

*SUMMER SCHOOL AND CONFERENCE
ON DYNAMICAL SYSTEMS*

**Interval exchange maps, renormalisation and
continued fractions**

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The cohomological equation for Roth type interval exchange maps

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0. Introduction

Let α be an irrational number, $(q_n)_{n \in \mathbb{N}}$ be the sequence of the denominators of its continued fraction expansion and $(a_n)_{n \in \mathbb{N}}$ be the sequence of its partial quotients. Roth type irrationals have several equivalent arithmetical characterizations:

- in terms of the rate of approximation by rational numbers: for all $\varepsilon > 0$ there exists a positive constant C_ε such that $|q\alpha - p| \geq C_\varepsilon q^{1+\varepsilon}$ for all rationals p/q ;
- in terms of the growth rate of the denominators of the continued fraction: $q_{n+1} = O(q_n^{1+\varepsilon})$ for all $\varepsilon > 0$;
- in terms of the growth rate of the partial quotients: $a_{n+1} = O(q_n^\varepsilon)$ for all $\varepsilon > 0$.

In addition to these purely arithmetical characterizations an equivalent definition arises naturally in the study of the cohomological equation associated to the rotation $R_\alpha : x \mapsto x + \alpha$ on the circle $\mathbb{T} = \mathbb{R}/\mathbb{Z}$: α is of Roth type if and only if for all $r, s \in \mathbb{R}$ with $r > s + 1 \geq 1$ and for all functions Φ of class C^r on \mathbb{T} with zero mean $\int_{\mathbb{T}} \Phi dx = 0$ there exists a unique function Ψ of class C^s on \mathbb{T} and with zero mean such that $\Psi - \Psi \circ R_\alpha = \Phi$.

The class of Roth type irrationals enjoys several nice properties: by the celebrated theorem of Roth all algebraic irrationals are of Roth type. Moreover the set of Roth type numbers has full measure and is invariant under the natural action of the modular group $SL(2, \mathbb{Z})$.

The goal of this paper is to characterize a class of interval exchange maps (i.e.m.'s) with similar properties (especially for the solutions of the associated cohomological equation and the fact of being a full measure class).

0.1 Interval exchange maps

Let \mathcal{A} denote an alphabet with $d \geq 2$ elements. Let I be an interval and $(I_\alpha)_{\alpha \in \mathcal{A}}$ a partition of I into d subintervals. An interval exchange map T is an invertible map of I which is a translation on each I_α . Thus T is orientation-preserving and preserves Lebesgue measure.

When $d = 2$ then T is just a rotation (modulo identification of the endpoints of I). It can be thought as the first return map of a linear flow on a two-dimensional torus on a transversal circle. Analogously when $d \geq 3$ by singular suspension any i.e.m. is related to the linear flow on a suitable translation surface (see, e.g. [V1] for details, or section 3 below) typically having genus higher than 2. A well-known dictionary between translation surfaces and Riemann surfaces relates i.e.m.'s to the theory of measured foliations on

surfaces (see, e.g. [FLP] for a introduction to measured foliations). Finally i.e.m.'s are related to the study of rational polygonal billiards (see [Ar], [Ta] and [KH], Chapter 14, for a general introduction to i.e.m.'s, flows on surfaces and polygonal billiards).

Typical i.e.m.'s are minimal (this is guaranteed by a condition due to Keane [Ke1] which is automatically dealt with if the intervals lengths are rationally independent) but note that ergodic properties of minimal i.e.m.'s can differ substantially from those of circle rotations: first they need not be uniquely ergodic [Ke2, KN, Co], and second, being ergodic they can be weakly mixing [KS, V3,V4]. On the other hand uniquely ergodic i.e.m.'s are generic [KR] and Keane's conjecture that almost every i.e.m. is uniquely ergodic was proven independently by Masur and Veech [Ma, V2], see also [Ker, Re].

One of the most important consequences for us of Keane's condition is that it allows to introduce and to iterate indefinitely continued fraction algorithms that generalize the classical algorithm (corresponding to the choice $d = 2$) [Ra, V2, Z1]. Both the Rauzy–Veech continued fraction algorithm and its accelerated version due to Zorich are ergodic w.r.t. an absolutely continuous invariant measure in the space of i.e.m.'s. However in the case of the Rauzy–Veech continued fraction the measure has infinite mass whereas the invariant measure for the Zorich algorithm has finite mass. The ergodic properties of the continued fraction map and of the related Teichmüller flow (see Section 4.2 for its definition) have been studied in detail [V5, V6, V7, Z2, Z4, Fo2].

0.2 The cohomological equation

Our study of the cohomological equation for i.e.m.'s has been prompted by Forni's [Fo1] celebrated paper on the cohomological equation associated to linear flows on surfaces of higher genus. Let us first state our main theorem.

We will denote $BV(\sqcup I_\alpha)$ (resp. $BV_*(\sqcup I_\alpha)$) the space of functions φ whose restriction to each of the intervals I_α is a function of bounded variation (resp. the hyperplane of $BV(\sqcup I_\alpha)$ made of functions whose integral on the disjoint union $\sqcup I_\alpha$ vanishes). We will also denote $BV_*^1(\sqcup I_\alpha)$ the space of functions φ which are absolutely continuous on each I_α and whose first derivative belongs to $BV_*(\sqcup I_\alpha)$.

Our first main result can be stated as follows:

Theorem A. *Let T be an interval exchange map with the Keane property and of Roth type. Let $\Phi \in BV_*^1(\sqcup I_\alpha)$. There exists a function χ constant on each interval I_α and a bounded function Ψ such that*

$$\Psi - \Psi \circ T = \Phi - \chi .$$

To make the above statement precise we need to define Roth type i.e.m.'s. This is the subject of section 1.3 below. For the time being we will content ourselves with briefly describing the three conditions which a Roth type i.e.m. must satisfy:

- (a) The first condition is a growth rate condition for the matrices appearing in an accelerated version of the Zorich continued fraction algorithm (see Section 1.2.4 for details). This condition is the precise analogue of the third of the equivalent arithmetical characterizations of Roth type irrational numbers given above.
- (b) The second condition is a spectral condition which guarantees unique ergodicity of Roth type i.e.m.'s. This condition does not follow from condition (a) (see Appendix B for a counterexample, and also [Ch]) but is automatically satisfied if the i.e.m. is of *constant type* (i.e. the matrices considered in (a) have bounded norm).
- (c) The third and last condition is a coherence condition.

