united nations educational, scientific and cultural organization

international atomic energy agency

the
abdus salam
international centre for theoretical physics

ICTP 40th Anniversary

SMR.1573 - 2

SUMMER SCHOOL AND CONFERENCE ON DYNAMICAL SYSTEMS

Flat surfaces, Interval exchange Transformations & Moduli Spaces of Abelian Differentials

(Lecture 2)

Anton Zorich
Universite de Rennes 1
I.R.M.A.R.
Rennes, France
& MPI,Bonn, Germany

These are preliminary lecture notes, intended only for distribution to participants

RECOMMENDED LITERATURE

- S. Kerckhoff, H. Masur, J. Smillie
  "Ergodicity of billiard flows and quadratic differentials", ANNALS OF MATH., <u>124</u> (1986), 293 - 311
  Extremely accessible introduction

- H. Masur, S. Tabachnikov
  "Flat structures and rational billiards", HANDBOOK ON DYNAMICAL SYSTEMS, VOL 1A, 1015 – 1089, North-Holland, Amsterdam, 2002
  Accessible serious SURVEY

- J. Smillie
  "The dynamics of billiard flows in rational polygons"; in "Dynamical Syst.", (Ed. by Sinai), Encyclopedia of Math. Sciences; Vol 100, Math. Phys. 1, Springer, 2000
  Accessible serious SURVEY

- A. Eskin, G. Forni, P. Hubert, H. Masur, T. Schmidt, A. Zorich
  Collection of papers to appear in HANDBOOK ON DYN. SYSTEMS, 1B
  TO APPEAR SOON: accessible elementary survey

- A. ZORICH NOTES OF RELATED LECTURES
  TO APPEAR SOON: elementary survey

Open problems

BILLIARDS

- Existence of a closed billiard trajectory in (almost) any triangle (polygon).
- Ergodicity of the billiard flow in (almost) any triangle

"RATNER THEOREM"

- Closure of any $GL(2;\mathbb{R})$-orbit is a nice complex "suborbifold"
- Classification of these suborbifolds
- Given a flat surface find its closure
- When a sequence of closures of $SL(2;\mathbb{R})$-orbits fill better and better an invariant manifold, the corresponding measures converge to the invariant measure of the manifold.

