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SUMMER SCHOOL AND CONFERENCE ON DYNAMICAL SYSTEMS

Polynomial Diffeomorphisms of C^2

(Lecture 5)

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We are assuming that $(a,b) \in \overline{\mathcal{H}\mathcal{A}}$ which implies that J is real. We want to show that when J is real we either have hyperbolicity or a quadratic tangency.

In fact we will show that when $f_{a,b}$ is not hyperbolic it is still very close to being hyperbolic.

The behavior of this diffeomorphism will be analogous to that of polynomial maps with preperiodic critical points.

Fix a periodic saddle point p.
Let $\varphi_p : \mathbb{C} \longrightarrow W^u_p$ be a parametrization of the unstable manifold.

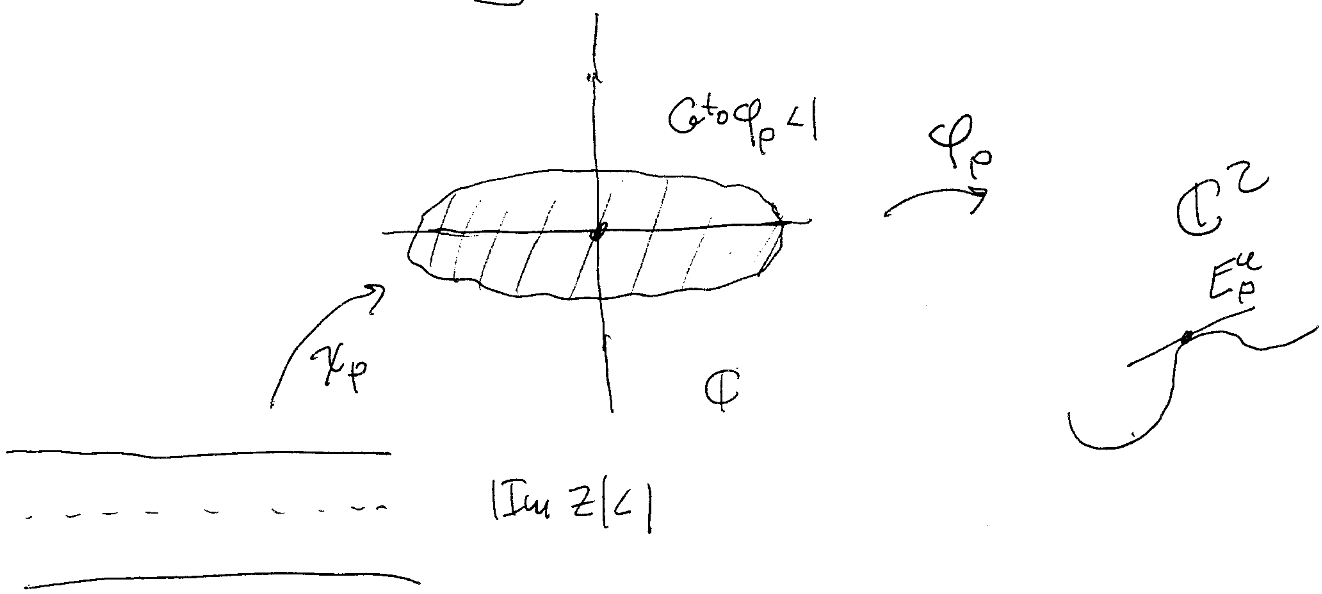
Recall that G^+ is p.s.h and pluriharmonic outside of J^+.
Since φ_p is holomorphic we see that $G^+ \circ \varphi_p$ is subharmonic and harmonic outside of

$$\varphi_p^{-1}(W^u_p \cap J^+) \subset \varphi_p^{-1}(J^- \cap J^+) = \varphi_p^{-1}(J) \subseteq \varphi_p^{-1}(\mathbb{R}^2).$$

If we choose our parametrizations to be "real" i.e. $\varphi_p(\bar{z}) = \overline{\varphi_p(z)}$ then $G^+ \circ \varphi_p$ is harmonic outside of some subset of the real axis.

What does this geometric condition mean dynamically?

We define a metric on E^u_p by defining a metric on W^u_p.