The second main result of this paper is

Theorem B. *Roth type interval exchange maps form a full measure set in the space of all interval exchange maps.*

Obviously, Theorem A is closely connected to Forni's fundamental theorem [Fo1] on the cohomological equation for area-preserving vector fields on surfaces. By singular suspension ("zippered rectangles", see Section 3), one obtains from an interval exchange map an area-preserving flow on a singular flat surface. Forni develops some Fourier analysis tools in this context, which allows him to solve the cohomological equation for almost every direction; our methods are completely different. He works in the Sobolev scale and his methods allow to lose no more than $3 + \varepsilon$ derivatives (for every $\varepsilon > 0$) [Fo3]. Our loss is smaller and we get an explicit Diophantine condition. On the other hand, given a singular flat surface, we do not know if almost every direction leads to a Roth type interval exchange map.

The connection with singular flat surfaces explains the type of regularity we introduce when we consider the cohomological equation for more regular data: we still allow discontinuities for Φ at the endpoints for the I_α ; on the other hand, we require the solution Ψ to be continuous on all of I . New linear conditions on Φ appear by integration of the cohomological equation. See Section 3 below for the precise statements.

When the singular suspension of an i.e.m. T is an invariant foliation for a pseudo-Anosov diffeomorphism, the continued fraction expansion of T is eventually periodic. This implies a strong version of condition (a).

Conditions (b) and (c) are also satisfied. Hence T is of Roth type (even of "bounded type") and Theorem A applies. This answers positively a question

raised by Forni ([Fo1], p. 342).

0.3 Summary of the contents

In the first section we introduce interval exchange maps and we develop the continued fraction algorithms to an extent which allows us to introduce Roth type i.e.m.'s. The Keane property (see 1.1.6) does not only guarantee that an i.e.m. is minimal but it also implies that the Rauzy–Veech continued fraction algorithm (described in 1.2.1–1.2.3) can be iterated indefinitely. Accelerating the Rauzy–Veech map by grouping together arrows with the same name in the Rauzy diagram leads to the Zorich continued fraction algorithm (described in 1.2.4) which has the advantage of having a finite mass a.c.i.m.. On the other hand, since every name is taken infinitely many times in the sequence of arrows in the Rauzy diagram corresponding to a given i.e.m. one can produce a further acceleration of the scheme by grouping together all arrows which take all possible names but one: this leads to the algorithm we will use in the definition of Roth type i.e.m.'s given in section 1.3 and already briefly described above. The notations and the presentation of the Rauzy–Veech–Zorich algorithms follow closely the expository paper [Y].

Section 2 is devoted to the study of the cohomological equation and to the proof of our main theorem A. When T is a minimal homeomorphism of a compact space X , we know from a theorem of Gottschalk and Hedlund [GH] that a continuous function on X is a T -coboundary of some continuous function as soon as its Birkhoff sums at some point of X are bounded (see Section 2.1.1). An i.e.m. with the Keane property is minimal but not continuous. Nevertheless, a Denjoy-like construction (see Section 2.1.2) allows to apply Gottschalk–Hedlund's theorem and conclude that a continuous function whose Birkhoff sums at some point are bounded is the T -coboundary of a bounded function. The next step in the proof is the reduction of the control of a general Birkhoff sum to the control of those *special* Birkhoff sums which are obtained by considering the return times of the point under iteration of the map (Section 2.2). These can be conveniently analyzed using the continued fraction. The estimates of these special Birkhoff sums for functions of bounded variation are given in Section 2.3 and the proof of the theorem is completed in Section 2.4.

In Section 3 we first recall how to construct a linear flow on a translation surface starting from an i.e.m. and certain suspension data (Sections 3.1–3.3). Then we relate the discrete cohomological equation for i.e.m.'s to the continuous one for the vertical (area-preserving) vector field constructed by suspension: this allows us to consider more regular data (i.e. belonging to the space BV_*^r of functions whose r -th derivative has bounded variation on each I_α and all intermediate derivatives have zero mean on $\sqcup I_\alpha$). We prove that for those the

loss of differentiability in solving the cohomological equation is the same as for functions in BV_*^1 (Section 3.4).

Section 4 is devoted to the proof of theorem B, i.e. that Roth type i.e.m.'s have full measure. To this purpose we need to describe how the Rauzy–Veech map acts at the level of the suspension data (Section 4.1). Then we combine the continued fraction algorithm (in Zorich form) with the Teichmüller flow in order to get a version which is normalized w.r.t. scales (Section 4.2). A careful comparison between the a.c.i.m. for the continued fraction map and the Lebesgue measure is carried out in Section 4.3 whereas in Section 4.4, following Zorich [Z1] we prove the integrability condition on the matrices needed to apply Oseledets multiplicative ergodic theorem. Then conditions (b) and (c) in the definition of Roth type i.e.m.'s have full measure (Section 4.5) by Oseledets theorem and the almost sure existence of a spectral gap proved by Veech in [V3]. Showing that condition (a) also has full measure requires more work and more precise informations on the combinatorics of the continued fraction map. This is summarized in a Proposition stated in Section 4.6 and proved in Section 4.8 whereas in Section 4.3 we show how to conclude the proof of theorem B by putting together the results of Sections 4.3 and 4.7 and applying a Borel–Cantelli argument.

The two appendices are devoted to the construction of concrete examples of Roth type i.e.m.'s and to the construction of non–uniquely ergodic i.e.m.'s satisfying condition (a) in Roth type (but of course not condition (b)).

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1. The continued fraction algorithm for interval exchange maps

1.1 Interval exchange maps

1.1.1

An interval exchange map (i.e.m.) is determined by combinatorial data on one side, length data on the other side.

The combinatorial data consists of a finite set \mathcal{A} of names for the intervals and of two bijections (π_0, π_1) from \mathcal{A} onto $\{1, \dots, d\}$ (where d is the cardinality of \mathcal{A}): these indicate in which order the intervals are met before and after the map.