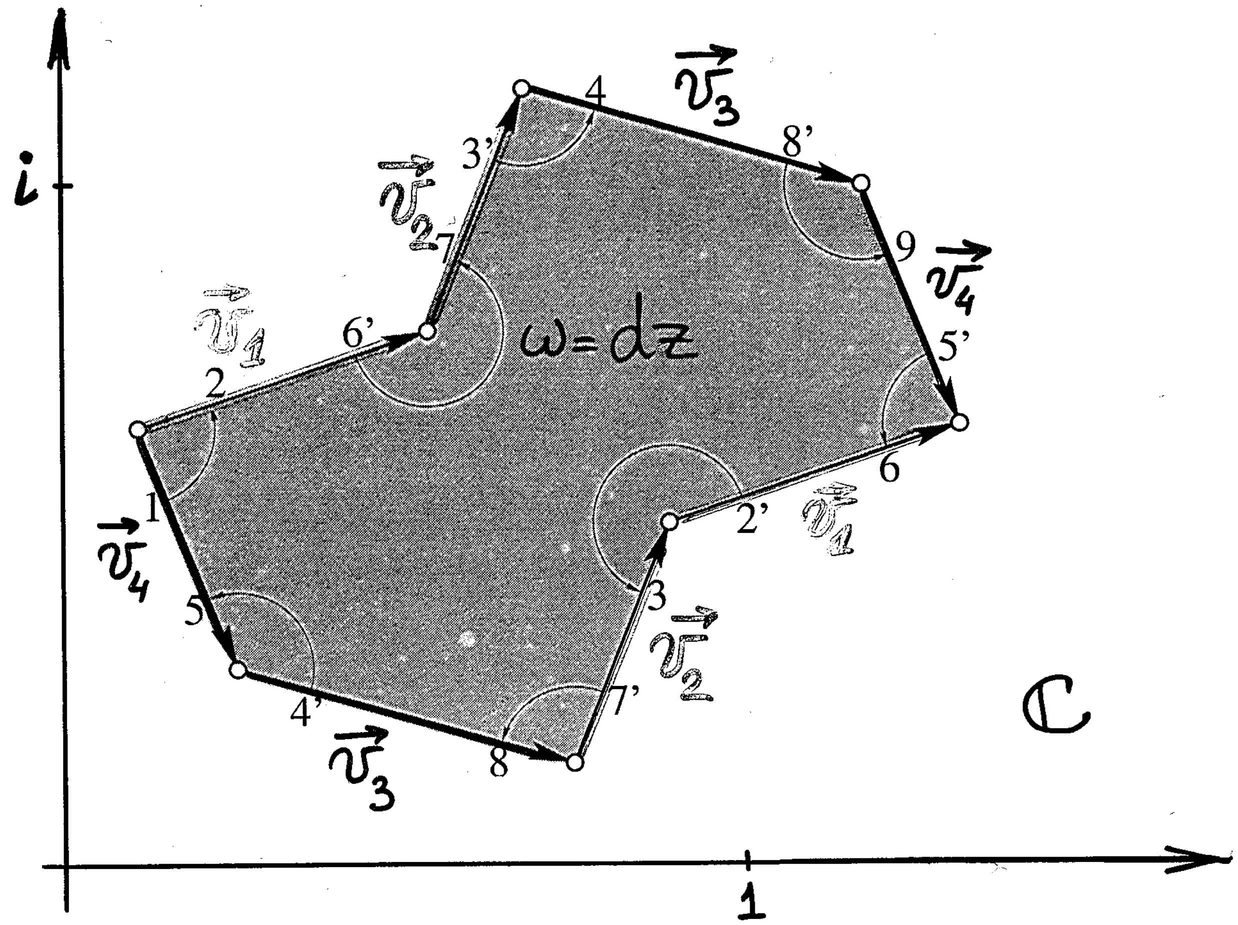
STRATA

- Is any connected component of any stratum of Abelian (quadratic) differentials a $K(\pi,1)$ - space? (M. Kontsevich)

LYAPUNOV EXPONENTS

- Prove that the first g Lyapunov exponents of the Teichmüller geodesic flow have simple spectrum*

VERY FLAT (TRANSLATION) SURFACES



- Vectors $\vec{v}_1, \ldots, \vec{v}_n \in \mathbb{R}^2 \cong \mathbb{C}$ are free parameters determining our flat surface (local coordinates in a stratum)

- Fancy interpretation:

$$\mathbb{C} \ni v_i = \int_{P_i}^{P_{i+1}} dz \quad \left(\begin{array}{c}\text{relative period} \\ \text{of } \omega = dz\end{array}\right)$$

- Flat surface + choice of vertical direction = Point $[\omega] \in H^1(M, \{P_1, \ldots, P_n\}; \mathbb{C})$ of the moduli space of Abelian differentials

BABY THEOREM *Chose any connected component of any stratum. Almost any flat surface in this component can be unwrapped to a polygon of some fixed combinatorics.*

PROOF:

1) It is easy to find SOME surface which can be unwrapped to a polygon

2) Deforming the vectors we get an open domain of surfaces which can be unwrapped

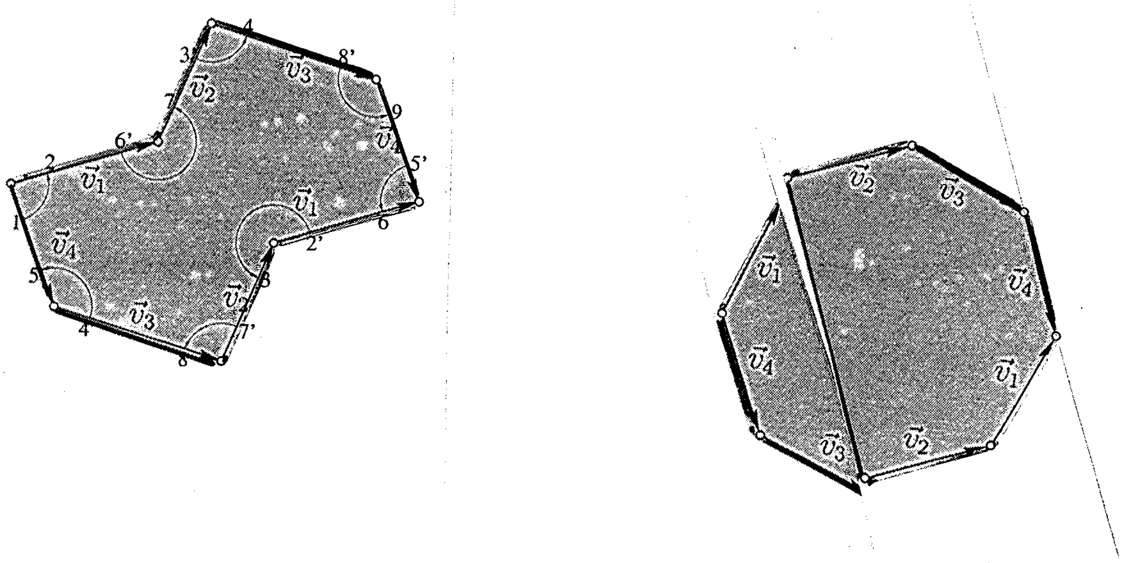
3) Applying ergodicity of $SL(2;\mathbb{R})$-action we get an open domain of full measure. ☒

HOW CAN IT BE TRUE?