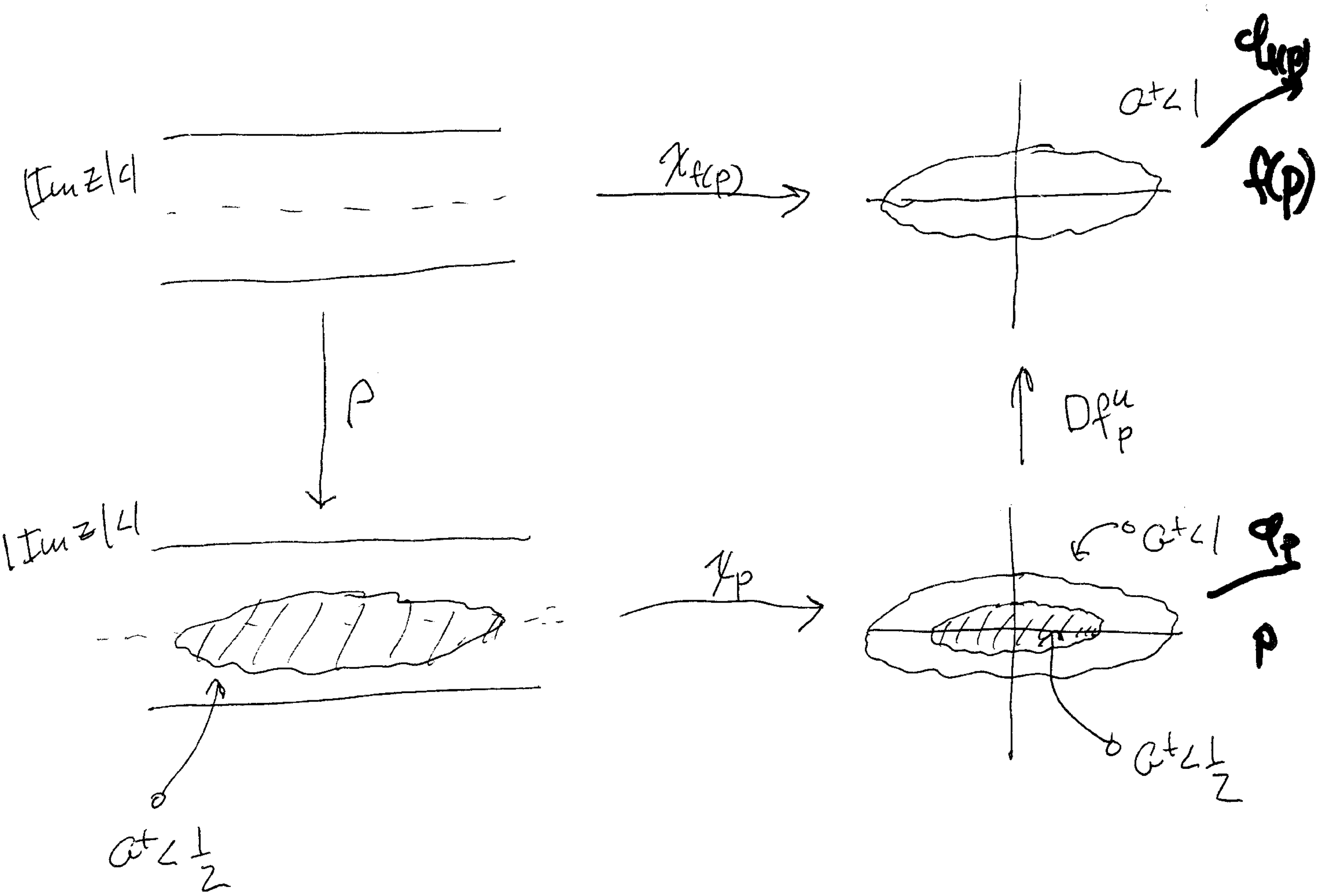


Define a metric on $\mathbb{C}$ and on E^u_p so that $\|D\chi_p\| = \|D\varphi_p\| = 1$.

We have $Df^u_p : E^u_p \longrightarrow E^u_{f(p)}$

Lemma. $\| Df^u_p \| \geq 2$.

Proof.



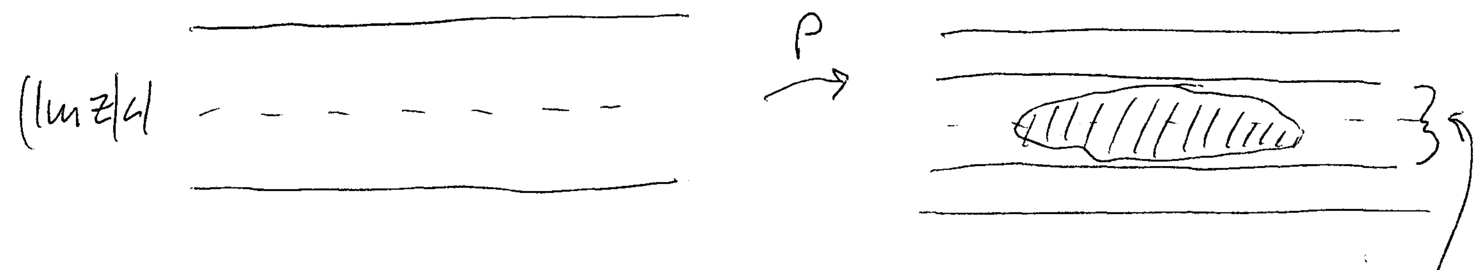
$\rho = \chi_p^{-1} \circ (Df^u_p)^{-1} \circ \chi_{f(p)}$.

The image of ρ is $\{ G^+ < \frac{1}{2} \}$.

We compare the functions $G^+ \circ \chi_p$ and $|\operatorname{Im} z|$.

Both functions take the value 1 on the set $|\operatorname{Im} z| = 1$. Both functions are harmonic outside of the real axis. On the real axis $G^+ \circ \chi_p \geq |\operatorname{Im} z|$. By the maximum principle $G^+ \circ \chi_p \geq |\operatorname{Im} z|$ on the strip. In particular the set where

$$\{ G^+ \circ \chi_p < \tfrac{1}{2} \} \subset \{ |\operatorname{Im} z| < \tfrac{1}{2} \}.$$



Note that the image of $z \mapsto 2\cdot\rho(z) \subset \{|\mathrm{Im}\, z|<1\}$ so $\|D(2\rho)\| \le 1$ by the Schwarz Lemma.

Thus

$$2\cdot\|D\rho\| \le 1 \quad \text{and} \quad \|D\rho\| \le \tfrac{1}{2}.$$

But $\quad D\rho = (D\chi_p)^{-1}\circ(Df^u_p)^{-1}\circ D\chi_{f(p)}$

so $\quad \|D\rho\| = \|Df^u_p\|^{-1}$ and

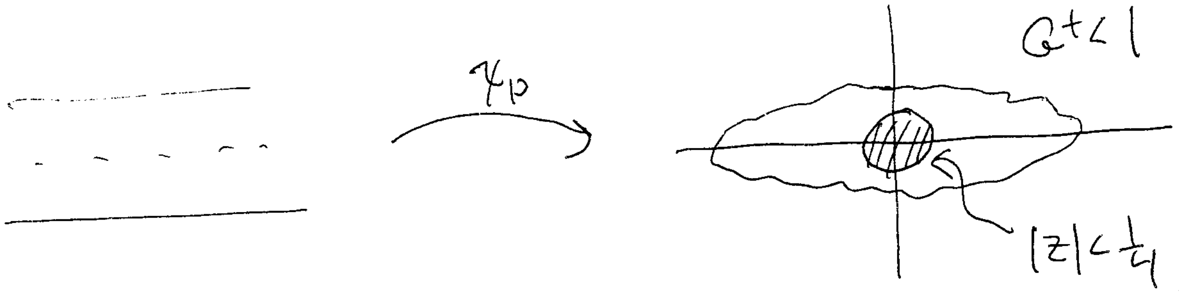
$$\|Df^u_p\| \ge 2.$$

Corollary. If p is a periodic saddle point of period n then the absolute value of the larger eigenvalue of Df_p^n is at least 2^n.

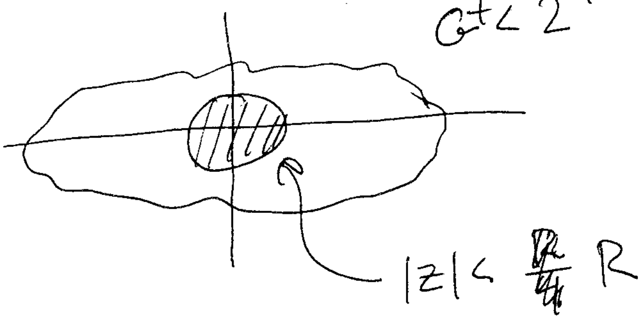
How do we move from information about periodic points to information about all points of J?

Claim that $\mathcal{X}_p$ is a normal family.

