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SMR.1573 - 1

SUMMER SCHOOL AND CONFERENCE ON DYNAMICAL SYSTEMS

Flat surfaces, Interval exchange Transformations & Moduli Spaces of Abelian Differentials

(Lecture 1)

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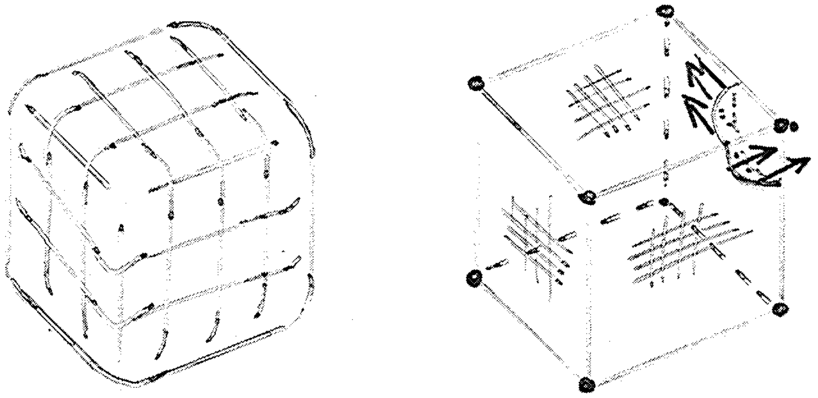
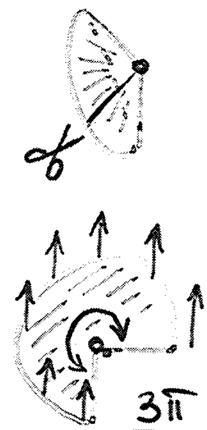
FLAT SURFACES.

Flat surface: all curvature is collapsed to several singular points (cone-singularities).





Example: absolutely flat sphere (with 8 cone-singularities with angles $\frac{3\pi}{2}$).
and with nontrivial $SO(2)$-holonomy

PROBLEMS: Behaviour of
- typical geodesics;
- closed geodesics.

RESULTS: ???

Even in the simplest case of a flat sphere with three generic cone-singularities it is not known whether
- the geodesic flow is ergodic (for a.a. cases)*;
- there exists at least one closed regular geodesic?

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*

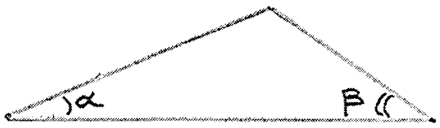

10.000

Billiards in polygons

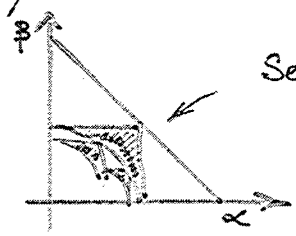
Billiard in a polygon $\sim$ Geodesic flow on the corresponding flat sphere

Exercise. Find a periodic billiard trajectory in an acute triangle

Existence of periodic trajectory is proved for 1/3 of obtuse triangles; for the remaining 2/3 it is an open problem.

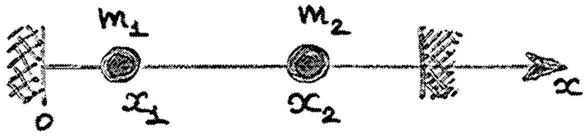


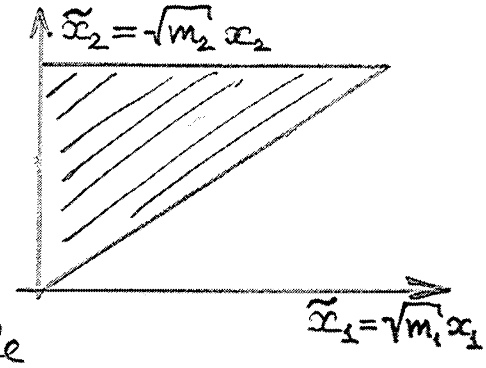
$\alpha + \beta \leq \frac{\pi}{2}$



Set of the measure 1/3 in the space of parameters (G. Galperin)