THEOREM (H. Masur, 1982; W.A. Veech, 1982)
$SL(2;\mathbb{R})$ (linear transformations) and even
$\begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}$ (expansion in horizontal direction, contraction in vertical direction)
act ergodically on any connected component of any family of flat surfaces (= any stratum $\mathcal{H}_1(n_1, n_2, \ldots, n_k)$ of Abelian differentials).

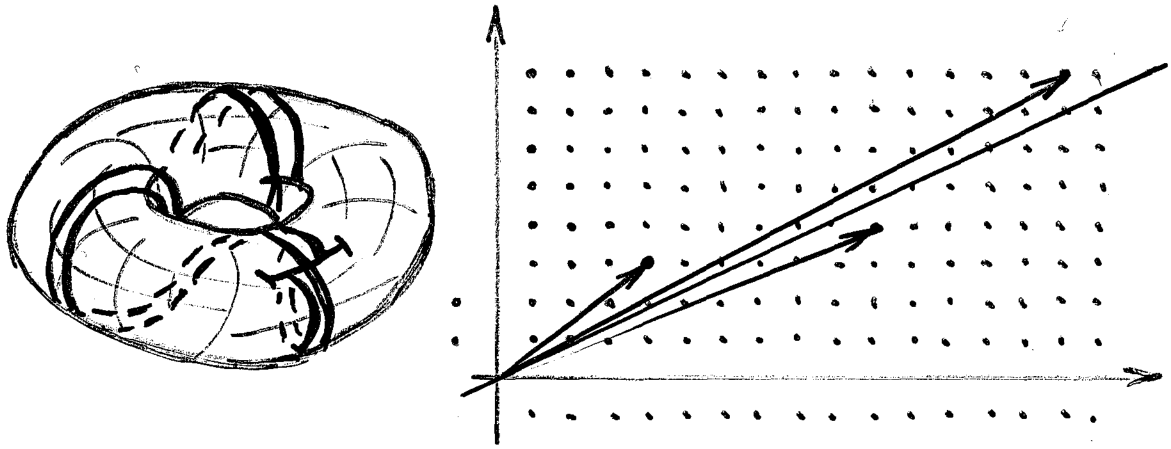
"CONTRADICTION"

How can expansion-contraction bring an irregular nonconvex octagon arbitrarily close to a nice regular convex octagon?



BEHAVIOR OF TYPICAL ("IRRATIONAL") GEODESICS. ASYMPTOTIC CYCLE.

Model case: a torus. Generic geodesic is dense. Joining the endpoints of a long piece of a geodesic we get a closed loop $c(n)$.