The length data $(\lambda_\alpha)_{\alpha \in \mathcal{A}}$ give the length $\lambda_\alpha > 0$ of the corresponding interval. More precisely, we set

$$\begin{aligned} I_\alpha &:= [0, \lambda_\alpha) \times \{\alpha\}, \\ \lambda^* &:= \sum_{\alpha \in \mathcal{A}} \lambda_\alpha, \\ I &:= [0, \lambda^*). \end{aligned}$$

We then define, for $\varepsilon = 0, 1$, a bijection j_ε from $\sqcup_{\alpha \in \mathcal{A}} I_\alpha$ onto I :

$$j_\varepsilon(x, \alpha) = \sum_{\pi_\varepsilon(\beta) < \pi_\varepsilon(\alpha)} \lambda_\beta.$$

The i.e.m. T associated to these data is the bijection $T = j_1 \circ j_0^{-1}$ of I .

1.1.2 If $\mathcal{A}, \pi_0, \pi_1, \lambda_\alpha$ are as above and $\chi : \mathcal{A}' \rightarrow \mathcal{A}$ is a bijection, we can define a new set of data by

$$\begin{aligned} \pi'_\varepsilon &= \pi_\varepsilon \circ \chi, \quad \varepsilon = 0, 1, \\ \lambda'_{\alpha'} &= \lambda_{\chi(\alpha')}, \quad \alpha' \in \mathcal{A}'. \end{aligned}$$

Obviously, the new i.e.m. T' determined by these data is the same, except for names, than the old one. In particular, we could restrict to consider *normalized combinatorial data* characterized by

$$\mathcal{A} = \{1, \dots, d\}, \quad \pi_0 = \text{id}_{\mathcal{A}}.$$

However, this leads to later to more complicated formulas in the continued fraction algorithm because the basic operations on i.e.m. 's do not preserve normalization.

1.1.3 Given combinatorial data $(\mathcal{A}, \pi_0, \pi_1)$, we set, for $\alpha, \beta \in \mathcal{A}$

$$\Omega_{\alpha, \beta} = \begin{cases} +1 & \text{if } \pi_0(\beta) > \pi_0(\alpha), \pi_1(\beta) < \pi_1(\alpha), \\ -1 & \text{if } \pi_0(\beta) < \pi_0(\alpha), \pi_1(\beta) > \pi_1(\alpha), \\ 0 & \text{otherwise.} \end{cases}$$

The matrix $\Omega = (\Omega_{\alpha, \beta})_{(\alpha, \beta) \in \mathcal{A}^2}$ is antisymmetric.

Let $(\lambda_\alpha)_{\alpha \in \mathcal{A}}$ be the length data and let T be the associated i.e.m. . For $\alpha \in \mathcal{A}$, $y \in j_0(I_\alpha)$, we have

$$T(y) = y + \delta_\alpha,$$

where the *translation vector* $\delta = (\delta_\alpha)_{\alpha \in \mathcal{A}}$ is related to the *length vector* $\lambda = (\lambda_\alpha)_{\alpha \in \mathcal{A}}$ by:

$$\delta = \Omega \lambda.$$

1.1.4 There is a *canonical involution* \mathcal{I} acting on the set of combinatorial data which exchange π_0 and π_1 . For any set $(\lambda_\alpha)_{\alpha \in \mathcal{A}}$ of length data, the intervals I_α, I are unchanged, but j_0 and j_1 are exchanged and T is replaced by T^{-1} . The matrix Ω is replaced by $-\Omega$ and the translation vector δ by $-\delta$. Observe that \mathcal{I} does not respect the combinatorial normalization.

1.1.5 In the following, we will always consider only combinatorial data $(\mathcal{A}, \pi_0, \pi_1)$ which are *admissible*, meaning that for all $k = 1, 2, \dots, d-1$, we have

$$\pi_0^{-1}(\{1, \dots, k\}) \neq \pi_1^{-1}(\{1, \dots, k\}).$$

Indeed, if we had $\pi_0^{-1}(\{1, \dots, k\}) = \pi_1^{-1}(\{1, \dots, k\})$ for some $k < d$, for any length data $(\lambda_\alpha)_{\alpha \in \mathcal{A}}$, the interval I would decompose into two disjoint invariant subintervals and the study of the dynamics would be reduced to simpler combinatorial data.

1.1.6 The Keane property Let T be an i.e.m. defined by combinatorial data $(\mathcal{A}, \pi_0, \pi_1)$ and length data $(\lambda_\alpha)_{\alpha \in \mathcal{A}}$.

Definition A connexion for T is a triple (α, β, m) where $\alpha, \beta \in \mathcal{A}$, $\pi_0(\beta) > 1$, m is a positive integer, and

$$T^m(j_0(0, \alpha)) = j_0(0, \beta).$$

We say that T has the Keane property if there is no connexion for T .

It turns out that this property is the appropriate notion of irrationality for i.e.m. . The following results are due to Keane ([Ke1]):

- An i.e.m. with Keane's property is minimal (i.e. all orbits are dense).
- If the length data are rationally independent (and the combinatorial data are admissible) then T has Keane's property.

1.2 The continued fraction algorithm

1.2.1 The basic operation. (Rauzy [Ra], Veech [V2]) Let T be an i.e.m. defined by combinatorial data $(\mathcal{A}, \pi_0, \pi_1)$ and length data $(\lambda_\alpha)_{\alpha \in \mathcal{A}}$. We assume as always that the combinatorial data are admissible.

We denote by α_0, α_1 the (distinct) elements of \mathcal{A} such that

$$\pi_0(\alpha_0) = \pi_1(\alpha_1) = d.$$

Observe that if $\lambda_{\alpha_0} = \lambda_{\alpha_1}$, the triple $(\alpha_0, \alpha_1, 1)$ is a connexion and T has not the Keane property.

We now assume that $\lambda_{\alpha_0} \neq \lambda_{\alpha_1}$ and define $\varepsilon \in \{0, 1\}$ by

$$\lambda_{\alpha_\varepsilon} = \text{Max}(\lambda_{\alpha_0}, \lambda_{\alpha_1}).$$

We set

$$\begin{aligned} \hat{\lambda}^* &= \lambda^* - \lambda_{\alpha_{1-\varepsilon}}, \\ \hat{I} &= [0, \hat{\lambda}^*) \subset I, \end{aligned}$$

and define $\hat{T} : \hat{I} \rightarrow \hat{I}$ to be the first return map of T in \hat{I} .

