Singularities & Scale Hierarchy Created in Flowing Plasmas

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1) Scale hierarchy in plasmas

2) Creation of small-scale structurs
(singularities)

3) Singular perturbation and universality class

4) Thermodynamics of plasma turbulence
— maximum entropy production

1) Scale hierarchy in plasmas.

- A plasma may have a very wide dynamic range of time and length scales.
- Plasmas can overcome scale hierarchy by creating structures.

How?

- wave-particle interactions through various resonances.
- chaos-induced entropy production.
- collisional dissipation at Kolmogorov scales.

Where?

- reconnection with "anomalous resistance"
- accretion disc and jet with "anomalous viscosity"
- acceleration at shock fronts
- heating through wave-particle interactions, etc.

Numata & Yoshida, PRL 88 (2002), 045003

Chaotic Orbit

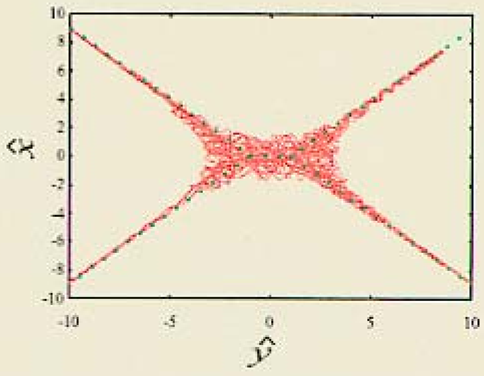
single particle orbit in 2D inhomogeneous magnetic field

$$\boldsymbol{B} = \begin{cases} (B_0(y \mp \ell_y)/\ell_x, B_0 x/\ell_x, 0) & (|y| > \ell_y) \\ (0, B_0 x/\ell_x, 0) & (|y| \le \ell_y) \end{cases}, \tag{65}$$

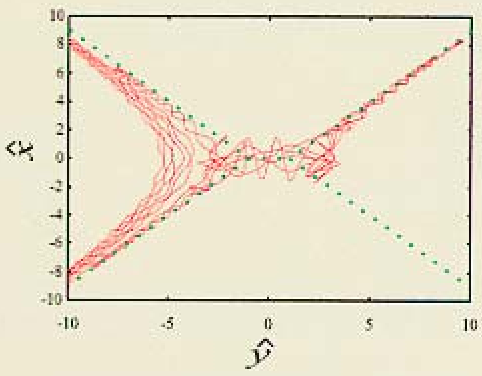
where B_0, ℓ_x and ℓ_y are constant numbers.

$$\frac{\delta_i}{\ell_x}\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} = m_A \boldsymbol{e} + \boldsymbol{v} \times \boldsymbol{b} \tag{66}$$

where δ_i is the ion collisionless skin depth, m_A is the Alfvén Mach number.