def: (S. Shwartzman) Asymptotic cycle

$$c = \lim_{n \to +\infty} \frac{1}{n} c(n) \in H_1(T^2; \mathbb{R})$$

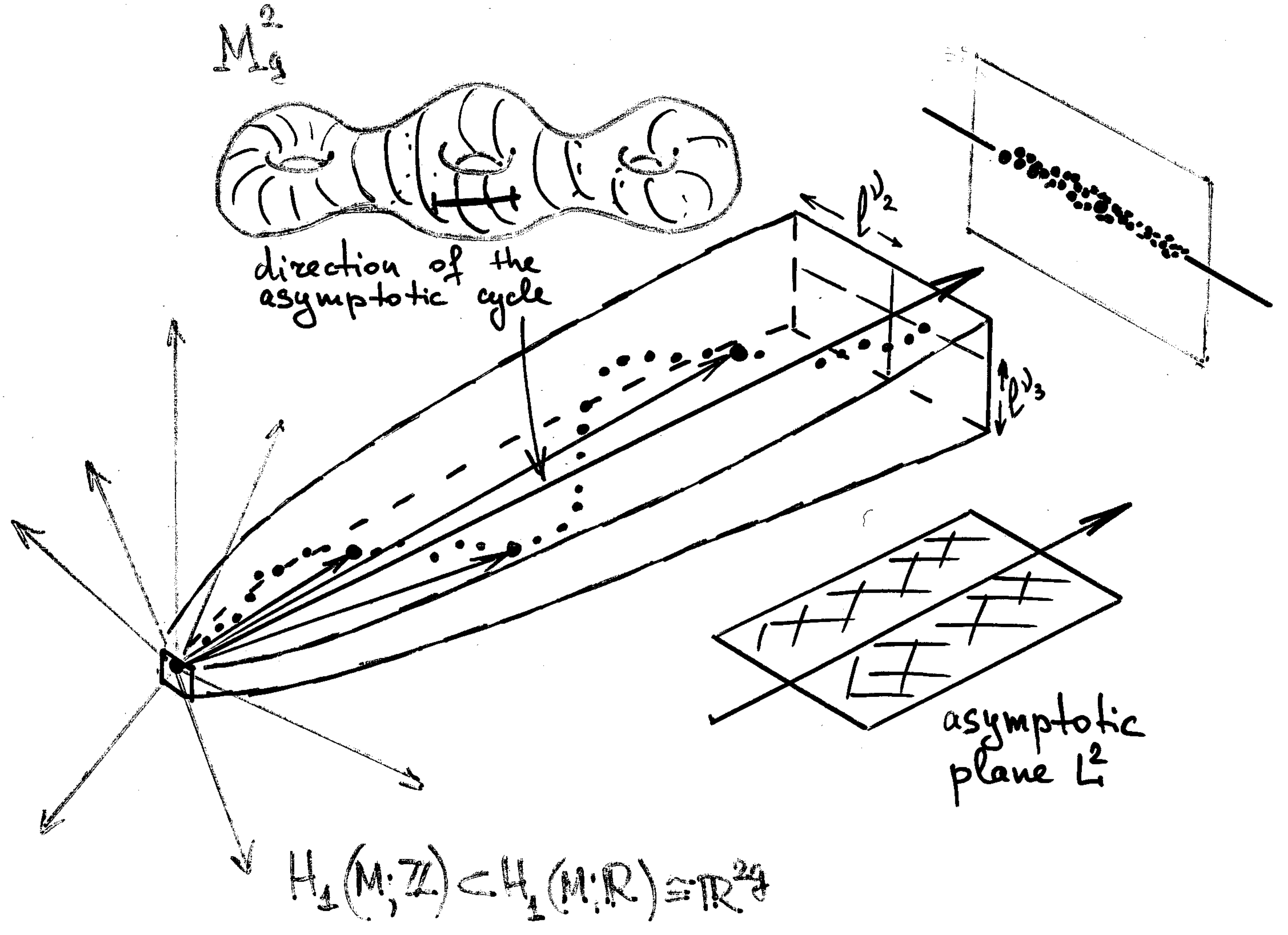
GENERAL CASE: Translation surface of genus $g \geq 2$.

THEOREM (Kerckhoff, Masur, Smillie, 1986) For any flat surface almost any direction is ergodic

COROLLARY For any flat surface a geodesic going in a typical (a.a.) direction is dense and has an asymptotic cycle.

DEVIATION FROM ASYMPTOTIC CYCLE

QUESTION: How do the cycles $c(n) \in H_1(M;\mathbb{Z})$ deviate from the direction of asymptotic cycle?