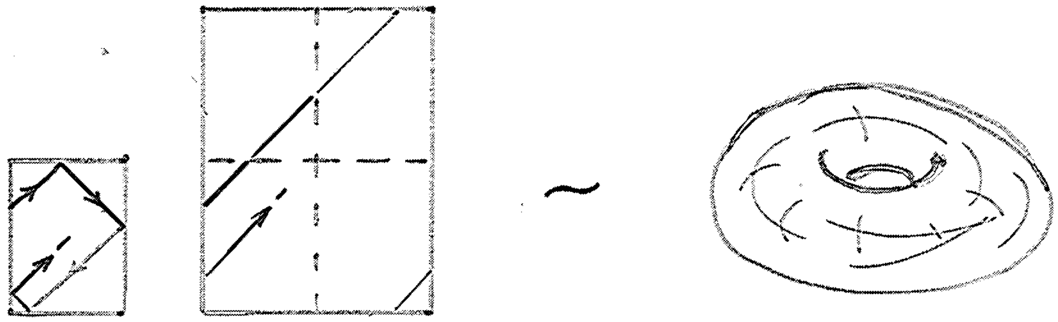
Pair of ellastic balls on a segment $\sim$ Billiard in a triangle



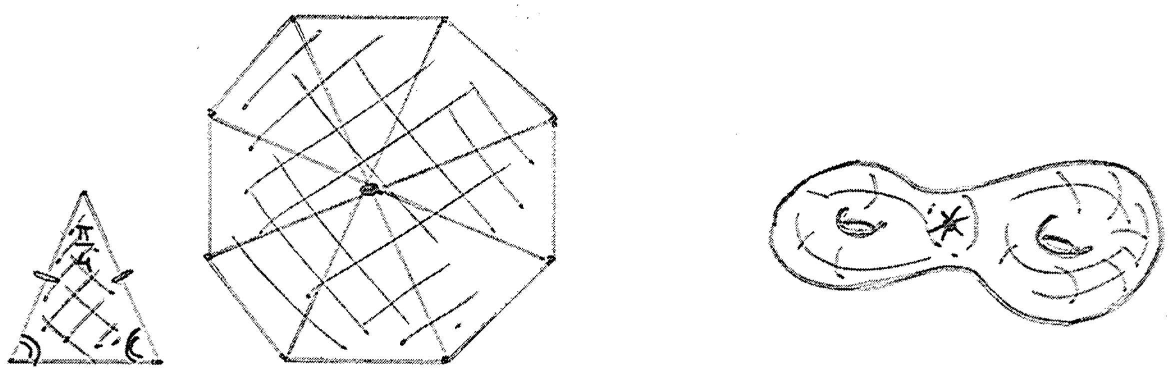


Configuration space = triangle
After rescaling dynamics = dynamics of the billiard

BILLIARDS IN RATIONAL POLYGONS



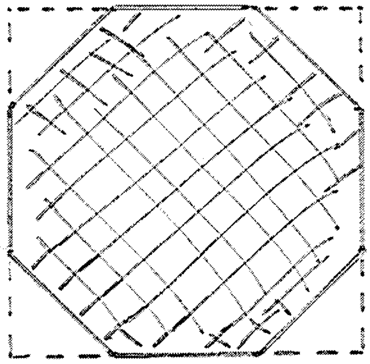
Billiard in a rectangle → unfolding → Directional flow on a torus (glued from 4 copies of the rectangle)

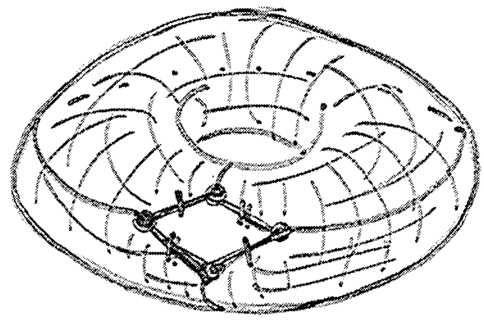


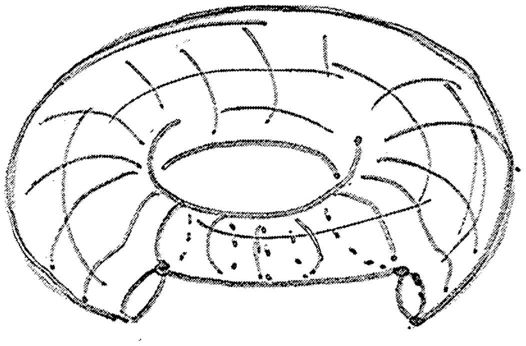
Billiard in a triangle $\left(\frac{\pi}{4}; \frac{3\pi}{8}; \frac{3\pi}{8}\right)$ → unfolding → Directional flow on a "flat" surface of genus two