The Koebe $\frac{1}{4}$ Theorem gives us the fact that $|z|<\frac{1}{4} \Rightarrow G^+\circ\varphi_p<1$

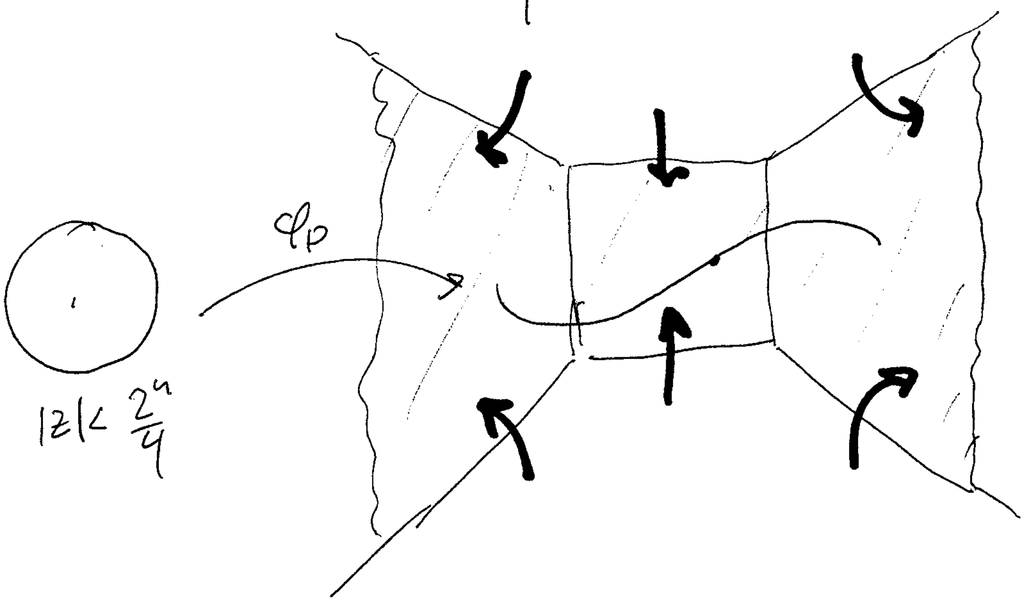


If we apply f^n we have



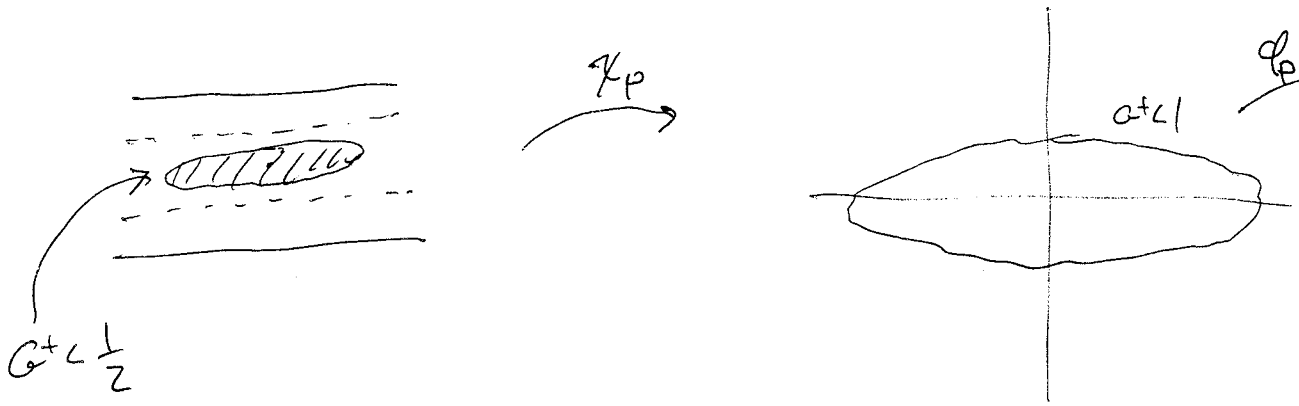
where $R\geq\frac{2^n}{4}$

So $|z|<\frac{2^n}{4}$ implies that $G^+<2^n$.



$G^+(x,y)\sim\log|x|$

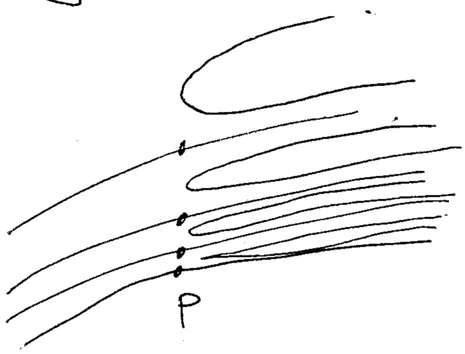
The theory of normal families tells us that every sequence has a convergent subsequence. In fact none of these limiting functions is constant.



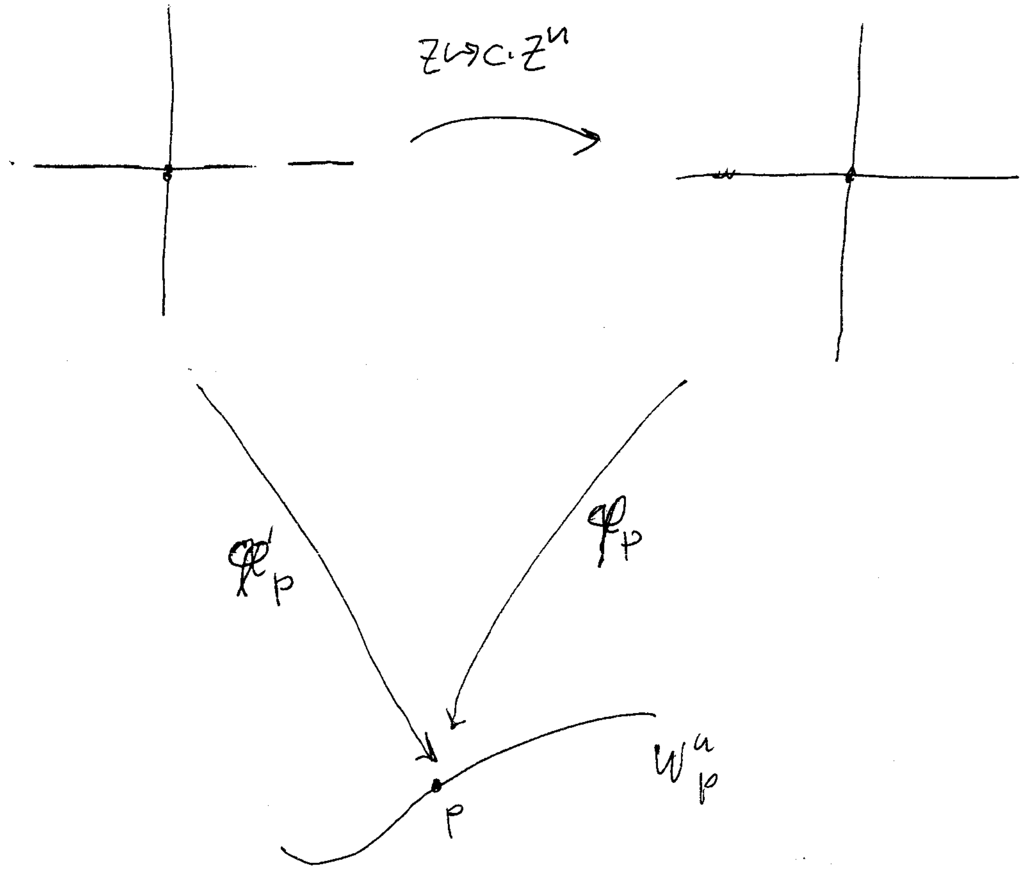
So for each $p \in J$ we have a $\varphi_p : \mathbb{C} \to \mathbb{C}^2$ passing through p and a metric which is uniformly expanded.