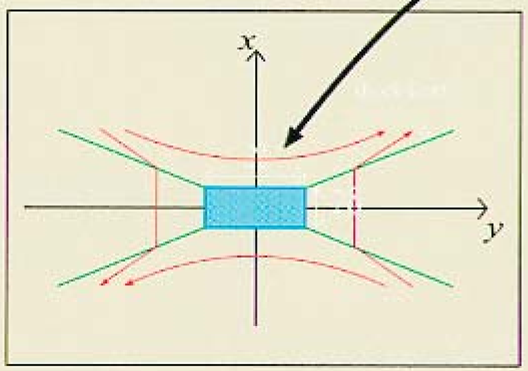


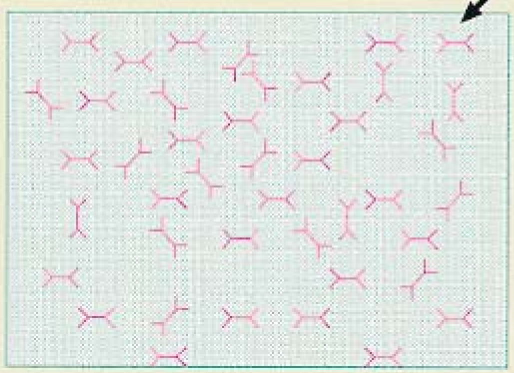
(a) m_A=0.001

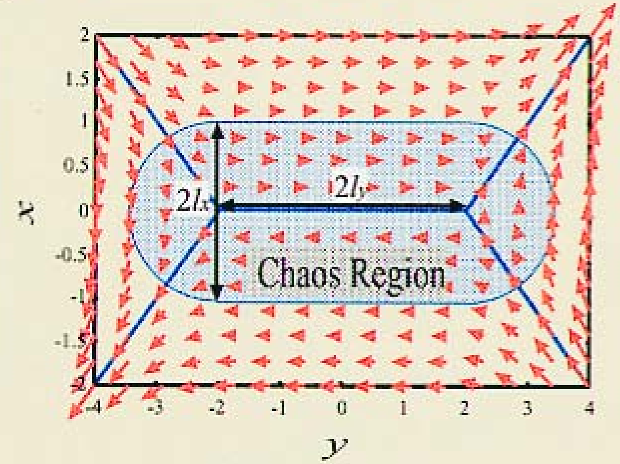


(b) m_A=0.01

Hierarchical Structure







Macroscopic — Global Scale

Mesoscopic — Diffusion Region

Microscopic — ~ Ion skin depth

Double Beltrami Field ?

Fractal current sheet (Shibata and Tanuma 2001)
Patchy reconnection (Kaw 1988)

2) Creation of singularities

— small-scale structurs

scale parameter : ε

limit $\varepsilon \to 0$ $\Rightarrow$ ideal model
(scale-less)

small scale $\Rightarrow$ "singurality"
$\sim O(\varepsilon)$

$\boxed{\text{Can an ideal plasma create singularilies within a finite time?}}$

Known facts:

- Euler eqs.

$$\partial_t \mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{v} = -\nabla p, \quad \nabla\cdot\mathbf{v} = 0$$

$$\begin{cases} \nabla\times \Rightarrow \partial_t \Omega - \nabla\times(\mathbf{v}\times\Omega) = 0 \\ 2\mathrm{D} \Rightarrow \partial_t \Omega + \{\phi, \Omega\} = 0 \\ \qquad\qquad (\Omega = -\Delta\phi, \quad \mathbf{v} = \nabla_\perp \phi) \end{cases}$$

- Navier-Stokes eqs.

$$\partial_t \mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{v} = -\nabla p + \underline{\varepsilon\Delta \mathbf{v}}, \quad \nabla\cdot\mathbf{v} = 0$$

◎ 2D Euler and NS have regular solutions, i.e., if initial flow is smooth, then the solution remains smooth forever.

T. Kato, Arch. Rational Mech. Anal. **25**, 188 (1967).

◎ 3D NS has a regular solution for $t < M(\mathbf{v}_0)$.

3D Euler — unknown.

Consider 2D plasmas :

(3D must be more singular)

. Ideal MHD eqs.

$$\begin{cases} \partial_t \mathbb{B} - \nabla\times(\mathbb{V}\times\mathbb{B}) = 0 \\ \partial_t \mathbb{V} + (\mathbb{V}\cdot\nabla)\mathbb{V} = -\nabla p + \underline{(\nabla\times\mathbb{B})\times\mathbb{B}}\,) , \quad \nabla\cdot\mathbb{V} = 0 \end{cases}$$

$$2D \Rightarrow \begin{cases} \partial_t \psi + \{\varphi, \psi\} = 0 \\ \partial_t(-\Delta\varphi) + \{\varphi, -\Delta\varphi\} + \underline{\{\psi, -\Delta\psi\}} = 0 \end{cases} \quad (\mathrm{MHD})$$

⇐ Singularities can be created within a finite time

. Generalized Ohm's law ⇒

$$\begin{cases} \partial_t \Omega_1 + \{\varphi, \Omega_1\} + \rho_s^2\{\psi, \Omega_2\} = 0 \\ \partial_t \Omega_2 + \{\varphi, \Omega_2\} + \{\psi, \delta_e^{-2}\Omega_1\} = 0 \end{cases}$$

$$\left(\begin{aligned} \Omega_1 &= (1-\delta_e^2\Delta)\psi \\ \Omega_2 &= -\Delta\varphi \end{aligned} \right.$$

defining
$$\begin{cases} \phi_\pm = \varphi \pm (\rho_s/\delta_e)\psi \\ \Omega_\pm = \Omega_1 \pm \rho_s\delta_e\Omega_2 \end{cases}$$

$$\Rightarrow \qquad \partial_t \Omega_\pm + \{\phi_\pm, \Omega_\pm\} = 0 \qquad (\text{G-MHD})$$

⇐ regularity continues.

