotentials for SQCD

Consider SQCD with $N_f < N_c$ flavors. Symmetries and holomorphy determine the superpotential to have the form

$$W = C(N_c, N_f) \left(\frac{\Lambda^{3N_c - N_f}}{\det M} \right)^{1/(N_c - N_f)}, \tag{8.1}$$

for some constants $C(N_c, N_f)$. Consider the case $N_f = N_c - 1$, by giving the flavors appropriate vevs, we can Higgs at a very high scale to $SU(2)$ with $N_f = 1$. Accounting for matching the running coupling, we find that to recover (5.2), we need $C(N_c, N_f = N_c - 1) = 1$ for all N_c. Then we can add mass terms and integrate out the massive flavors to determine $C(N_c, N_f)$ in general, e.g. add $W_{tree} = m_{N_f} M_{N_f N_f}$ and integrate out $M_{N_f N_f}$ to obtain the result for $N_f - 1$ flavors. The final result is that

$$W = (N_c - N_f)\left(\frac{\Lambda^{3N_c - N_f}}{\det M}\right)^{1/(N_c - N_f)}, \tag{8.2}$$

For $N_f < N_c - 1$, the superpotential (8.2) can be understood as coming from gaugino condensation in the low energy unbroken $SU(N_c - N_f)$, upon using the matching relation.

In obtaining (8.2), we use the following matching relations on the running gauge coupling, which are useful to remember:

$$\text{Higgsing; } \Lambda_{N_c-1,N_f-1}^{3(N_c-1)-(N_f-1)} = \frac{\Lambda_{N_c,N_f}^{3N_c-N_f}}{M_{N_f N_f}}. \tag{8.3}$$

$$\text{masses; } \Lambda_{N_c,N_f-1}^{3N_c-(N_f-1)} = m_{N_f N_f}\Lambda_{N_c,N_f}^{3N_c-N_f}. \tag{8.4}$$

In general, Higgsing makes the theory less asymptotically free, so more weakly coupled. And adding masses makes the theory more asymptotically free (since fewer flavors), so the theory is more strongly coupled. We see these effects in (8.3) and (8.4): when we Higgs at a high scale, $\langle M_{N_f N_f}\rangle$ is large in (8.3), so the low-energy scale Λ_{N_c-1,N_f-1} is smaller than Λ_{N_c,N_f}, corresponding to a low-energy theory that is more weakly coupled. On the other hand, for masses, we see from (8.4) that the factor of m in the numerator makes the low energy scale on the LHS larger than the high energy scale, corresponding to stronger coupling.

For $N_f \geq N_c$, we know from the observation at the end of sect. 4 that there can be no dynamically generated superpotential that lifts the moduli space, these theories have an exactly degenerate quantum moduli space of vacua. It's interesting then, to ask about the classical singularities at the origin. One way to determine some information about the theory for general N_f is to add mass terms. So let's consider SQCD with general N_f massive flavors, coming from addition of $W_{tree} = m^{i\tilde{j}} Q_i \widetilde{Q}_{\tilde{j}}$. The low-energy superpotential is completely fixed here by symmetries and holomorphy and recovering known limits to be

$$W_{low} = N_c\left(\det m\Lambda^{3N_c - N_f}\right)^{1/N_c}. \tag{8.5}$$

In particular, this is obtained from (8.2) upon adding $W_{tree} = m^{i\tilde{j}} M_{i\tilde{j}}$, upon integrating out M. The result (8.5) can be interpreted as coming from $SU(N_c)$ gaugino condensation, using the generalization of the matching relation (8.4). With this interpretation, we see that the result (8.5) applies for general N_f, even for $N_f > N_c$.

If, for $N_f > N_c$, we blindly Legendre transform (8.5), ignoring the fact that $\det M = 0$ classically (since the rank of M is at most N_c), we obtain that

$$W_{dyn} = -(N_f - N_c)\left(\frac{\det M}{\Lambda^{3N_c - N_f}}\right)^{1/(N_f - N_c)}, \tag{8.6}$$

which is just the naive continuation of (8.2). We can't take (8.6) seriously in the phase where there are no masses and a quantum moduli space of degenerate vacua, e.g. it doesn't recover the classical moduli space limit for large $\langle M \rangle$. But it does give the right result in the massive phase, where we add $W_{tree} = mM$ masses for all flavors. Seiberg duality gives a way to interpret the result (8.6) as coming from gaugino condensation in a $SU(N_f - N_c)$ dual theory!