Does this mean that f is hyperbolic? Not quite.

We can have one point and two different sequences converging to it.

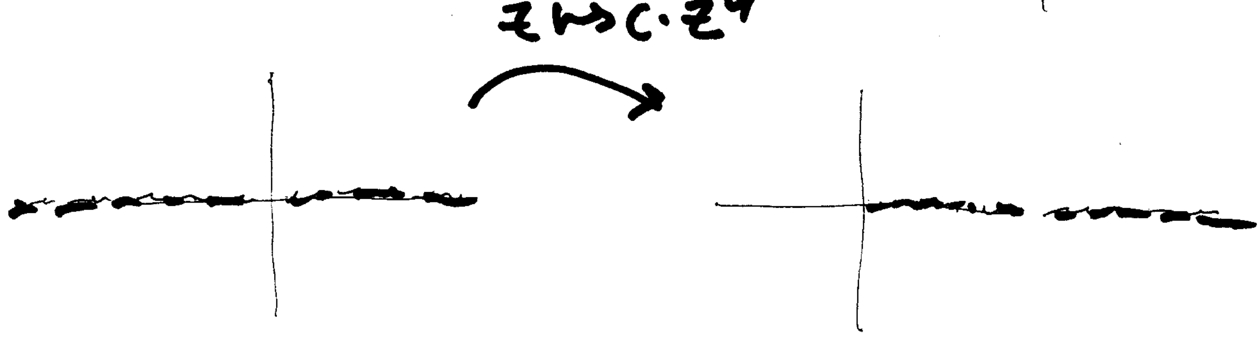


When this happens one function φ_p parametrizes the unstable manifold at p and the other φ'_p n-fold covering of the unstable manifold.



What values can n take?

The reality condition implies that $n=2$. This can only happen when p is a one-sided periodic point.



In fact the picture must look like:

This is how we see that the tangencies are quadratic, if they exist. If there are no tangencies then we conclude that the diffeo. is hyperbolic.

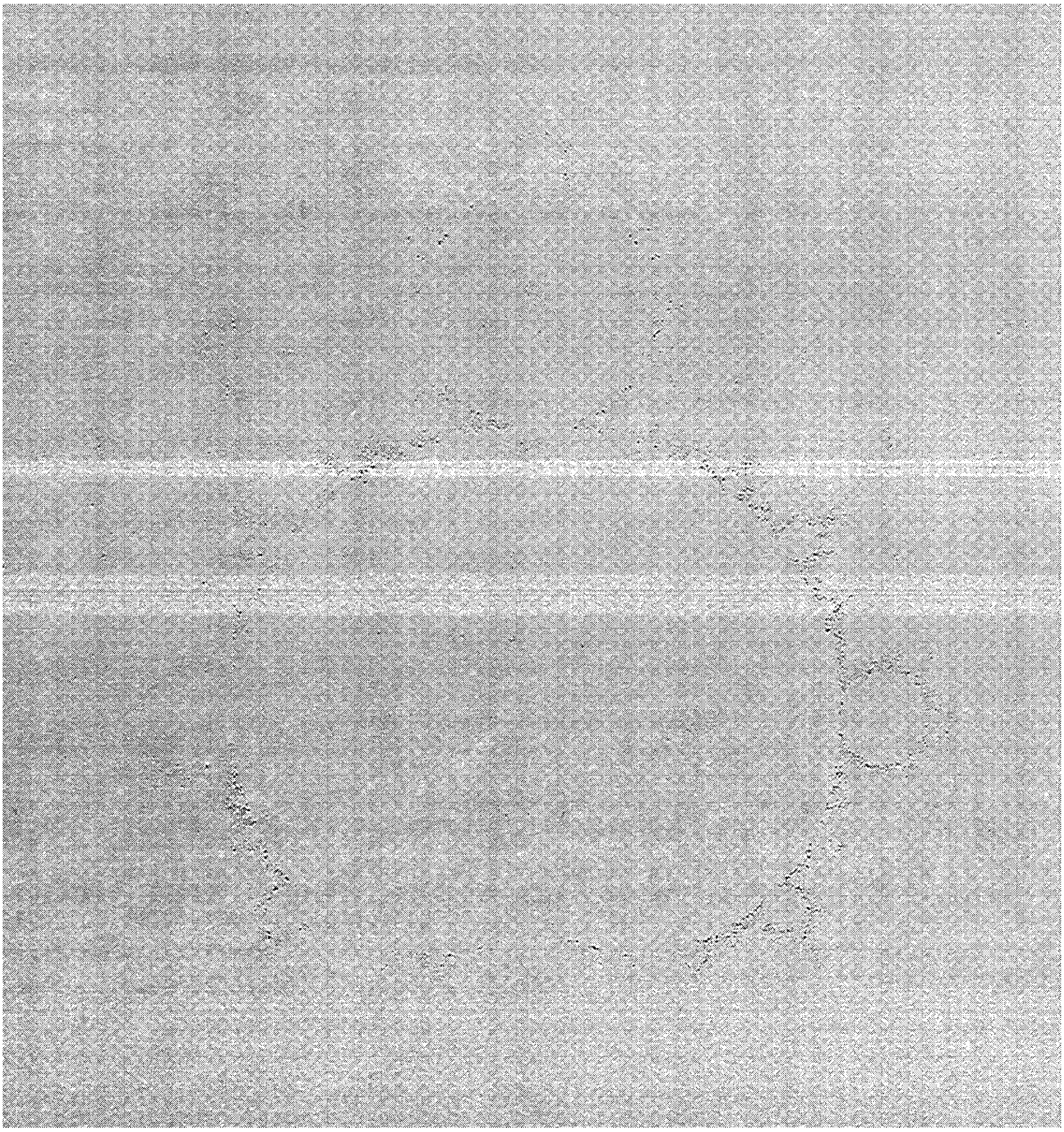
What next?