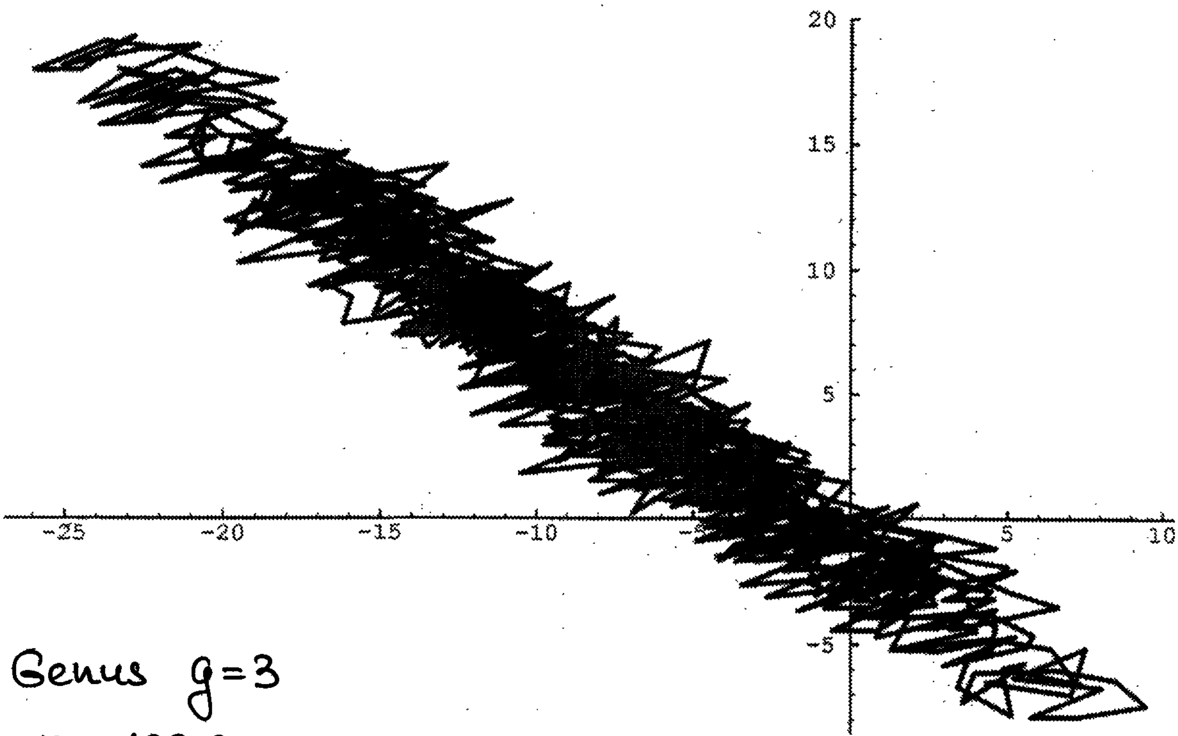


PROJECTION D'UNE TRAJECTOIRE

Projection d'une ligne brisée joignante

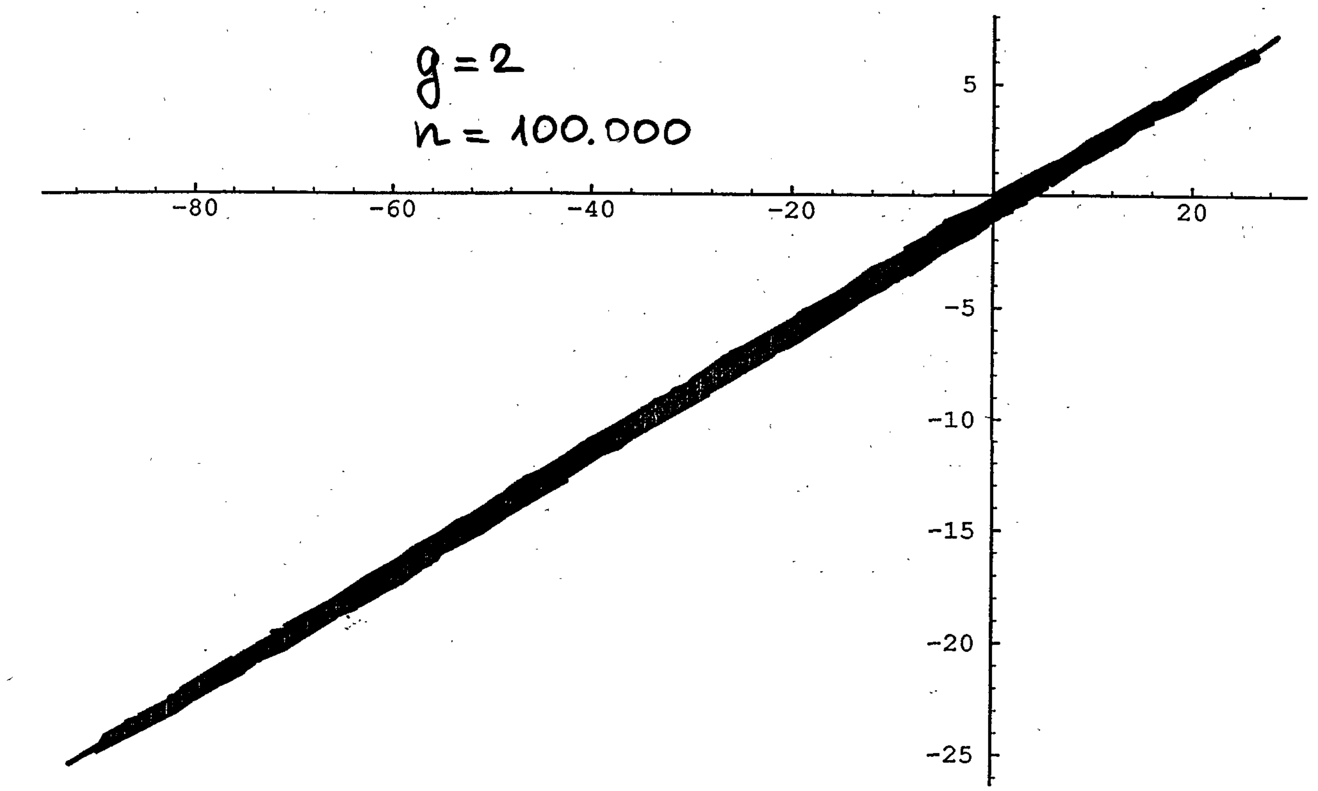
$$c(0),\ c(1),\ \ldots,\ c(n) \in H_1(S;\mathbb{Z})$$

sur un plan orthogonal à la direction asymptotique.