9. Seiberg duality

There is an interesting story to be told for SQCD for the cases $N_f = N_c$ (quantum deformed moduli space) and $N_f = N_c + 1$ (extra massless fields at the origin). Because of time limitations, we'll cut to the chase and describe Seiberg duality. The above phenomena can be recovered nicely from the Seiberg dual description.

Seiberg duality relates the original "electric" SQCD theory, $SU(N_c)$ with N_f fundamental flavors Q_f and $\widetilde{Q}_{\tilde{f}}$, to another "magnetic" theory, with group $SU(N_f - N_c)$, having N_f fundamental flavors of dual quarks q^f and $\tilde{q}^{\tilde{f}}$, and N_f^2 singlets $M_{f\tilde{g}}$, with superpotential

$$W = M_{fg} q^f \tilde{q}^{\tilde{g}}. \tag{9.1}$$

For $N_f < 3N_c$ the electric theory is asymptotically free. For $N_f > \frac{3}{2}N_c$ the magnetic dual is asymptotically free. In the range $\frac{3}{2}N_c < N_f < 3N_c$, where both theories are asymptotically free, the interpretation of the duality is that the two theories, which look very different in the UV, flow to the same interacting SCFT in the IR. On the other hand, when $N_f < \frac{3}{2}N_c$, the interpretation is that the electric theory, which appears strongly coupled in the IR, is actually an IR free theory in another set of variables, since the magnetic dual flows to a free theory in the IR. So the electric theory is then in a free magnetic phase, with IR free $SU(N_f - N_c)$ gluons and quarks, which are some composite solitonic objects of the original $SU(N_c)$ theory.

The duality map of chiral primary operators is as follows:

$$Q_f \widetilde{Q}_g \leftrightarrow M_{f\tilde{g}}, \tag{9.2}$$

$$Q_{f_1} \dots Q_{f_{N_c}} \leftrightarrow \epsilon_{f_1 \dots f_{N_c} f_{N_c+1} \dots f_{N_f}} q^{f_{N_c+1}} \dots q^{f_{N_f}}, \tag{9.3}$$

where we omit, for simplicity, factors related to the dynamical scales Λ of the electric theory and $\widetilde{\Lambda}$ of the magnetic theory, that are needed on dimensional grounds.

A very non-trivial check of the duality is that both theories have the same $SU(N_f) \times SU(N_f) \times U(1)_B \times U(1)_R$ flavor symmetry, and that all of the 't Hooft anomalies match. To illustrate this matching, consider

$$\mathrm{Tr} SU(N_f)^2 U(1)_R = N_c \left(-\frac{N_c}{N_f}\right) = -(N_f - N_c)\left(-\frac{N_f - N_c}{N_f}\right) + N_f \left(1 - 2\frac{N_c}{N_f}\right),$$

with the contribution on the LHS coming from the fields Q_f and those on the RHS coming from q^f and $M_{f\tilde{g}}$. One can likewise check that all of the other 't Hooft anomalies, e.g. $\mathrm{Tr}SU(N_f)^3$, $\mathrm{Tr}U(1)_B^2 U(1)_R$, $\mathrm{Tr}U(1)_R$, $\mathrm{Tr}U(1)_R^3$ all match (you'll see that the matching e.g. of $\mathrm{Tr}U(1)_R^3$ is quite non-trivial).

As a check of the duality, consider Higgsing in the electric theory, by giving $\langle Q_{N_f} \widetilde{Q}_{N_f} \rangle$ a nonzero expectation value. This Higges $SU(N_c) \to SU(N_c - 1)$, with $N_f \to N_f - 1$ since one field is eaten, on the electric side. The dual of that low-energy electric theory should be $SU(N_f - N_c)$ again, with $N_f - 1$ light flavors. And that's exactly what we get starting in the magnetic dual of the original theory: taking $\langle M_{N_f \tilde{N}_f} \rangle \neq 0$ has the effect of giving a mass to the dual quark flavor $q^{N_f} \tilde{q}^{\tilde{N}_f}$. So Higgsing of the electric theory corresponds to a mass term in the magnetic dual. Conversely, adding a mass term to the electric theory has the effect of Higgsing in the magnetic dual: the electric mass term maps to an additional superpotential term $W_{tree} = mM_{N_f \tilde{N}_f}$ in the magnetic dual. Then, integrating out the $M_{N_f \tilde{N}_f}$ massive field in the dual theory, the EOM force $\langle q^{N_f} \tilde{q}^{\tilde{N}_f} \rangle \neq 0$, Higgsing the dual theory to $SU(N_f - N_c - 1)$, with $N_f - 1$ light flavors remaining uneaten. That low-energy theory is precisely that expected, as it's the dual of the low-energy electric theory with group $SU(N_c)$ and $N_f - 1$ light flavors.