[$\rho_s \to 0$, $\delta_e \to 0$ ⇒ (MHD)]

3) Singular perturbation and universality class

Interacting hierarchy ⇒ complexity

○ nonlinearity → energy transfer to different scales

"universality class" may have a "deep structure"

○ singular perturbation → "anchor" = intrinsic scale

ε

I. Singular Perturbation

Example

- Ideal model

$$\partial_t v + (v\cdot\nabla)v = -\nabla p, \quad \nabla\cdot v = 0$$

  - scale invariant

$$\hat{x} = \frac{x}{L}, \quad \hat{u} = \frac{u}{U}, \quad \hat{t} = \frac{t}{L/U}, \quad \hat{p} = \frac{p}{U^2}$$

  - energy conservation

$$E(t) = \|v(t)\|^2 = \int_\Omega |v|^2\, dx = \text{const.}$$

- add dissipation (singular perturbation)

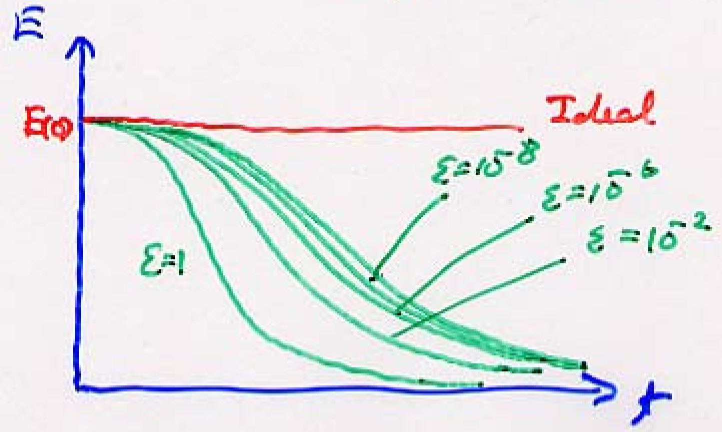
$$\partial_t v + (v\cdot\nabla)v = -\nabla p + \underline{\varepsilon\Delta v}, \quad \nabla\cdot v = 0$$

  - ε : scale parameter

$$L \to L' \Rightarrow \varepsilon \to (L/L')^2\varepsilon$$

  - energy decay

$$E(t) = E(0) - \varepsilon\int_0^t \|\nabla\times v\|^2\, dt$$



Nonlinearity selects the scale

formal understanding of singular perturbation

- **0-model**

  ideal, scale invariant

  solution : f_0

- **ε-model** = 0-model + $\varepsilon \cdot \boxed{\text{high deriv.}}$

  non-ideal, scale dependent

  solution : f_ε

(case 1) $\lim_{\varepsilon \to 0} f_\varepsilon \in \{f_0\}$

(case 2) $\lim_{\varepsilon \to 0} f_\varepsilon$: singular

If $f_\varepsilon = f_\varepsilon^{(-)} + f_\varepsilon^{(+)}$

$\lim_{\varepsilon \to 0} f_\varepsilon^{(-)} \in \{f_0\}$

$\lim_{\varepsilon \to 0} f_\varepsilon^{(+)}$: singular

$f_\varepsilon^{(-)}$ is the universality class.

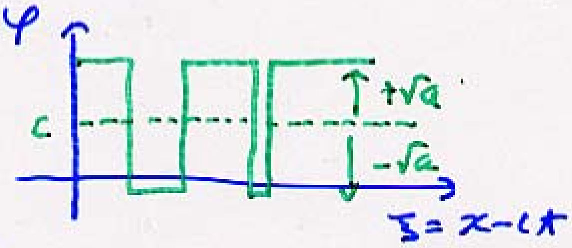
Example of case 1 [entropy solution of shock]

- 0-model $\partial_t u + u\,\partial_x u = 0$

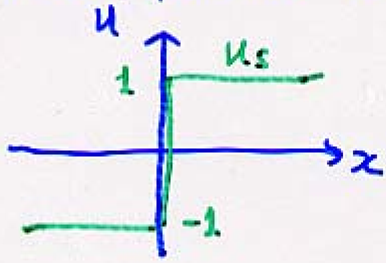
  propagating wave $u(x,t) = \varphi(x-ct)$

  $$\to -c\varphi' + \varphi'\varphi = 0 \to \frac{1}{2}\left[(\varphi-c)^2\right]' = 0$$