When $\varepsilon = 0$ we have

$$\hat{T}(y) = \begin{cases} T(y) & \text{if } y \notin j_0(I_{\alpha_1}), \\ T^2(y) & \text{if } y \in j_0(I_{\alpha_1}). \end{cases}$$

When $\varepsilon = 1$ we have similarly

$$\hat{T}^{-1}(y) = \begin{cases} T^{-1}(y) & \text{if } y \notin j_1(I_{\alpha_0}), \\ T^{-2}(y) & \text{if } y \in j_1(I_{\alpha_0}). \end{cases}$$

In both cases, it appears that \hat{T} is again an interval exchange map which can be defined using the same alphabet \mathcal{A} . The length data for \hat{T} are given by

$$\begin{aligned} \hat{\lambda}_\alpha &= \lambda_\alpha \text{ if } \alpha \neq \alpha_\varepsilon, \\ \hat{\lambda}_{\alpha_\varepsilon} &= \lambda_{\alpha_\varepsilon} - \lambda_{\alpha_{1-\varepsilon}}. \end{aligned}$$

The combinatorial data $(\hat{\pi}_0, \hat{\pi}_1)$ for \hat{T} are given by

$$\hat{\pi}_\varepsilon = \pi_\varepsilon$$

and

$$\hat{\pi}_{1-\varepsilon}(\alpha) = \begin{cases} \pi_{1-\varepsilon}(\alpha) & \text{if } \pi_{1-\varepsilon}(\alpha) \leq \pi_{1-\varepsilon}(\alpha_\varepsilon), \\ \pi_{1-\varepsilon}(\alpha) + 1 & \text{if } \pi_{1-\varepsilon}(\alpha_\varepsilon) < \pi_{1-\varepsilon}(\alpha) < d, \\ \pi_{1-\varepsilon}(\alpha_\varepsilon) + 1 & \text{if } \pi_{1-\varepsilon}(\alpha) = d. \end{cases}$$

We rewrite the relation between old and new length data as

$$\lambda = V\hat{\lambda},$$

where

$$V = \mathbf{I} + E_{\alpha_\varepsilon \alpha_{1-\varepsilon}}$$

has now non negative integer coefficients and belongs to the group $SL(\mathbb{Z}^A)$. We also write

$$(\hat{\pi}_0, \hat{\pi}_1) = R_\varepsilon(\pi_0, \pi_1)$$

and observe that these new combinatorial data are admissible.

1.2.2 Rauzy diagrams Let \mathcal{A} be an alphabet. We define an oriented graph as follows. The vertices are the admissible pairs (π_0, π_1) . Each vertex (π_0, π_1) is the starting point of exactly two arrows with endpoints at $R_0(\pi_0, \pi_1)$ and $R_1(\pi_0, \pi_1)$. The arrow connecting (π_0, π_1) to $R_\varepsilon(\pi_0, \pi_1)$ is said to be of *type* ε .

The operations R_0, R_1 are obviously invertible. Therefore each vertex is also the endpoint of exactly two arrows, one of each type.

To each arrow in the graph, we associate a *name* in \mathcal{A} : it is the element α_ε such that $\pi_\varepsilon(\alpha_\varepsilon) = d$ (where (π_0, π_1) is the starting point of the arrow and ε is its type). The element $\alpha_{1-\varepsilon}$ will then be called the *secondary name* of this arrow.

A *Rauzy diagram* is a connected component of this oriented graph.

Obviously, the *Rauzy operations* R_0, R_1 commute with change of names (see 1.2). Up to change of names, there is only one Rauzy diagram with $d = \text{card } \mathcal{A} = 2$, and one with $d = \text{card } \mathcal{A} = 3$.

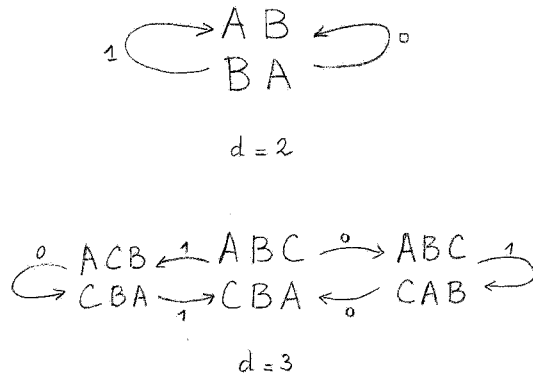


Fig. 1 Rauzy diagrams

In the diagrams above the pair (π_0, π_1) is denoted by the symbol $\pi_0^{-1}(1) \dots \pi_0^{-1}(d)$ $\pi_1^{-1}(1) \dots \pi_1^{-1}(d)$. For $d = \text{card } \mathcal{A} = 4$ there are 2 distinct Rauzy diagrams:

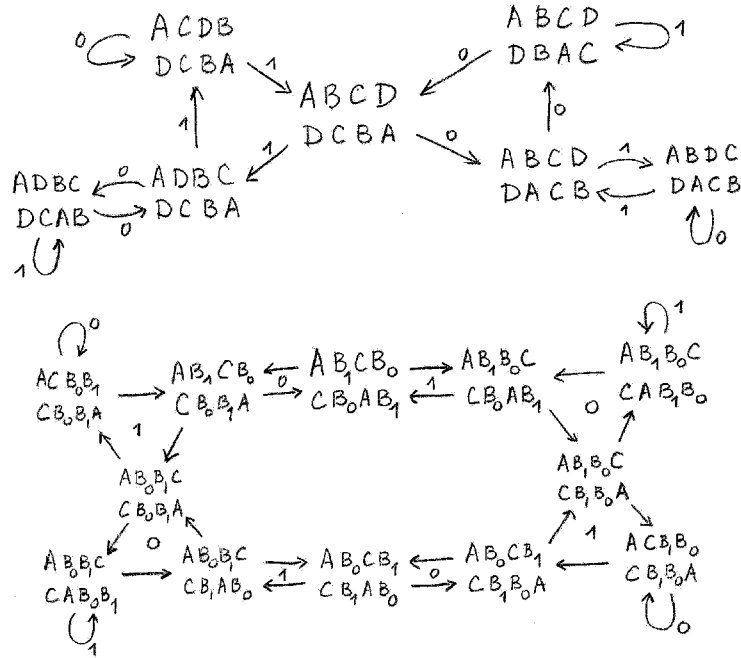


Fig. 2 Rauzy diagrams, $d=4$

In each of these diagrams, the symmetry with respect to the vertical axis corresponds to the action of the canonical involution.