So the duality is preserved by the addition of mass terms and Higgsing, and actually exchanges the two deformations. Recall that masses take the theory to stronger coupling, while Higgsing takes the theory to weaker coupling, so we see that making one side more strongly coupled makes the dual weaker, and visa versa. This is how electric-magnetic duality should behave. In some contexts (e.g. the $SO(N_c)$ generalization, with N_f flavors) we can really connect the dual quarks to magnetic monopoles, justifying more precisely the name "magnetic" for the dual theory.

The superpotential (8.6) is recovered in the dual description upon taking $\langle M \rangle$ to have generic expectation values. Then $\langle M \rangle$ acts as masses for all dual quarks and, accounting for the matching relation, we obtain (8.6) from $SU(N_f - N_c)$ gaugino condensation in the dual theory. This, again, is appropriate when masses have been added to all flavors on the electric side.

Suppose, on the other hand, that the electric flavors are all massless. It's interesting to see, then, how the magnetic dual reproduces the constraint that $rank(M) \leq N_c$, as is clear in the electric theory from $M = Q\widetilde{Q}$. This classical electric constraint comes from a non-perturbative effect in the dual theory: if $rank(M) > N_c$, the dual $SU(N_f - N_c)$ theory has fewer than $N_f - N_c$ light flavors. Then a non-perturbatively generated superpotential, which is analogous to (8.2) in the dual, implies that there is no supersymmetric vacuum in this case.

Suppose, by adding appropriate masses, we flow down to electric $SU(N_c)$, with $N_f = N_c + 1$ massless flavors. In the magnetic dual, as mentioned above, this flow corresponds to Higgsing, and for the case where there are only $N_f = N_c + 1$ flavors left massless, the dual $SU(N_f - N_c)$ theory is completely Higgsed. The superpotential thus obtained from the magnetic theory is

$$W = \frac{1}{\Lambda^{2N_c - 1}_{N_f = N_c + 1}} \left(M_{i\tilde{j}} B^i \widetilde{B}^{\tilde{j}} - \det M \right), \tag{9.4}$$

where the first term is the remaining superpotential coming from (9.1), with the dual quarks q^i and $\widetilde{q}^i$ replaced with electric baryons via (9.3). The last term in (9.4) comes from an instanton in the Higgsed $SU(N_f - N_c)$ theory, with the factor of $\det M$ coming from scale matching relations. The superpotential (9.4), was obtained by Seiberg in a separate work, that predates Seiberg duality. It's EOM gives the classical moduli space constraints, so it doesn't lift the classical moduli space and is hence OK, even though there are powers of Λ in the denominator. The magic behind this is that (9.4) includes additional fields, corresponding to not imposing the classical constraints. These extra fields are massive on the classical moduli space, but become massless at the origin of the moduli space, explaining the singularity of the classical moduli space at the origin. The existence of these additional massless fields at the origin satisfies a non-trivial check, t' Hooft anomaly matching, as shown first by Seiberg.

Adding a mass term for one of the flavors, (9.4) leads to an interesting result for the theory with $N_f = N_c$ massless flavors:

$$W = 0 \qquad \text{with} \quad \det M - B\tilde{B} = \Lambda^{2N_c}_{N_f=N_c}. \tag{9.5}$$

The classical moduli space constraint is quantum deformed by the term on the RHS. The classical singularity at the origin is neatly eliminated, since it's not on the quantum deformed moduli space.

Adding $W = mM_{N_f N_f}$ to (9.5), with the constraint imposed, properly recovers the superpotential (8.2) for the theory with $N_f < N_c$ flavors. So the various dynamics and effective superpotentials of the electric theory are properly recovered in the dual description.