In the work I just described we translated a topological dynamical property into a geometrical property and then into a smooth dynamical property.

In one dimensional complex dynamics there is a "machine" for doing this: the Branner-Hubbard-Yoccoz theory of puzzles.

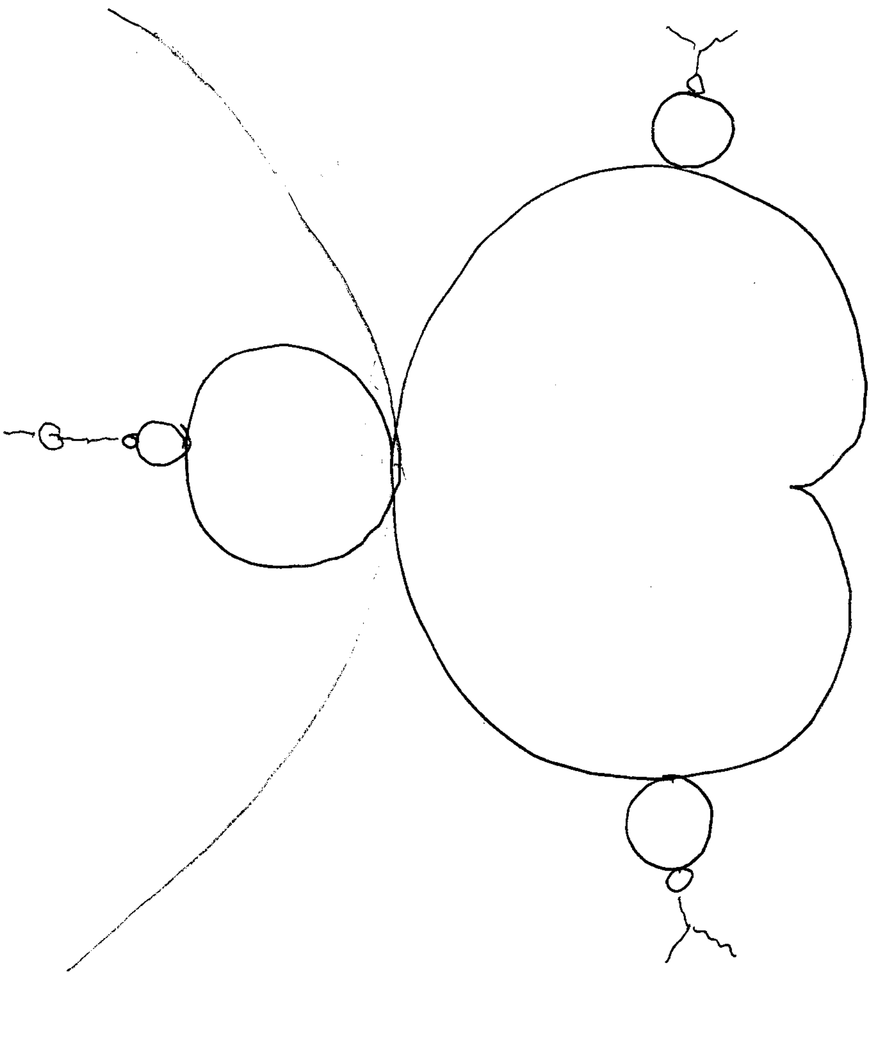
What are puzzles in one dimensional complex dynamics and what might they be in two dimensional complex dynamics?

We can think of a puzzle as a type of Markov partition with one significant difference.

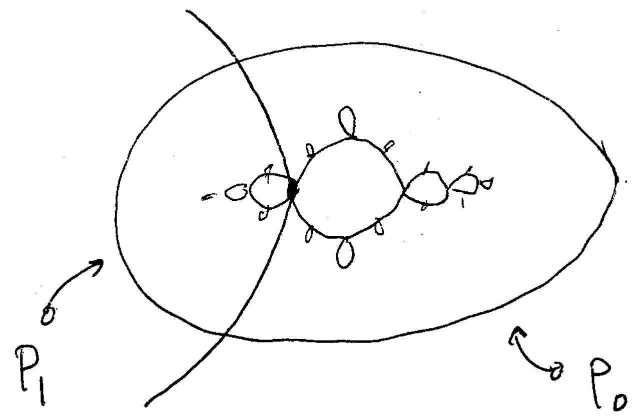


M = {c : J_c is connected}

I want to look at one particular puzzle the puzzle associated with the $\frac{1}{2}$-limb of the Mandelbrot set.



The $\frac{1}{2}$-limb in parameter space corresponds to c values (such as $c=-1$) for which we have a particular partition of the Julia set



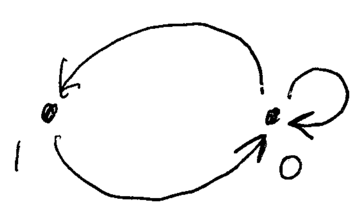
Two "external rays" meet at a fixed point and f_c interchanges these external rays.

This partition gives us a coding of points.

Let z be a point with a bounded orbit. We assign z to the symbol sequence $s = s_0 s_1 s_2 \ldots$ where