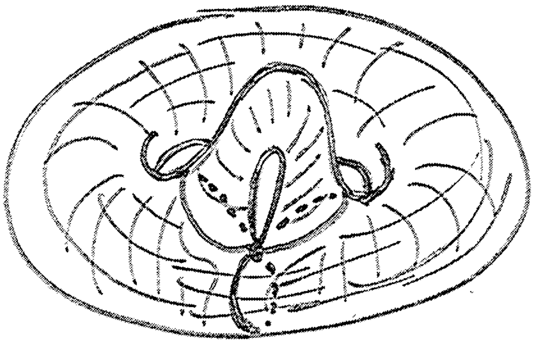
Billiards in rational polygons → Geodesics on VERY FLAT (translation) surfaces

VERY FLAT SURFACE OF GENUS TWO









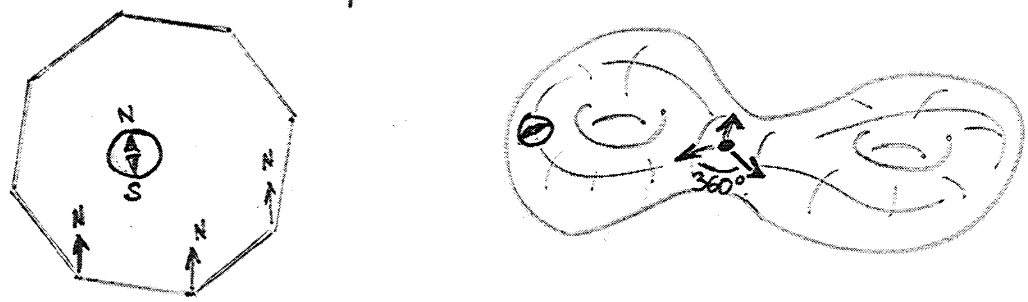
ANTON ZORICH

Object of study: VERY flat surfaces

def: VERY flat (= translation) surface :=
flat + trivial holonomy

Example 1. Flat torus

Example 2. Flat pretzel



Properties of a translation structure:

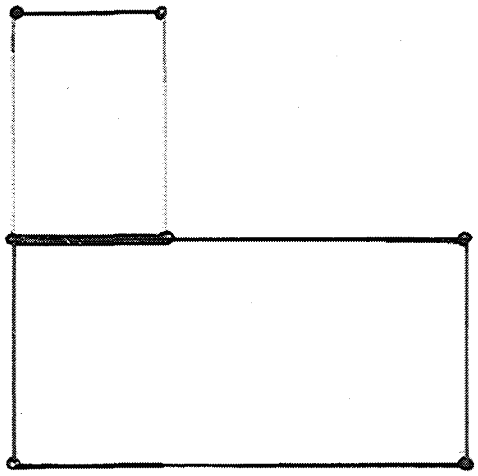
- Haven chosen direction to the North at one point we can parallely transport it to any other point
- Any geodesic goes in a fixed direction; in particular, it never intersects itself

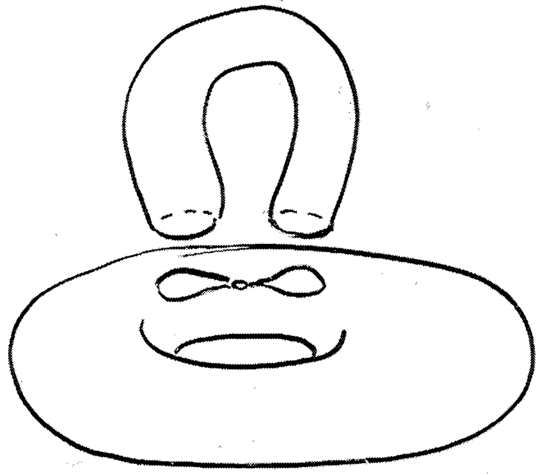
Problems:

- Behavior of generic geodesics
- Behavior of closed geodesics
- Ergodicity and unique ergodicity of the directional flow
- What is going on when genus g tends to ∞?

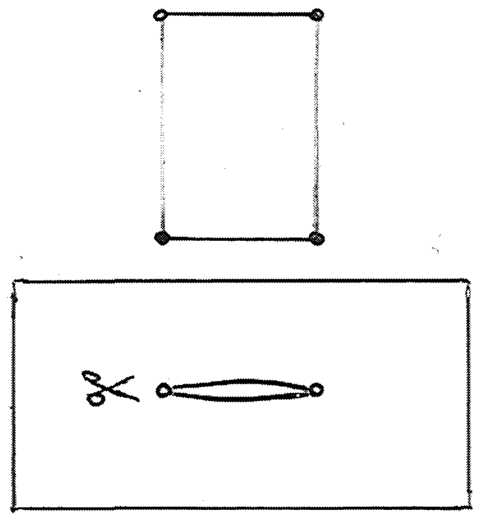
Surfaces de translation

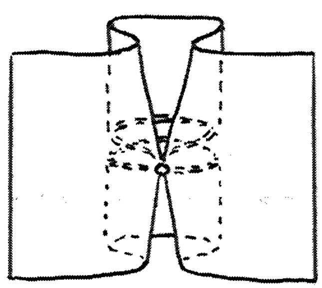
déf : Surface de translation = plate + holonomie triviale





Nous avons collé une anse.