Genus $g=3$

$n = 100.000$


g = 2
n = 100.000
-80
-60
-40
-20
20
5
-5
-10
-15
-20
-25

ASYMPTOTIC FLAG OF SUBSPACES

THEOREM (A. Z. 1997) For a typical (a.e.) pair (flat surface, direction) there is a flag of subspaces $L^1 \subset L^2 \leq \ldots \leq L^g \leq H_1(M, \mathbb{R})$ such that for every geodesic in this direction the distance from the cycle $c(n)$ to L^k, $k<g$, is about $\|c(n)\|^{\nu_{k+1}}$ and the distance from $c(n)$ to L^g is bounded.

Here $2 > 1+\nu_2 \geq 1+\nu_3 \geq \ldots \geq 1+\nu_g \geq 1$ are the Lyapunov exponents of the Teichmüller geodesic flow.

THEOREM (G. FORNI, 2002)

$$\nu_g > 0 \quad , \text{ so } \dim L^g = g$$

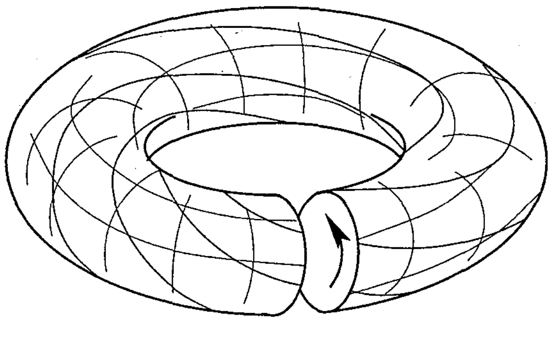
Conjecture $\nu_2 > \nu_3 > \ldots > \nu_g$

Remark. The fact that $\nu_2 > 0$ is one of the key points in the recent (June 2004)

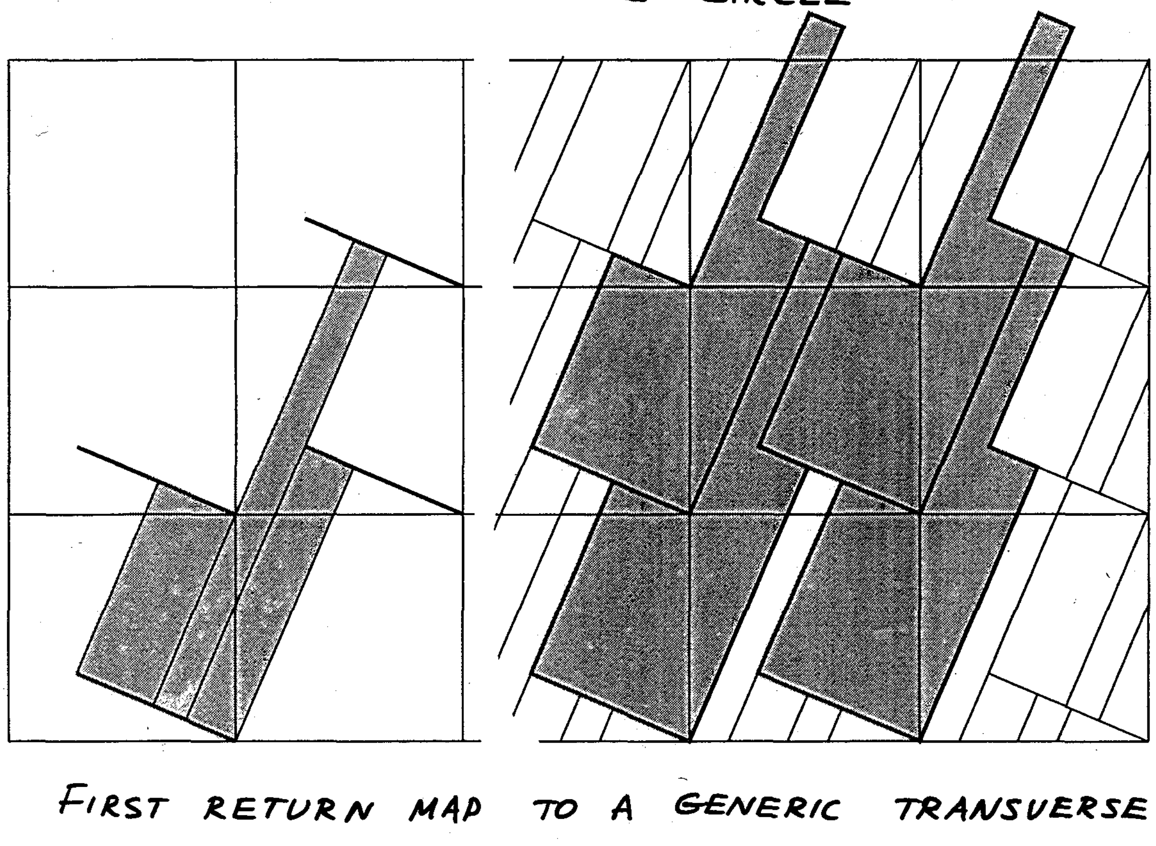
THEOREM (A. AVILA* + G. Forni ; June 2004)
Almost all interval exchange transformations are weakly mixing