In the last diagram, there is a further symmetry with respect to the center of the diagram, which corresponds to the exchange of the names B_0, B_1 . This is a monodromy phenomenon: to each admissible pair (π_0, π_1) , one can associate the permutation $\pi := \pi_1 \circ \pi_0^{-1}$ of $\{1, \dots, d\}$, which is invariant under change of names. When we identify vertices with the same permutation, we obtain a *reduced Rauzy diagram* and we have a covering map from the Rauzy diagram onto the reduced Rauzy diagram.

In the first three examples above, the covering map is an isomorphism. In the last example, the degree of the covering map is 2 and the reduced Rauzy diagram is

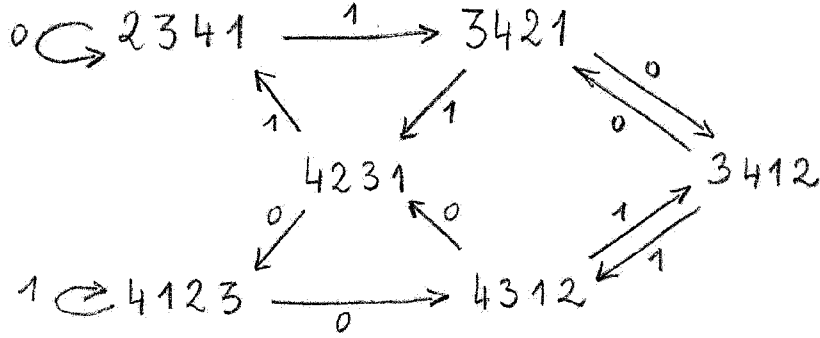


Fig. 3 Reduced Rauzy diagram, $d=4$.

where π is denoted by $(\pi^{-1}(1), \dots, \pi^{-1}(d))$.

1.2.3 The Rauzy–Veech algorithm Let T be an i.e.m. with admissible combinatorial data. If T has Keane’s property, the basic operation is defined for T and it is immediate to check that the new i.e.m. \hat{T} again has Keane’s property. Therefore we can iterate the basic operation and generate a sequence $(T^{(n)})_{n \geq 0}$ of i.e.m. ’s (with $T^{(0)} = T$). We will denote $(\pi_0^{(n)}, \pi_1^{(n)})$ the combinatorial data of $T^{(n)}$, by $(\lambda_\alpha^{(n)})_{\alpha \in \mathcal{A}}$ its length data, by $\gamma^{(n)}$ the arrow in the Rauzy diagram connecting $(\pi_0^{(n-1)}, \pi_1^{(n-1)})$ to $(\pi_0^{(n)}, \pi_1^{(n)})$, by $V^{(n)}$ the matrix relating $\lambda^{(n-1)}$ to $\lambda^{(n)}$ through

$$\lambda^{(n-1)} = V^{(n)} \lambda^{(n)} .$$

Conversely, it is not difficult to check that when T has a connexion, the algorithm has to stop because one runs at some point in the equality case $\lambda_{\alpha_0} = \lambda_{\alpha_1}$ in the basic operation.

Proposition Each name in \mathcal{A} is taken infinitely many times by the sequence of arrows $(\gamma^{(n)})_{n > 0}$.

Proof. Let \mathcal{A}' be the set of names which are taken infinitely many times and let $\mathcal{A}'' = \mathcal{A} \setminus \mathcal{A}'$. Replacing T by some $T^{(N)}$, we can assume that names in \mathcal{A}'' are not taken at all. Then the lengths $\lambda_\alpha^{(n)}$, $\alpha \in \mathcal{A}''$, do not depend on n . But then

elements $\alpha \in \mathcal{A}''$ can only appear as secondary names at most finitely many times. Replacing again T by some $T^{(N)}$, we can assume that secondary names are never in \mathcal{A}'' . Then the sequences $(\pi_\varepsilon^{(n)}(\alpha))_{n>0}$, for $\varepsilon \in \{0, 1\}$, $\alpha \in \mathcal{A}''$, are non decreasing and we can assume (replacing once again T by some $T^{(N)}$) that they are constant.

We now claim that we must have $\pi_\varepsilon^{(0)}(\alpha'') < \pi_\varepsilon^{(0)}(\alpha')$ for all $\alpha'' \in \mathcal{A}''$, $\alpha' \in \mathcal{A}'$ and $\varepsilon \in \{0, 1\}$. Because the pair $(\pi_0^{(0)}, \pi_1^{(0)})$ is admissible, this implies $\mathcal{A}' = \mathcal{A}$. To prove the claim, assume that there exist $\alpha' \in \mathcal{A}'$, $\alpha'' \in \mathcal{A}''$, $\varepsilon \in \{0, 1\}$ with $\pi_\varepsilon^{(0)}(\alpha') < \pi_\varepsilon^{(0)}(\alpha'')$. As $\pi_\varepsilon^{(n)}(\alpha'') = \pi_\varepsilon^{(0)}(\alpha'')$ for all $n \geq 0$, we can never have $\pi_\varepsilon^{(n)}(\alpha') = d$ for some $n > 0$. By definition of \mathcal{A}' , there must exist $n \geq 0$ such that $\pi_{1-\varepsilon}^{(n)}(\alpha') = d$; but then $\pi_\varepsilon^{(n+1)}(\alpha'') \neq \pi_\varepsilon^{(0)}(\alpha'')$, which gives a contradiction. \square

1.2.4 The Zorich algorithm and its accelerations

When $d = 2$, setting $x = \lambda_B/\lambda_A$, the basic operation reduces to the well-known map

$$g(x) = \begin{cases} \frac{x}{1-x} & \text{for } 0 < x < 1/2, \\ \frac{1-x}{x} & \text{for } 1/2 < x < 1, \end{cases}$$

with a parabolic fixed point at 0. There is a unique absolutely continuous invariant measure, namely dx/x , but this measure is infinite. On the other hand, the Gauss map generating the continued fraction algorithm has $dx/(1+x)$ as a finite a.c.i.m. .

For i.e.m.'s with more intervals, identifying i.e.m.'s with proportional length data (and the same combinatorial data), Veech has shown [V2] that there exists again for the basic operation a unique absolutely continuous invariant measure. Again this measure is infinite. Zorich has discovered ([Z1]) how to concatenate several steps of the basic operations in order to get a finite a.c.i.m. .