PLANE SECTIONS OF $\mathbb{Z}^3$ - PERIODIC SURFACES IN $\mathbb{R}^3$

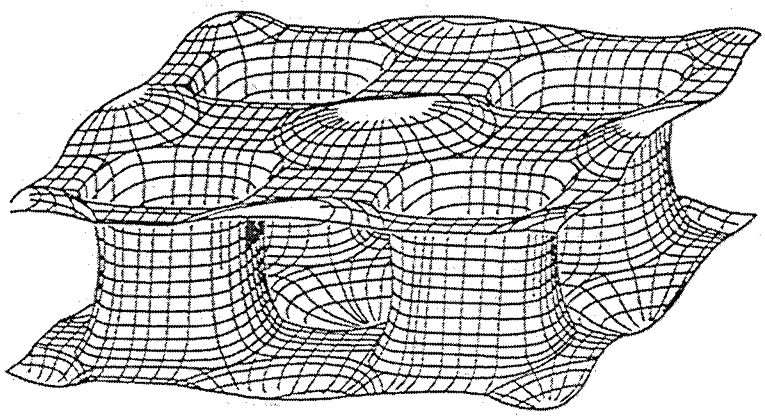


Fig. 66. Fermi surface of tin based on the results reported in [41].

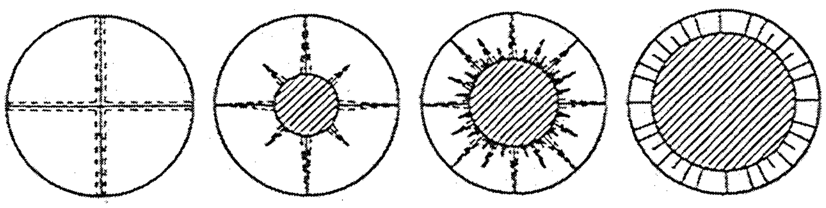
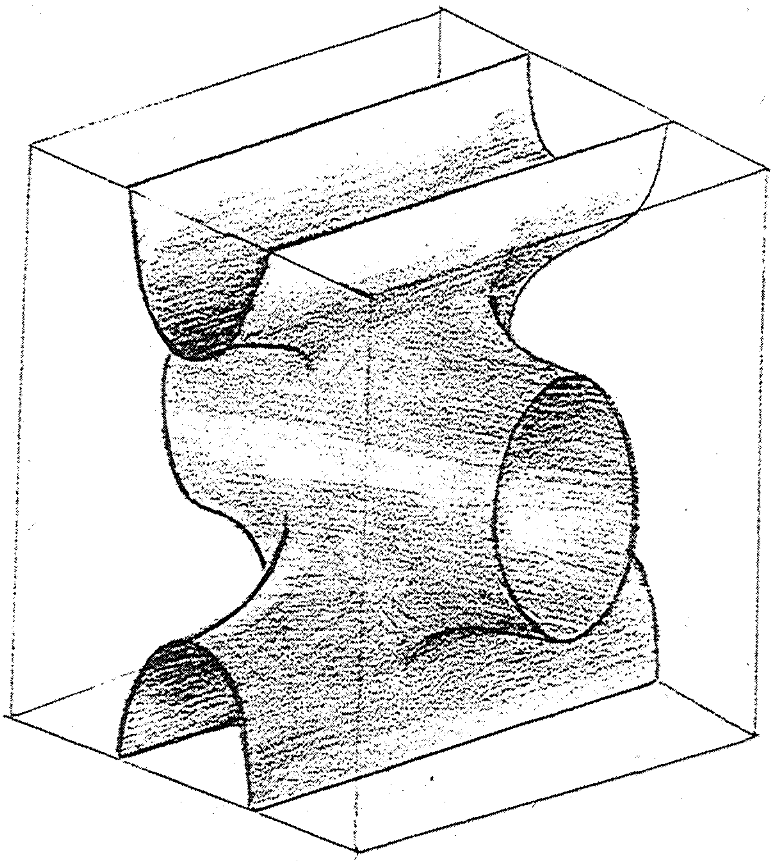


Fig. 67. Stereographic projections of the special directions of the magnetic field for constant-energy surfaces of the two-dimensional network type.