* TALK ON Tuesday, 20 July 14^{00}

FIRST RETURN MAP OF A DIRECTIONAL FLOW ON A TORUS TO A TRANSVERSAL

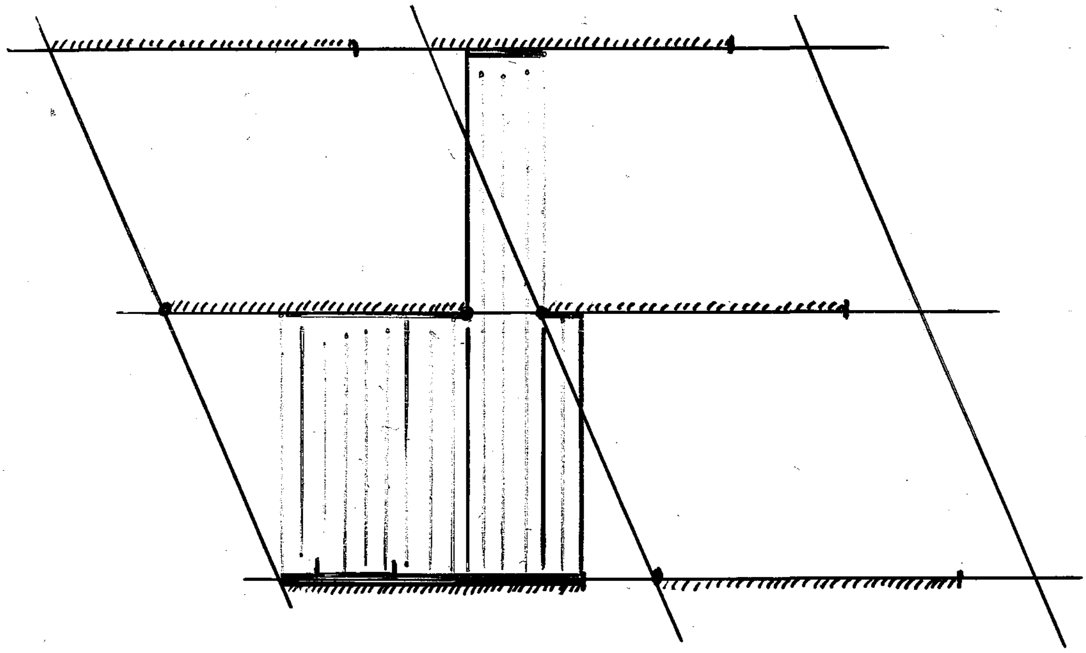


FIRST RETURN MAP TO THE MERIDIAN IS A ROTATION OF THE CIRCLE

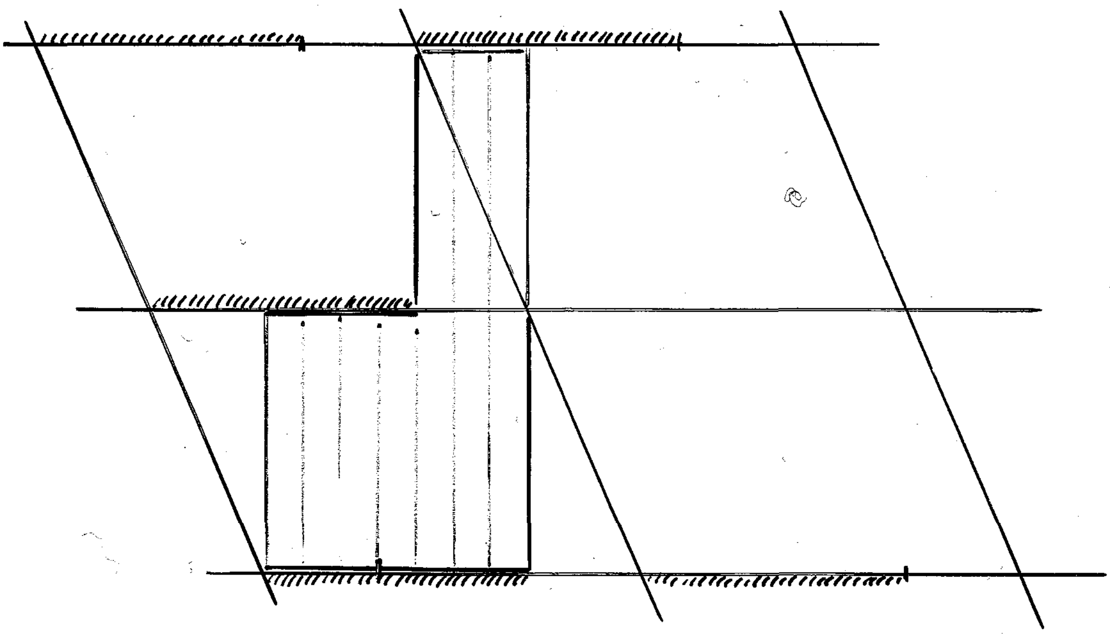


FIRST RETURN MAP TO A GENERIC TRANSVERSE INTERVAL IS AN INTERVAL EXCHANGE TRANSFORMATION OF THREE SUBINTERVALS. IT DEFINES A DECOMPOSITION OF THE TORUS INTO A "BUILDING OF RECTANGLES".

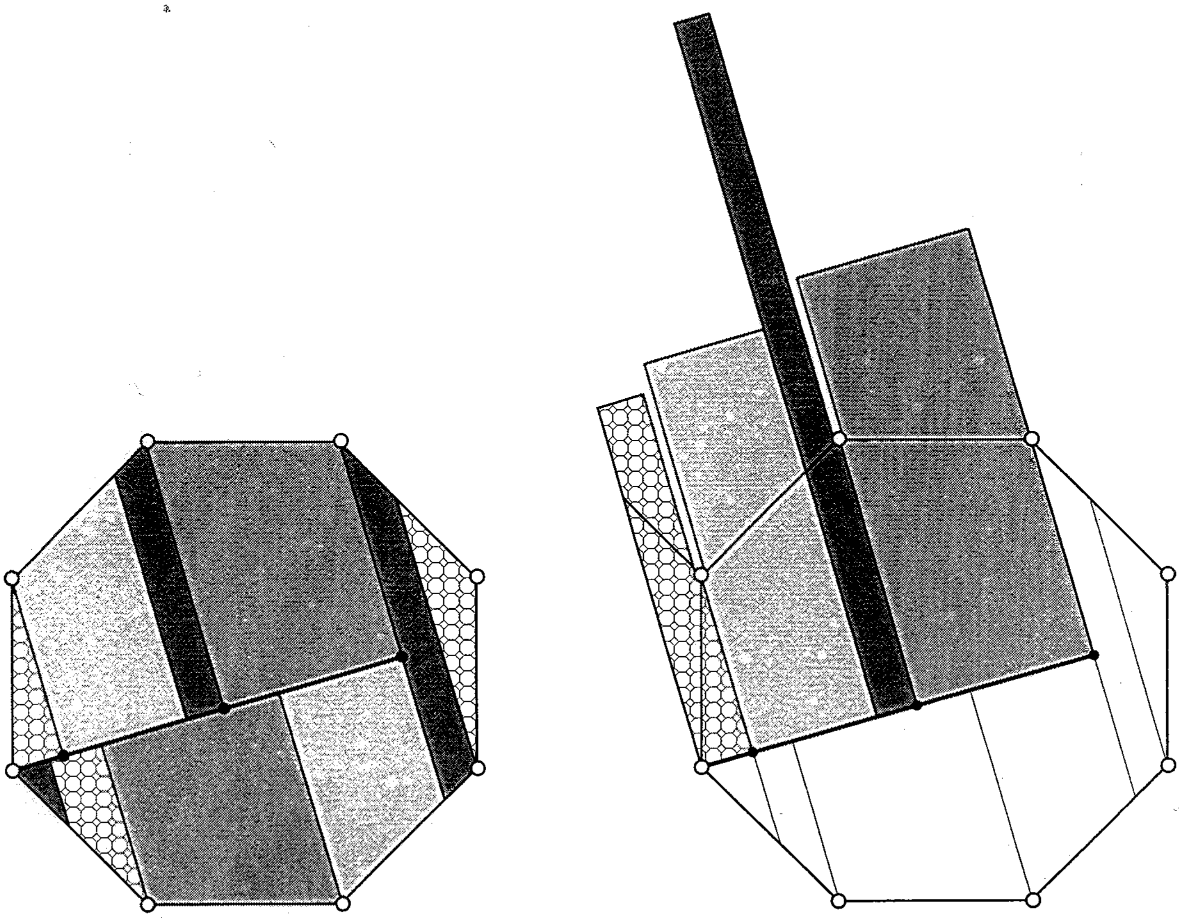
Generic choice of a subinterval induces a tiling of the torus by three rectangles



Smart choice of a transverse interval gives a tiling by only two rectangles :



FIRST RETURN MAP FOR SURFACES OF HIGHER GENERA



First return map to a transverse interval (properly chosen) on this surface of genus two is an interval exchange transformation of four intervals. It defines a decomposition of the surface into a building of zippered rectangles