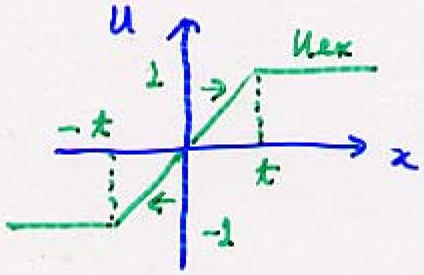
  $$\varphi = c \pm \sqrt{a} \quad (a:\text{const})$$



Riemann problem ($c=0$)

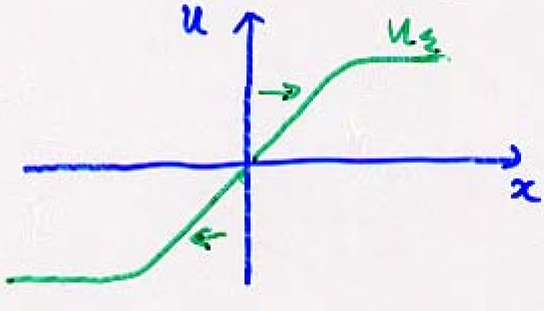


standing shock



expansion wave

- ε-model $\partial_t u + u\,\partial_x u = \varepsilon\,\partial_x^2 u$



$u_\varepsilon \to u_{ex}$

Example of Case 2 [double Beltrami]

- 0-model = ideal MHD
- ε-model = H-MHD (two-fluid MHD)

$$\varepsilon = \frac{(c/\omega_{pi})}{L}$$

(i) Formulation

- electron eq. of motion

$$\mathbb{E} + \mathbb{V}_e \times \mathbb{B} + \frac{1}{en} \nabla p_e = 0$$

- ion eq. of motion

$$\partial_t \mathbb{V} + (\mathbb{V} \cdot \nabla) \mathbb{V} = \frac{e}{M} (\mathbb{E} + \mathbb{V} \times \mathbb{B}) - \frac{1}{Mn} \nabla p_i$$

$$\mathbb{V}_e = \mathbb{V} - \mathbb{j}/en = \mathbb{V} - \nabla \times \mathbb{B}/\mu_0 e n$$

$$\mathbb{E} = -\partial_t \mathbb{A} - \nabla \phi$$

normalization

$$\hat{x} = x/L_0 \;,\; \hat{\mathbb{B}} = \mathbb{B}/B_0 \;,\; \hat{n} = n/n_0$$

$$\hat{t} = t/(L_0/V_A) \;,\; \hat{\mathbb{V}} = \mathbb{V}/V_A \;,\; \hat{p} = p/(B_0^2/\mu_0)$$

normalized eq. (assume $\hat{n} = 1$)

$$(\varepsilon\text{-model}) \begin{cases} \partial_{\hat{t}} \hat{\mathbb{A}} = (\hat{\mathbb{V}} - \varepsilon \hat{\nabla} \times \hat{\mathbb{B}}) \times \hat{\mathbb{B}} - \hat{\nabla}(\hat{\phi} - \varepsilon p_e) \\ \partial_{\hat{t}} (\varepsilon \hat{\mathbb{V}} + \hat{\mathbb{A}}) = \hat{\mathbb{V}} \times (\hat{\mathbb{B}} + \varepsilon \hat{\nabla} \times \hat{\mathbb{V}}) - \nabla\left(\frac{\varepsilon \hat{V}^2}{2} + \varepsilon p_i + \hat{\phi}\right) \end{cases}$$

• $\varepsilon \to 0$

$$
(0\text{-model})\begin{cases} \partial_t \mathbb{B} + (\mathbb{V}\cdot\nabla)\mathbb{B} - (\mathbb{B}\cdot\nabla)\mathbb{V} = 0 \\ \partial_t \mathbb{V} + (\mathbb{V}\cdot\nabla)\mathbb{V} - (\mathbb{B}\cdot\nabla)\mathbb{B} = -\nabla\left(p + \frac{B^2}{2}\right) \end{cases}
$$