Let T be an i.e.m. with Keane's property, $T^{(n)}$, $\gamma^{(n)}$, $V^{(n)}$ the data generated by the iteration of the basic operation. Let also $1 \leq D < d$. We define inductively an increasing sequence $n_D(k) = n_D(k, T)$ by setting $n_D(0) = 0$ and: $n_D(k+1)$ is the largest integer such that no more than D names are taken by the $\gamma^{(n)}$, for $n_D(k) < n \leq n_D(k+1)$.

The sequence is well defined because of the Proposition above.

Obviously, for $1 < D < d$, $(n_D(k))_{k \geq 0}$ is a subsequence of $(n_{D-1}(l))_{l \geq 0}$.

We will define, for $k > 0$

$$Z_{(D)}(k) = V^{(n_D(k-1)+1)} \dots V^{(n_D(k))} .$$

The case $D = 1$ is the one considered by Zorich ([Z1]). We will on the other hand be interested in the case $D = d - 1$.

When the context is clear, we will simply write $Z(k)$ for $Z_{(d-1)}(k)$ and $T^{(k)}$ for $T^{(n_{d-1}(k))}$, $\lambda^{(k)}$ for its length data. With these notations, we have

$$\lambda^{(k)} = Z(k+1)\lambda^{(k+1)} .$$

We will also set, for $k < l$

$$Q(k, l) = Z(k+1) \cdots Z(l) ,$$

in order to have

$$\lambda^{(k)} = Q(k, l)\lambda^{(l)} .$$

We will also write $Q(l)$ for $Q(0, l)$. The coefficients $Q_{\alpha\beta}(k, l)$ have the following interpretation. Let $I^{(k)} = \sqcup_{\alpha \in \mathcal{A}} j_0(I_\alpha^{(k)})$ be the domain of $T^{(k)}$. For $l \geq k$, we have $I^{(l)} \subset I^{(k)}$ and $T^{(l)}$ is the first return map of $T^{(k)}$ in $I^{(l)}$. Then, the non negative integer $Q_{\alpha\beta}(k, l)$ is the time spent in $j_0(I_\alpha^{(k)})$ by any point of $j_0(I_\beta^{(l)})$ until it returns in $I^{(l)}$.

We will also introduce

$$Q_\beta(k, l) = \sum_{\alpha \in \mathcal{A}} Q_{\alpha\beta}(k, l) ,$$

which is the return time in $I^{(l)}$ for points in $I_\beta^{(l)}$.

The following Lemma is the main reason to choose $D = d - 1$ rather than $D = 1$.

Lemma. *Let T satisfy Keane's condition. Assume that*

$$l \geq \begin{cases} k + 2d - 3 & \text{if } d \geq 3 \\ k + 2 & \text{if } d = 2. \end{cases}$$

Then, for all $\alpha, \beta \in \mathcal{A}$, we have $Q_{\alpha\beta}(k, l) > 0$.

Proof. Replacing T by $T^{(k)}$, it is sufficient to consider the case $k = 0$. For $r \geq 0$, set

$$\hat{Q}(r) = V^{(1)} \cdots V^{(r)} ;$$

as the diagonal terms of the V matrices are equal to 1 (and all the terms are non negative) we have

$$\hat{Q}_{\alpha\beta}(r) > 0 \Rightarrow \hat{Q}_{\alpha\beta}(r+1) > 0 .$$

Fix $\alpha, \beta \in \mathcal{A}$. We will construct a sequence of *distinct* indices $\alpha_1 = \alpha, \alpha_2, \dots, \alpha_s = \beta$ and integers $r_1 = 0 < r_2 < \dots < r_s$ such that

$$\hat{Q}_{\alpha_1\alpha_j}(r) > 0 \text{ for } r \geq r_j .$$

If $\alpha = \beta$, $s = 1$, $r_1 = 0$ and the property is satisfied. Otherwise, let r_2 be the smallest positive integer such that the name of $\gamma^{(r_2)}$ is α_1 , and let α_2 be the secondary name of $\gamma^{(r_2)}$; we have $\alpha_2 \neq \alpha_1$ and $V_{\alpha_1\alpha_2}^{(r_2)} = 1$ hence $\hat{Q}_{\alpha_1\alpha_2}(r) > 0$ for $r \geq r_2$.

Assume that $\alpha_1, \dots, \alpha_j, r_1, \dots, r_j$ have been constructed, with $\beta \neq \alpha_l$ for $1 \leq l \leq j$. Let r'_j be the smallest integer $> r_j$ such that the name of $\gamma^{(r'_j)}$ does not belong to $\{\alpha_1, \dots, \alpha_j\}$ and let r_{j+1} be the smallest integer $> r'_j$ such that the name of $\gamma^{(r_{j+1})}$ belongs to $\{\alpha_1, \dots, \alpha_j\}$; let α_{j+1} be the secondary name of $\gamma^{(r_{j+1})}$. Then α_{j+1} is the name of $\gamma^{(r_{j+1}-1)}$ and therefore is distinct from $\alpha_1, \dots, \alpha_j$. By construction, we have, for some $1 \leq l \leq j$

$$V_{\alpha_l\alpha_{j+1}}^{(r_{j+1})} = 1 ,$$

and also

$$\hat{Q}_{\alpha_l\alpha_{j+1}}(r_{j+1} - 1) > 0$$

because $r_{j+1} > r_l$. We conclude that

$$\hat{Q}_{\alpha_l\alpha_{j+1}}(r) > 0 \text{ for } r \geq r_{j+1} .$$

At some point we will obtain $\alpha_s = \beta$. It remains to see how many steps of the accelerated Zorich algorithm (with $D = d - 1$) are needed to attain r_s . Obviously, we have $r_2 \leq n_{d-1}(1) + 1$. Then, for $2 \leq j < d - 1$, we have

$$\begin{aligned} r'_j &\leq n_{d-1}(2j - 2) , \\ r_{j+1} &\leq n_{d-1}(2j - 1) . \end{aligned}$$

Finally, when $s = d > 2$, we have

$$\begin{aligned} r'_{d-1} &\leq n_{d-1}(2d - 4) + 1 , \\ r_d &\leq n_{d-1}(2d - 3) . \end{aligned}$$

□

1.3 Roth-type interval exchange maps

Roth-type i.e.m. should satisfy Keane's condition so that the continued fraction algorithm is defined, and three further conditions which are now explained.