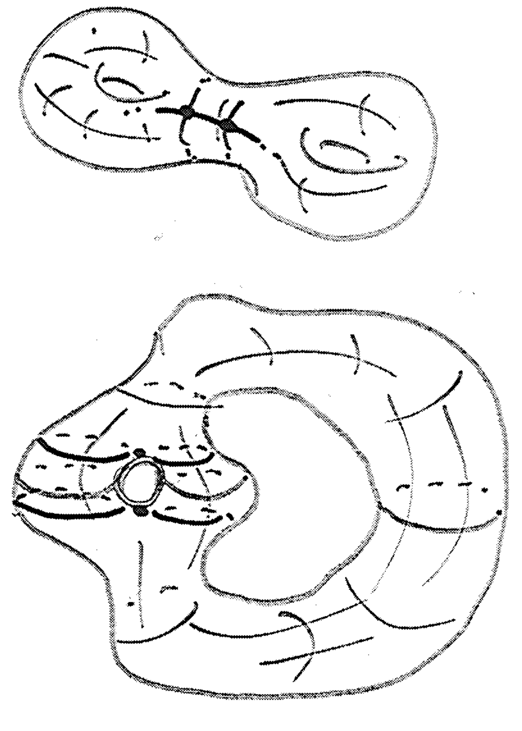


A piece of a $\mathbb{Z}^3$-periodic surface

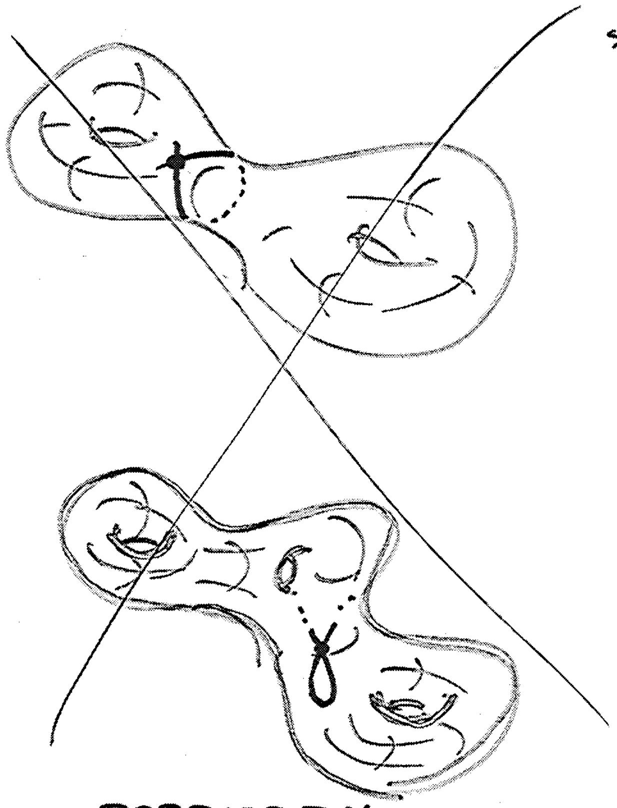
Sections of a $\mathbb{Z}^3$-periodic surface $\hat{M} \subset \mathbb{R}^3$ by hyperplanes $\alpha x + \beta y + \gamma z = \text{const}$ project to leaves of a closed 1-form $\omega = (\alpha \cdot dx + \beta \cdot dy + \gamma \cdot dz)\big|_M$ on a compact surface $M \subset T^3 = \mathbb{R}^3/\mathbb{Z}^3$

CALABI – HUBBARD-MASUR – KATOK – ... THEOREM

THEOREM *Consider a foliation on a closed surface of genus g defined by a close 1-form ω. One can find a flat metric with isolated singularities (and trivial holonomy) in which the foliation would become a foliation of parallel straight lines if and only if ω does not have closed oriented singular leaves homologous to zero*