(ii) vortex transport

curl of (ε-model)

$$
\partial_t \Omega_j = \nabla\times(\mathbb{U}_j \times \Omega_j) \quad (j=1,2)
$$

$$
(e)\begin{cases} \Omega_1 = \mathbb{B} \\ \mathbb{U}_1 = \mathbb{V} - \varepsilon\nabla\times\mathbb{B} \end{cases} \qquad (i)\begin{cases} \Omega_2 = \mathbb{B} + \varepsilon\nabla\times\mathbb{V} \\ \mathbb{U}_2 = \mathbb{V} \end{cases}
$$

(iii) Beltrami conditions

$$
\mathbb{U}_j = \underbrace{\mu_j}_{\text{constant}}\Omega_j \quad (j=1,2)
$$

Using (e) and (i), and $a = 1/\mu_1$, $b = 1/\mu_2$.

$$
\begin{cases} \mathbb{B} = a(\mathbb{V} \mp \varepsilon\nabla\times\mathbb{B}) \\ \mathbb{B} + \varepsilon\nabla\times\mathbb{V} = b\mathbb{V} \end{cases}
$$

Combining,

$$
\varepsilon^2 \nabla\times\nabla\times \mathcal{U} + \varepsilon(a^{-1} - b)\nabla\times\mathcal{U} + (1 - b/a)\mathcal{U} = 0
$$

for $\mathcal{U} = \mathbb{B}$ and $\mathbb{V}$.

Defining

$$\lambda_\pm = \frac{1}{2\varepsilon}\left[(b-a^{-1}) \pm \sqrt{(b-a^{-1})^2 - 4(1-b/a)}\right],$$

We may write

$$(\mathrm{curl} - \lambda_+)(\mathrm{curl} - \lambda_-)\,\mathbb{U} = 0 .$$

General solution!

$$\mathbb{U} = C_+ \mathbb{G}_+ + C_- \mathbb{G}_-$$

$$\left(\nabla\times \mathbb{G}_\pm = \lambda_\pm \quad : \text{Beltrami eigenfunction} \right)$$

The double Beltrain (DB) field:

$$\begin{cases} \mathbb{B} = C_+ \mathbb{G}_+ + C_- \mathbb{G}_- \\ \mathbb{V} = (a^{-1} + \varepsilon\lambda_+) C_+ \mathbb{G}_+ + (a^{-1} + \varepsilon\lambda_-) C_- \mathbb{G}_- \end{cases}$$

- Beltrami eigenfunctions span the Hilbert space of solenoidal (incompressible) vector fields.
- DB field is a "condensate" into 2.
- One is the universality class.
  the other is singular (intrinsic)

limit $\varepsilon \to 0$ and universality class.

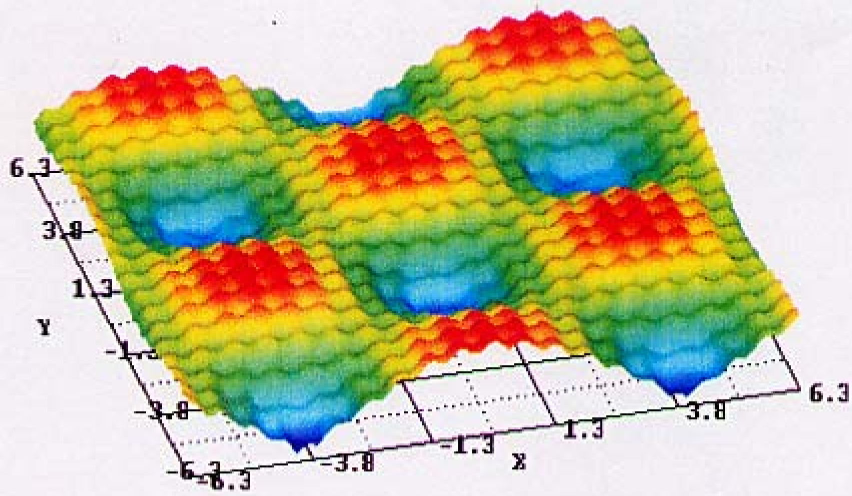
writing $b/a = 1 + \delta \quad (\delta = O(\varepsilon))$,

for small ε,

$$\lambda_- \simeq \frac{\delta}{\varepsilon}\left(\frac{1}{a} - a\right)^{-1} \quad \to \text{universal}$$

$$\lambda_+ \simeq -\frac{1}{\varepsilon}\left(\frac{1}{a} - a\right) \quad \to \pm\infty \ (\varepsilon \to 0)$$