1.3.1 Size of the Z matrices

Take $D = d - 1$ in 1.2.4. We will first ask for the Z matrices to be not too big in the following sense:

(a) *for every $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that for all $k \geq 0$ we have*

$$\|Z(k+1)\| \leq C_\varepsilon \|Q(k)\|^\varepsilon .$$

When $d = 2$, this amounts exactly to the classical Roth type approximation property for an irrational number θ : for all $\varepsilon > 0$, there exists $\gamma_\varepsilon > 0$ such that for all rational p/q one has

$$|\theta - p/q| \geq \gamma_\varepsilon q^{-2-\varepsilon} .$$

In terms of the convergents $(p_k/q_k)_{k \geq 0}$ of θ with partial quotients $(a_k)_{k \geq 1}$, this is equivalent to have, for all $\varepsilon > 0$

$$a_{k+1} = O(q_k^\varepsilon) ,$$

which explains our terminology.

We can reformulate (a) in terms of the lengths $\lambda_\alpha^{(k)}$. It is convenient here to take as norm of a matrix the sum of all coefficients (in absolute value; the matrices that we consider here have nonnegative entries).

Proposition *We have always, for $k \geq 0$*

$$\text{Max}_{\alpha \in \mathcal{A}} \lambda_\alpha^{(k)} \geq \lambda^* \|Q(k)\|^{-1} \geq \text{Min}_{\alpha \in \mathcal{A}} \lambda_\alpha^{(k)} .$$

Condition (a) is equivalent to the following converse estimate: for all $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that

$$\text{Max}_{\alpha \in \mathcal{A}} \lambda_\alpha^{(k)} \leq C_\varepsilon \text{Min}_{\alpha \in \mathcal{A}} \lambda_\alpha^{(k)} \|Q(k)\|^\varepsilon .$$

Proof. The first estimate follows from

$$\lambda^* = \sum_{\alpha \in \mathcal{A}} \lambda_\alpha^{(0)} = \sum_{\beta \in \mathcal{A}} Q_\beta(k) \lambda_\beta^{(k)} .$$

Assume (a) is satisfied. Let l be equal to $k + 2d - 3$ (if $d \geq 3$) or $k + 2$ (if $d = 2$) as in the Lemma in 1.2.4. We have

$$\|Q(k, l)\| \leq C'_\varepsilon \|Q(k)\|^\varepsilon$$

for all $\varepsilon > 0$ (with an appropriate constant C'_ε).

This gives

$$\text{Max}_{\alpha \in \mathcal{A}} \lambda_{\alpha}^{(k)} \leq C'_{\varepsilon} \|Q(k)\|^{\varepsilon} \text{Max}_{\alpha \in \mathcal{A}} \lambda_{\alpha}^{(l)} .$$

On the other hand, the Lemma 1.2.4 gives

$$\text{Min}_{\alpha \in \mathcal{A}} \lambda_{\alpha}^{(k)} \geq \text{Max}_{\alpha \in \mathcal{A}} \lambda_{\alpha}^{(l)} ,$$

giving the required estimate. Assume now that the estimate of the Proposition holds. We have always

$$\text{Max}_{\alpha \in \mathcal{A}} \lambda_{\alpha}^{(k)} \geq d^{-1} \|Z(k+1)\| \text{Min}_{\alpha \in \mathcal{A}} \lambda_{\alpha}^{(k+1)} .$$

On the other hand, by definition of the Z matrices, there exists $\alpha_0 \in \mathcal{A}$ such that

$$\lambda_{\alpha_0}^{(k)} = \lambda_{\alpha_0}^{(k+1)} .$$

But we have

$$\begin{aligned} \lambda_{\alpha_0}^{(k+1)} &\leq \text{Max}_{\alpha \in \mathcal{A}} \lambda_{\alpha}^{(k+1)} \\ &\leq C_{\varepsilon} \text{Min}_{\alpha \in \mathcal{A}} \lambda_{\alpha}^{(k+1)} \|Q(k+1)\|^{\varepsilon} \\ &\leq C_{\varepsilon} d \|Z(k+1)\|^{-1} \text{Max}_{\alpha \in \mathcal{A}} \lambda_{\alpha}^{(k)} \|Q(k+1)\|^{\varepsilon} \\ &\leq C_{\varepsilon}^2 d \|Z(k+1)\|^{-1} \text{Min}_{\alpha \in \mathcal{A}} \lambda_{\alpha}^{(k)} \|Q(k)\|^{\varepsilon} \|Q(k+1)\|^{\varepsilon} \\ &\leq C_{\varepsilon}^2 d \|Z(k+1)\|^{-1} \lambda_{\alpha_0}^{(k)} \|Q(k)\|^{\varepsilon} \|Q(k+1)\|^{\varepsilon} \end{aligned}$$

which implies

$$\|Z(k+1)\| \leq C_{\varepsilon}^2 d \|Q(k)\|^{\varepsilon} \|Q(k+1)\|^{\varepsilon}$$

and allows to conclude that (a) holds . □

Remark 1. Assume condition (a) is satisfied. Set $k_0 = 2d - 3$ if $d \geq 3$, $k_0 = 2$ if $d = 2$. Following the same lines that in the last Proposition, we see that for any $\varepsilon > 0$, there exists $C_{\varepsilon} > 0$ such that for $k \geq k_0$ we have

$$\text{Min}_{\alpha, \beta \in \mathcal{A}} Q_{\alpha\beta}(k) \geq C_{\varepsilon}^{-1} \|Q(k)\|^{1-\varepsilon} .$$

On the other hand it is easy to see that, even in the case of 3 intervals this estimate does not imply condition (a).

Remark 2. Boshernitzan has defined ([Bo]) another condition which generalizes Roth condition for irrational numbers. Namely, he asks that T satisfies

Keane's condition and that the minimum distance m_n between discontinuity points of the n -th iterate T^n of T should verify

$$m_n \geq \frac{\gamma_\varepsilon^{-1}}{n^{1+\varepsilon}} .$$

He proves that this condition has full measure.

The relation between Boshernitzan's condition and condition (a) above is however not clear.