$$s_n = j \text{ if } f^n(z) \in P_j.$$

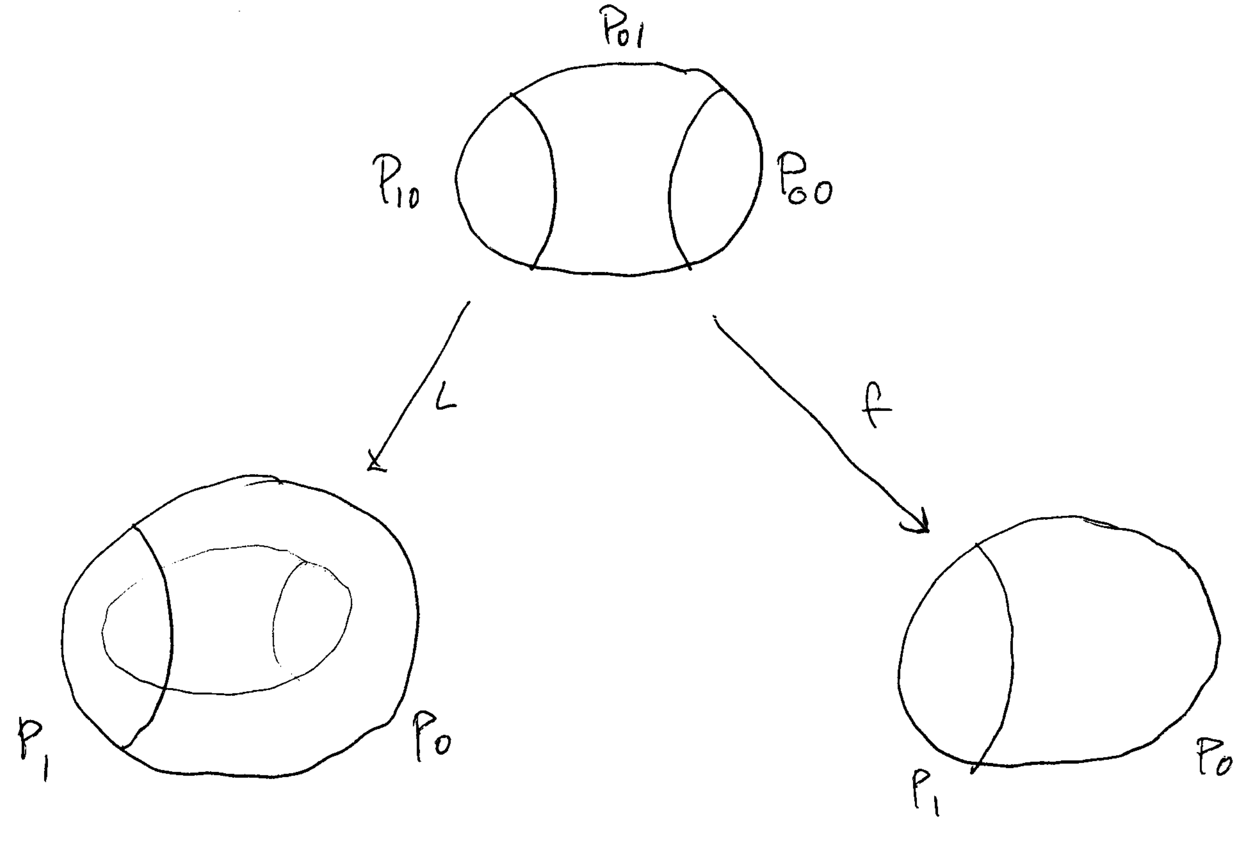
Only certain transitions are allowed. We can record the allowable transitions in the following graph:



What happens if $f^n(z) \in P_0 \cap P_1$ for some n?

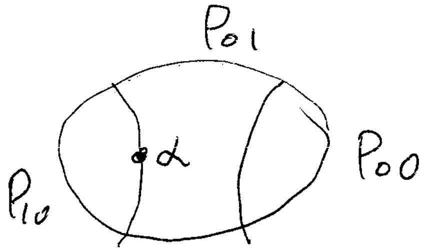
We would like a way to code all points with bounded orbits but we are forced to accept a certain amount of ambiguity.

Let P_{jk} be the set of $z \in P_j$ so that $f(z) \in P_k$



Definition. A coding of a point z is a sequence $s = s_0 s_1 \ldots$ such that $f^j(z) \in P_{s_j s_{j+1}}$.

Example.



The fixed point α has exactly two codings,

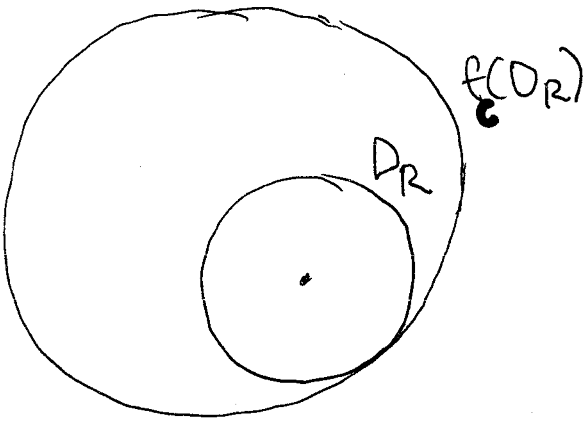
$s = 010101\ldots$ and

$s = 101010\ldots$

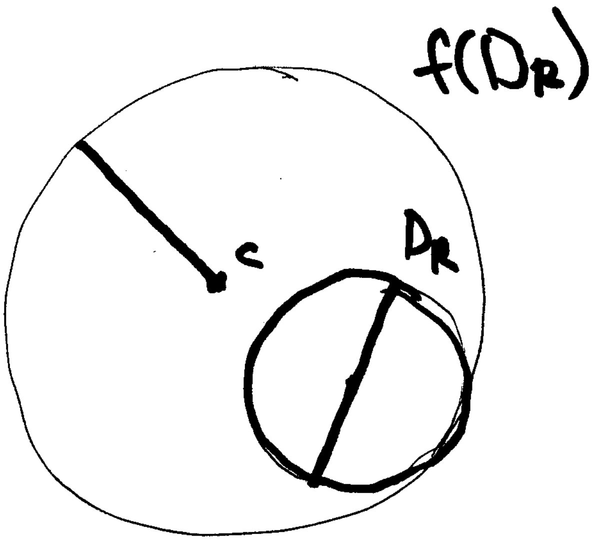
Proposition. Every z with bounded orbit has at least one and at most 2 codings.

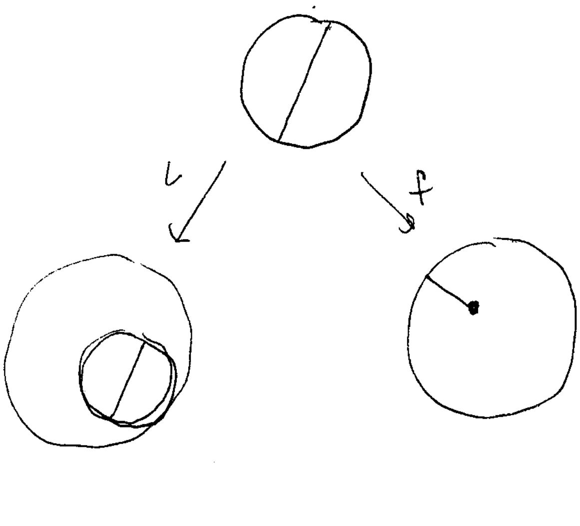
We would like to define an analogoos system of coding points for some region in the parameter space of complex Hénon diffeomorphisms

Recall that given c there exists an R so that D_R contains all bounded orbits.