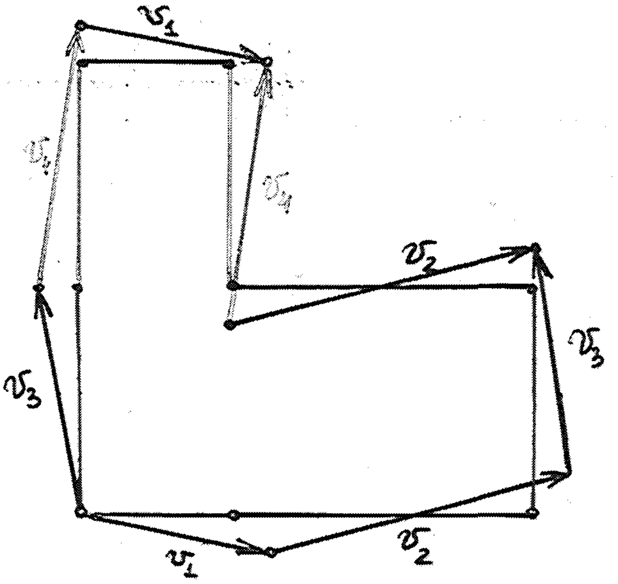


PERFECT

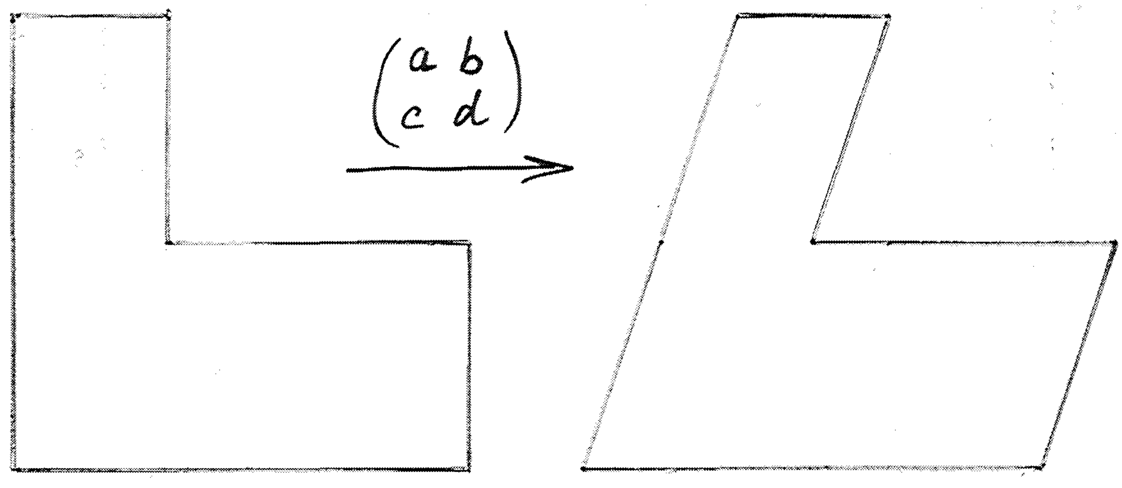


FORBIDDEN

ESPACES DE SURFACES DE TRANSLATION.

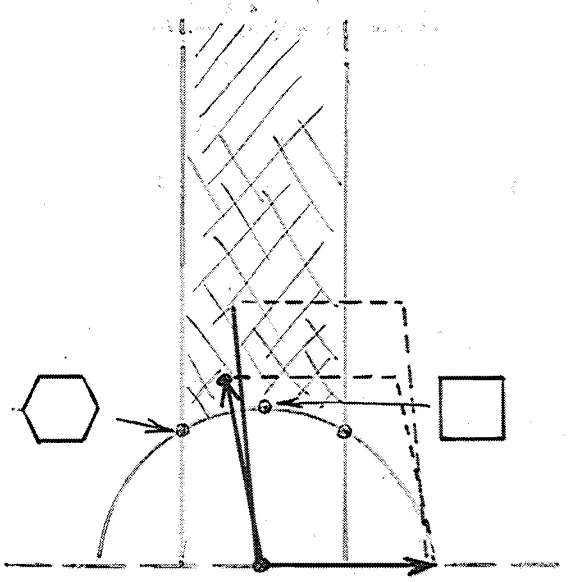


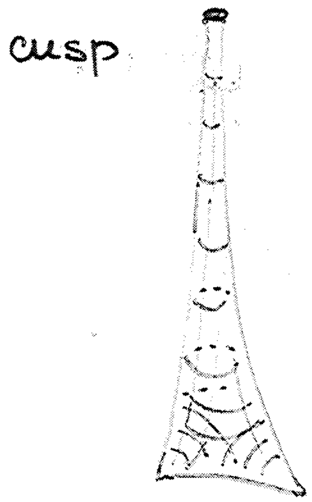
On peut déformer une surface de translation ; la déformation ne change pas les angles coniques