→ Z.Y - Mahajan - Ohsaki
Scale hierarchy created in plasma flow
Phys. Plasmas 11 (2004), 3660


6.3
3.8
1.3
Y
-3.8
-6.3
-6.3
-3.8
-1.3
X
1.3
3.8
6.3

Mechanism of the creation of the small-scale component of DB fields

revisit 2D

- ideal MHD (incompressible)

$$\begin{cases} \mathbf{B} = \nabla\psi \times \mathbf{e}_z + B_z \mathbf{e}_z \\ \mathbf{V} = \nabla\varphi \times \mathbf{e}_z + V_z \mathbf{e}_z \end{cases}$$

$$\Rightarrow \quad \varphi = \varphi(\psi), \quad V_z = V_z(\psi), \quad B_z = B_z(\psi)$$

$$\nabla\cdot\left[(1-\phi'^2)\nabla\psi\right] + \left(\frac{\phi'^2}{2}\right)' |\nabla\psi|^2 + G'(\psi) = 0$$

$$G(\psi) = \frac{B_z^2(\psi)}{2} - \frac{V_z^2(\psi)}{2} + P(\psi)$$

- flow-characteristics must align to mag.-characteristics

- H-MHD

$$\begin{cases} -\Delta\psi = \partial_\psi G(\psi, \phi) \\ -\Delta\phi = -\partial_\phi G(\psi, \phi) \end{cases}$$

$$G(\psi,\phi) = \frac{B_z^2(\psi,\phi)}{2} - \frac{V_z^2(\psi,\phi)}{2} + P(\psi,\phi)$$

$$= \frac{1}{\varepsilon^2}\left[\frac{(a^{-2}-1)\psi^2}{2} + (b-a^{-1})\psi\phi - \frac{(b^2-1)\phi^2}{2}\right]$$

- flow-characteristics can be independant to mag.-ch. $\longrightarrow$ DB

$\Leftarrow$ "Dispersive effect" heals the singularity

4) Thermodynamics of plasma turbulence

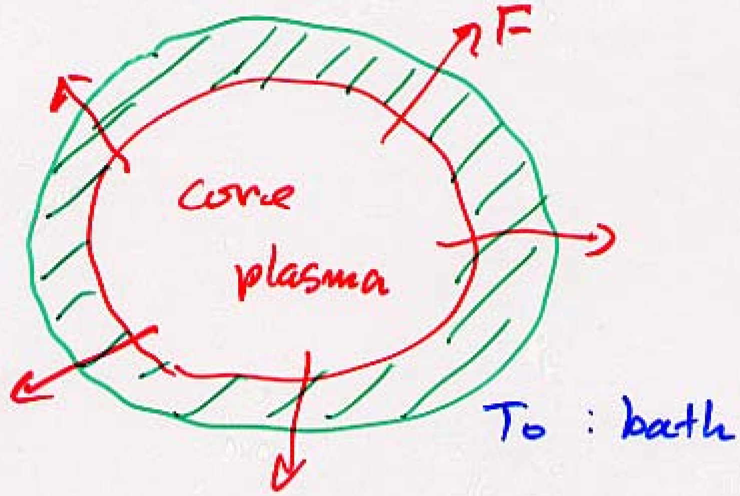
— maxiomum entropy production

Scale hierarchy

$\Rightarrow$ effective entropy production

flow $\nearrow$ ordered structure .... $\Delta T \uparrow$

flow $\searrow$ enhanced dissipation... $\delta S_i \uparrow$



Consider a cycle:

$$dU = \delta Q - p\,dV$$

(1st law)

$$W := \oint p\,dV = -\cancel{\oint dU} + \oint \delta Q$$

$$= \oint \delta Q - T_0 \underbrace{\oint dS}_{\leq 0}$$

$$\delta S_e = \frac{\delta Q}{T}, \quad dS = \delta S_e + \underbrace{\delta S_i}_{\geq 0}$$

(2nd law)

$$W = \underbrace{\oint \left(1 - \frac{T_0}{T}\right)\delta Q}_{\text{Carnot max. work}} - T_0 \oint \delta S_i$$

Fluid: sum of cycles

$$\dot{X} := \partial_t X + \nabla \cdot \mathbf{F}(X), \quad \delta X = \int_{\ell} \dot{X}\,dt$$

streamline

$$\dot{W} - T_0 \dot{S} = \left(1 - \frac{T_0}{T}\right)\dot{Q} - T_0 \dot{S}_i$$