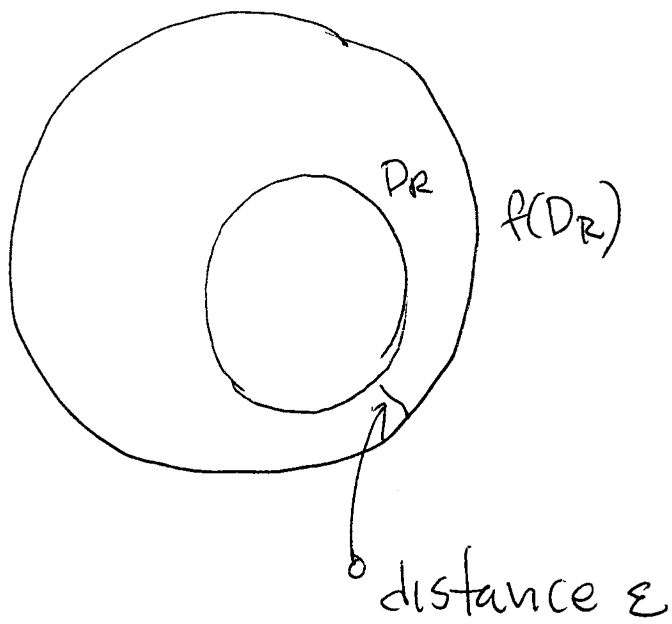
If $|c|>2$ then $f_c(0)=c$ lies outside of D_R and we have



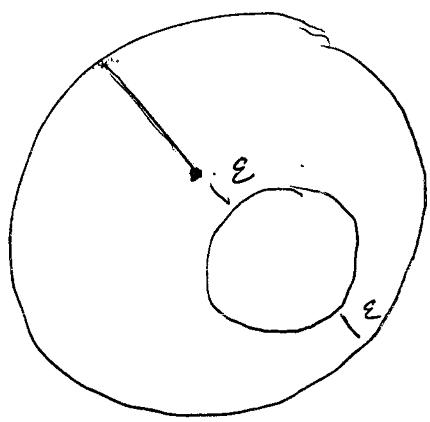


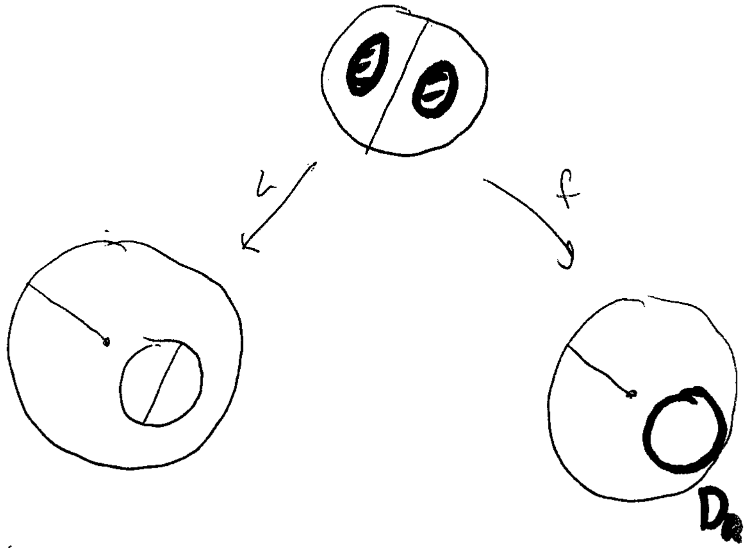
This picture allows us to code orbits. If we fatten our pieces we can use the same scheme to code ε-pseudo-orbits.

Given c and ε there is an R so that D_R contains all ε-pseudo-orbits.



If $|c|$ is sufficiently large then we can code ε-pseudo-orbits.



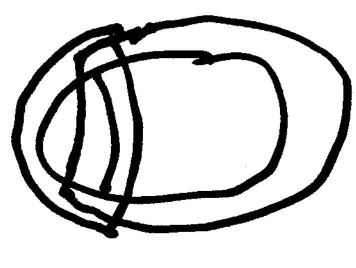


This coding is stable in the sense that δ-close ε-pseudo-orbits have the same coding.

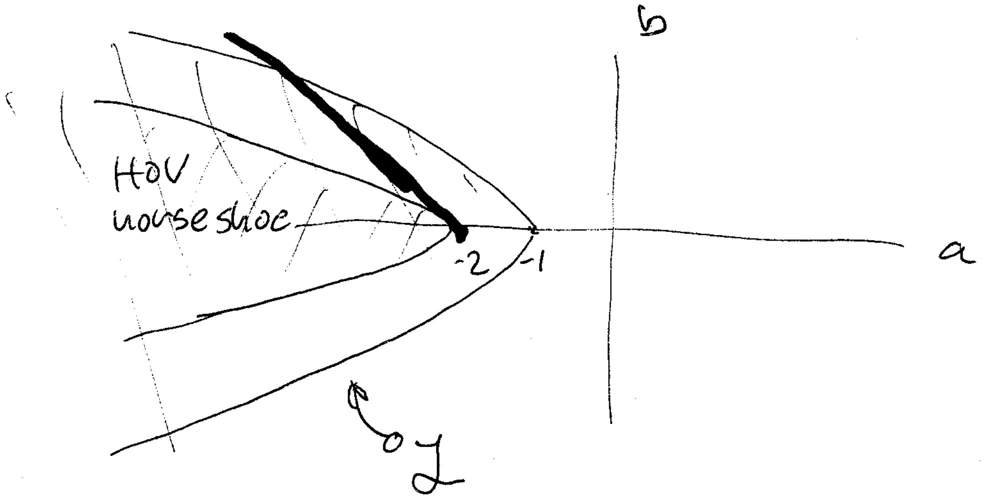
Fundamental trivial idea:

For (a,b) in the HOV horseshoe region we can recover the horseshoe coding of orbits by thinking about the orbits of $f_{a,b}$ as ε-pseudo orbits for the one dimensional map f_a.

This gives us a recipe for extending the $\frac{1}{2}$-limb coding to Hénon diffeos. We fatten up P_0 and P_1 so that we obtain a coding of ε-pseudo-orbits.



We obtain a fairly large region $\mathcal{L} \subset \{(a,b)\}$ of parameters for which we have a $\frac{1}{2}$-limb coding.