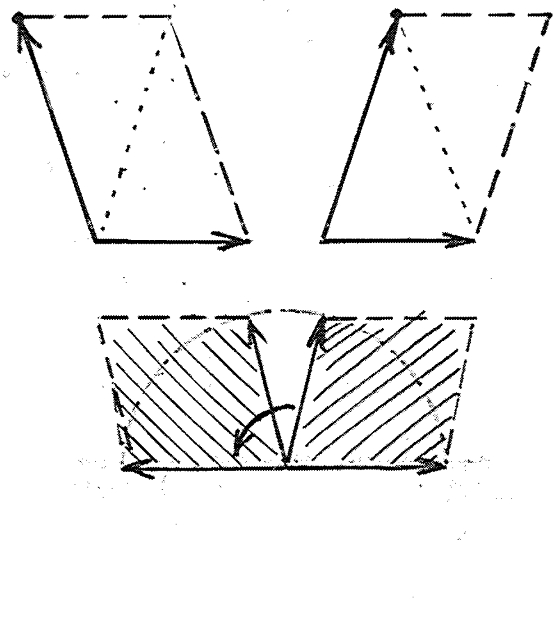
L'action du groupe $SL(2;\mathbb{R})$ sur l'espace de surfaces de translation $\mathcal{H}(d_1, \dots, d_n)$. Ici $2\pi(d_i+1)$ sont les angles coniques.

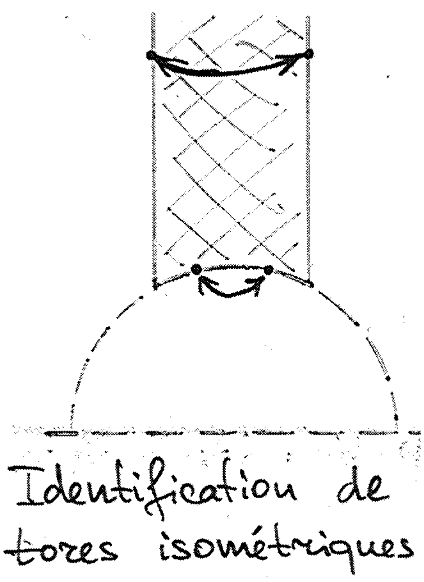
EXEMPLE LUDIQUE : ESPACE DES TORES PLATS





On choisit le patron du tore défini par ses deux géodésiques fermées les plus courtes. La plus courte des deux est normalisée à 1 et mise horizontalement. Le vecteur correspondant à la deuxième géodésique paramétrise la structure plate.