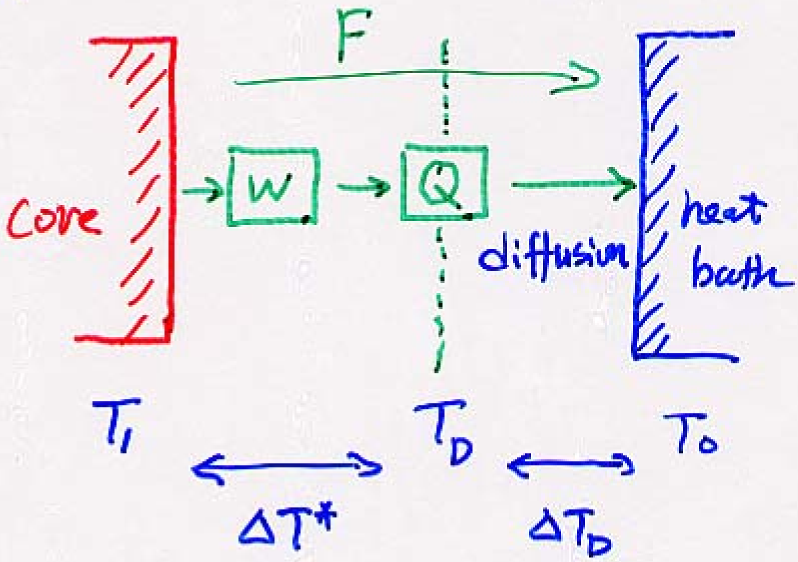
sum over fluid elements $\Rightarrow$

$$\int \dot{W}\,d\rho - T_0 \int \dot{S}\,d\rho = \int \left(1 - \frac{T_0}{T}\right)\dot{Q}\,d\rho - T_0 \int \dot{S}_i\,d\rho$$

Steady state $\Rightarrow \int \dot{W}\,d\rho = 0, \ \int \dot{S}\,d\rho = 0, \ \int \dot{Q}\,d\rho = 0$

$$\Rightarrow T_0 \int \frac{1}{T}\dot{Q}\,d\rho + \underbrace{T_0 \int \dot{S}_i\,d\rho}_{\text{entropy production}} = 0 \quad : \text{entropy balance}$$

Layer model



$$T_D = T_0 + \Delta T_D = T_0 + \eta_0 F$$

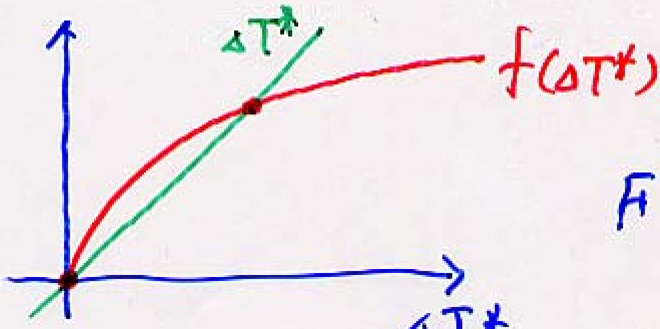
$$\dot{S}_D = \left(\frac{1}{T_0} - \frac{1}{T_D}\right) F$$

$$T_1 = T_0 + \underline{\eta(P)}\, F$$

Assume $\eta(P) = \eta_0 + aP$

$$P = \left(1 - \frac{T_D}{T_1}\right) F \leq \left(1 - \frac{T_0}{T_1}\right) F = P_{max}$$

$$\Delta T^* = \frac{a F^2 \Delta T^*}{T_0 + \eta_0 F + \Delta T^*} =: f(\Delta T^*)$$



$$F > F_{min} = \frac{1}{2}\left(\frac{\eta_0}{a} + \sqrt{\frac{\eta_0^2}{a^2} + 4\frac{T_0}{a}}\right)$$

Requirments

- Creation of large-scale ordered structure at almost ideal efficiency (instability) $\boxed{\dot{W}}$

- Efficient dissipation in a small-scale hierarchy $\boxed{\dot{Q}}$

$\Rightarrow$ F ($> F_{min}$) can sustain the ordered structure / enhanced $\dot{S}_i$.