This coding gives names to points and we can count the number of points with a given coding.

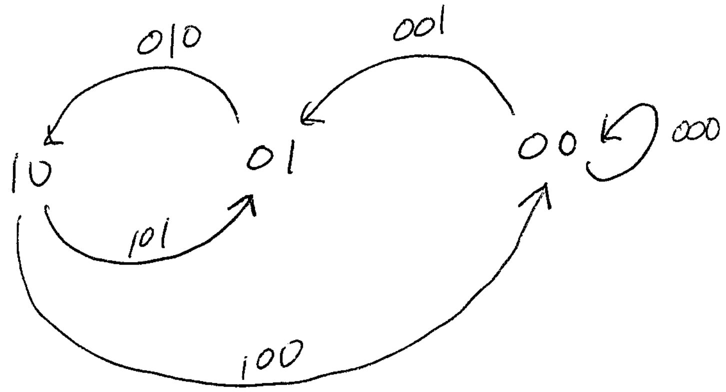
Theorem. (Bedford-S)

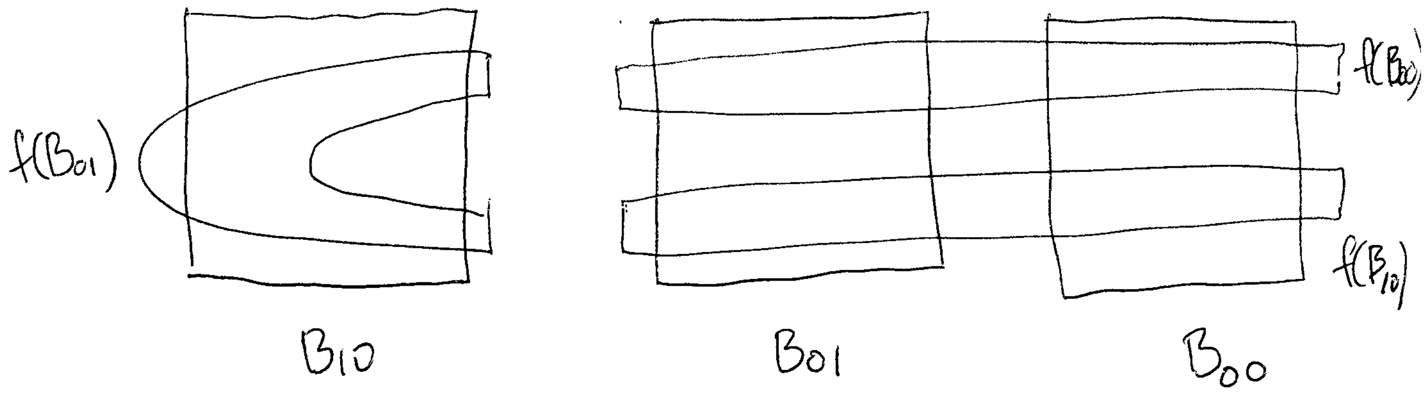
If $a, b \in \mathbb{R}^2 \cap \mathcal{L}$ and $b > 0$ then $f_{a,b}$ is a hyperbolic horseshoe if and only if there are 4 real points with the coding sequence

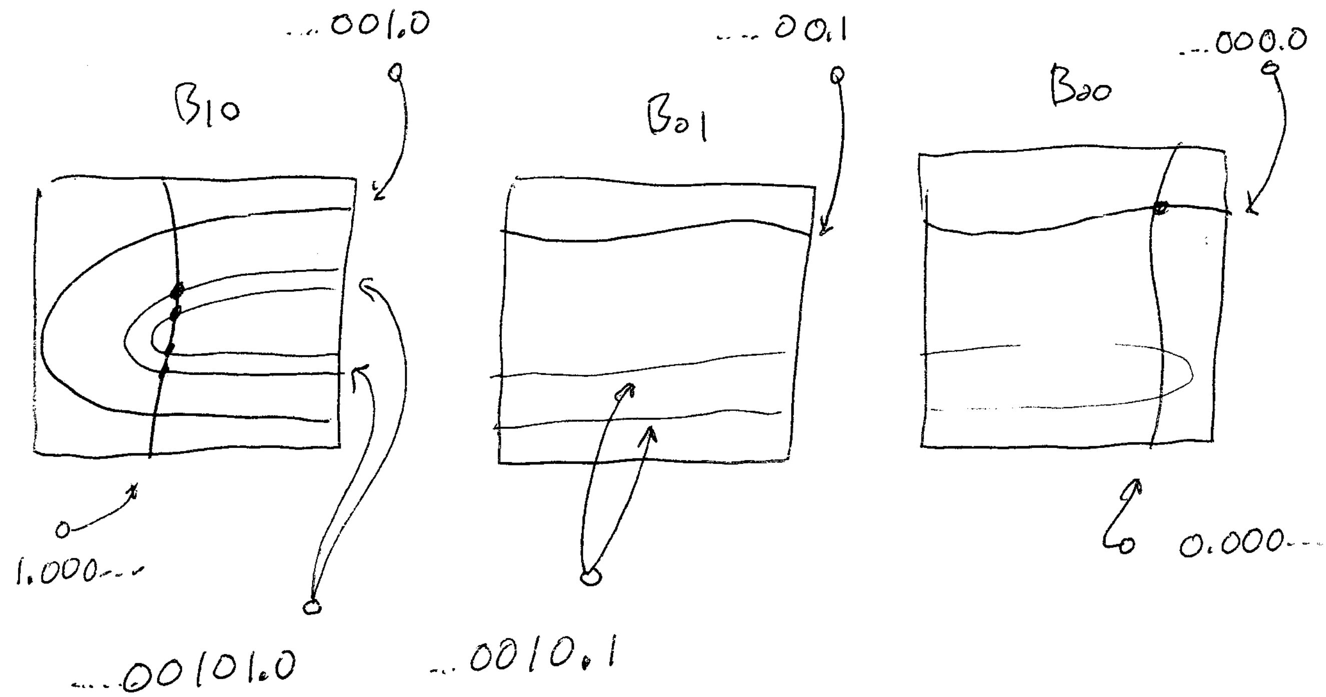
$$\ldots 0000101 0000 \ldots$$

If there are exacty 3 such points then $f_{a,b}|K_{a,b}$ is topologically conjugate to a full 2-shift with 2 homoclinic orbits identified.

The technique of proof is to use our fattened puzzle pieces to build boxes in $\mathbb{C}^2$ which have transitions determined by the graph







The points that we need to count are the intersections of the stable segment 1.000.... and the unstable segments with coding ....00101.0.