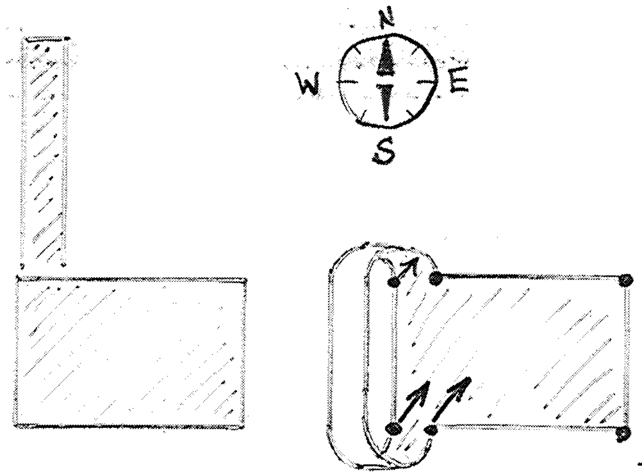


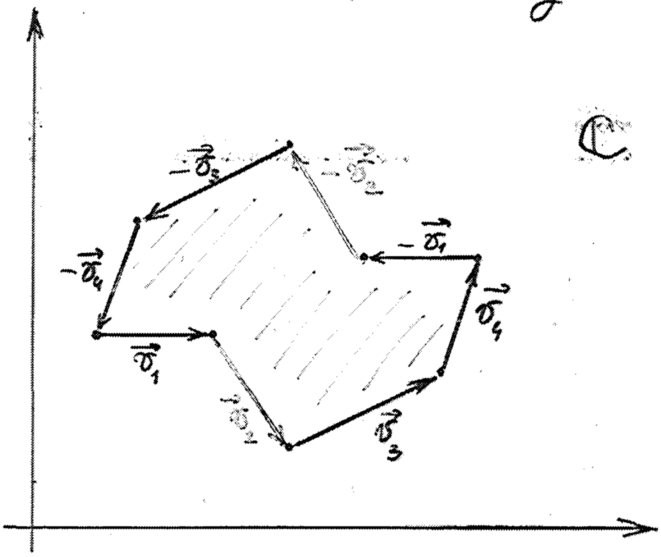


Identification de tores isométriques

TRANSLATION SURFACES

def Translation surface = flat + trivial $SO(2)$-holonomy





gluing rules: translations of the Euclidean plane	$z = z' + \text{const}$
(translation) flat structure + a choice of the vertical direction	complex structure + holomorphic 1-form $\omega = dz$ (Abelian differential)
conical point $2\pi(d+1)$	zero of holomorphic form ω of degree d (in local coordinates $\omega = w^d\,dw$
side $\vec{v}_i$ of a polygon	period $\int_{P_k}^{P_l} \omega = \int_{\vec{v}} \omega$ of ω
area of the surface	$\frac{i}{2}\int_{C_g} \omega \wedge \bar{\omega} = \frac{i}{2}\sum_{i=1}^{g}(A_i \bar{B}_i - \bar{A}_i B_i)$
family of the surfaces sharing the same collection of cone angles	moduli space of Abelian differentials $\mathcal{H}(d_1, \ldots, d_n)$ where $d_1, \ldots, d_n$ are degrees of zeroes, $\sum d_i = 2g-2$
coordinates in the family: vectors $\vec{v}_i$ defining the polygon	coordinates in $\mathcal{H}(d_1, \ldots, d_n)$: collection of relative periodes of ω = = cohomology class of ω in $H^1(C_g, \{P_1, \ldots, P_n\}; \mathbb{C})$

ERGODICITY OF $SL(2;\mathbb{R})$ - action.

Volume element in $\mathcal{H}(d_1, \ldots, d_n)$ =

volume element in the vector space $H^1(\ldots ; \mathbb{C})$
normalized by the lattice $H^1(\ldots ; \mathbb{Z} \oplus i\mathbb{Z})$

$$\Omega = d\vec{v}_1 \cdot d\vec{v}_2 \cdot \ldots \cdot d\vec{v}_m \quad \text{(for a reasonably chosen polygon)}$$

Unit hyperboloid $\mathcal{H}_1(d_1, \ldots, d_n) \subset \mathcal{H}(d_1, \ldots, d_n)$

Area (flat surface) $= 1$ or $\frac{i}{2} \sum_{i=1}^{g} (A_i \overline{B_i} - \overline{A_i} B_i) = 1$

We get the induced volume element $\dfrac{\Omega}{d\,\text{Area}} = \Omega(\text{grad Area}, \omega, \omega, \ldots \omega)$

$SL(2;\mathbb{R})$ - action is the action on coefficients $H^1(\ldots ; \mathbb{C})$.

Geometrically :

This action preserves – the volume element
– the unit hyperboloid (and the function "Area")

THEOREM (H. Masur 1980 ; W. Veech 1980)

- The total volume of $\mathcal{H}_1(d_1, \ldots, d_n)$ is finite ;
- The $SL(2;\mathbb{R})$ action is ergodic on every connected component of every stratum $\mathcal{H}(d_1, \ldots, d_n)$.
- The Teichmüller geodesic flow $\begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}$ is ergodic.

Remark $H^1(\ldots ; \mathbb{R}) \oplus H^1(\ldots ; i\mathbb{R})$ are "unstable" and "stable" directions, but Teichmüller geodesic flow is not uniformly hyperbolic.

MAIN CHALLENGE :

- Description of $SL(2;\mathbb{R})$-invariant submanifolds ;
- Description of $SL(2;\mathbb{R})$-invariant measures.
- Asymptotics of the dynamical characteristics when $g \to +\infty$

GENERAL PHYLOSOPHY. MOTIVATIONS.

GENERAL PHYLOSOPHY. Nontrivial geometric and dynamical properties of (almost any) individual flat surface are expressed in terms of simple geometric and dynamical properties of the corresponding family.

Hope. The closure of $GL(2;\mathbb{R})$-orbit of any flat surface is a nice complex algebraic orbifold. Its geometric and dynamical properties are responsible for geometry and dynamics of the corresponding flat surface.

MOTIVATIONS.

- Billiards in rational polygons $\rightarrow$
  geodesics on translation surfaces
- Measured foliations on surfaces $\rightarrow$
  can be "straightened" to straight line foliations on flat surfaces
- Interval exchange transformations; dynamical systems with "parabolic" dynamics

EXAMPLES OF HOW DOES THE "GENERAL PHYLOSOPH WORK.

THEOREM (H. Masur)
Let S be a flat surface. If the directional flow in the vertical direction is not uniquely ergodic then the "Teichmüller geodesic" $g_t S$ is divergent (i.e. it eventually leaves every compact set in the moduli space $\mathcal{M}_g$ of complex structures)

THEOREM (W. A. Veech)
Let S be a flat surface. If its $SL(2;\mathbb{R})$-orbit is closed, then the directional flow in each direction is either periodic or uniquely ergodic.