1.3.2 Spectral gap

As soon as $k \geq 2d - 3$ ($k \geq 2$ if $d = 2$), all entries in the matrix $Q(k)$ are strictly positive. It is therefore not unreasonable to expect that the positive cone is more expanded by $Q(k)$ than the other directions, in the spirit of Perron–Frobenius theorem.

However this is not automatic, as attested by the existence of minimal non uniquely ergodic i.e.m. 's (an i.e.m. satisfying Keane's condition is uniquely ergodic if and only if the image under $Q(k)$ of the positive cone converges to a ray as $k \rightarrow \infty$).

Our second condition ensures that this weird behaviour does not occur.

For each $k \geq 0$, let $\Gamma^{(k)}$ be a copy of \mathbb{R}^A . One should think of $\Gamma^{(k)}$ as the space of functions on $\sqcup_{\alpha \in A} I_\alpha^{(k)}$ which are constant on each $I_\alpha^{(k)}$. For $0 \leq k \leq l$, let $S(k, l)$ be the linear map from $\Gamma^{(k)}$ to $\Gamma^{(l)}$ whose matrix in the canonical basis is ${}^t Q(k, l)$. This can be interpreted as a special Birkhoff sum (see Section 2 below).

For $\varphi = (\varphi_\alpha)_{\alpha \in A} \in \Gamma^{(k)}$, define

$$I_k(\varphi) = \sum_{\alpha \in A} \lambda_\alpha^{(k)} \varphi_\alpha ;$$

we have then

$$I_l(S(k, l)\varphi) = I_k(\varphi) .$$

Denote by $\Gamma_*^{(k)}$ the kernel of the linear form I_k . We will ask the following:

(b) *There exists $\theta > 0, C > 0$ such that, for all $k \geq 0$, we have*

$$\|S(k) |_{\Gamma_*^{(0)}}\| \leq C \|S(k)\|^{1-\theta} = C \|Q(k)\|^{1-\theta} .$$

Observe that an i.e.m. satisfying Keane's condition and (b) must be uniquely ergodic.

In appendix B we construct i.e.m. 's which satisfy condition (a) but are *not* uniquely ergodic (see also [Ch]); therefore condition (b) is not a consequence of condition (a).

However, if instead of condition (a) we consider the stronger condition (reminding of bounded type irrational numbers):

(ã) *the sequence $Z(k)$ is bounded*

then condition (b) follows. Indeed, each $Q(k, k + 2d - 3)$ ($Q(k, k + 2)$ when $d = 2$) will contract by a definite factor < 1 the Hilbert metric of the projective positive cone.

1.3.3 Coherence

To define our third condition, we consider again the operators $S(k, l) : \Gamma^{(k)} \rightarrow \Gamma^{(l)}$. Let $\Gamma_s^{(k)}$ be the linear subspace of $\Gamma^{(k)}$ whose elements v satisfy the following: there exists $\sigma = \sigma(v) > 0$, $C = C(v) > 0$ such that, for all $l \geq k$, one has

$$\|S(k, l)v\| \leq C\|S(k, l)\|^{-\sigma}\|v\|.$$

We call $\Gamma_s^{(k)}$ the stable subspace of $\Gamma^{(k)}$. Obviously, one has $\Gamma_s^{(k)} \subset \Gamma_*^{(k)}$. On the other hand, $\Gamma_s^{(k)}$ is never reduced to 0 because it always contains the translation vector $(\delta_\alpha^{(k)})_{\alpha \in \mathcal{A}}$.

The operator $S(k, l)$ maps $\Gamma_s^{(k)}$ onto $\Gamma_s^{(l)}$. Therefore we can define a quotient operator

$$S_b(k, l) : \Gamma^{(k)}/\Gamma_s^{(k)} \rightarrow \Gamma^{(l)}/\Gamma_s^{(l)}.$$

As we have quotiented out the stable directions, it is not unreasonable to expect that the norm of the inverse of $S_b(k, l)$ is not too large. This is what our third condition is about:

(c) *for any $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that, for all $l \geq k$, we have*

$$\begin{aligned} \|[S_b(k, l)]^{-1}\| &\leq C_\varepsilon\|Q(l)\|^\varepsilon, \\ \|S(k, l)|_{\Gamma_s^{(k)}}\| &\leq C_\varepsilon\|Q(l)\|^\varepsilon. \end{aligned}$$

Remark. *The second estimate in (c) was wrongly omitted in [MMY].*

1.3.4 Roth-type interval exchange maps

We say that an i.e.m. T is of *Roth type* if it satisfies Keane's condition and conditions (a), (b), (c).

In the next Section, we will solve the cohomological equation for i.e.m. 's of Roth type. In Section 4 we will prove the following

Theorem. *Roth type interval exchange maps form a subset of full measure.*

We also observe that if an i.e.m. T satisfies Keane's condition, and its Rauzy-Veech continued fraction is eventually periodic (meaning that the path γ in

the Rauzy diagram is eventually periodic), then condition (a), (b) and (c) are automatically satisfied and therefore T is of Roth type.

2. The cohomological equation

2.1 The Theorem of Gottschalk and Hedlund

2.1.1 The statement

We recall the following theorem of Gottschalk and Hedlund. Let X be a compact topological space, f a minimal homeomorphism of X and ψ a real valued continuous function on X . Given $x_0 \in X$ and $n \geq 1$ we denote $S_n\psi(x_0)$ the Birkhoff sum $\sum_{j=0}^{n-1} \psi \circ T^j(x_0)$. Suppose that there exists a point $x_0 \in X$ and a positive constant C such that for all positive integer n one has $|S_n\psi(x_0)| \leq C$. Then the cohomological equation

$$\varphi \circ f - \varphi = \psi$$

has a continuous solution φ .

2.1.2 Application to interval exchange maps

Let T be an i.e.m. satisfying Keane's condition. Then T is minimal but not continuous. However, the following well-known construction, reminiscent of Denjoy counterexamples, allows to bypass this problem.

For $n \geq 0$, define

$$\begin{aligned} D_0(n) &= \{T^{-n}(j_0(0, \alpha)), \alpha \in \mathcal{A}, \pi_0(\alpha) > 1\}, \\ D_1(n) &= \{T^{+n}(j_1(0, \alpha)), \alpha \in \mathcal{A}, \pi_1(\alpha) > 1\}. \end{aligned}$$

It follows from the Keane property that these sets are disjoint from each other and do not contain 0.

Define an atomic measure μ by

$$\mu = \sum_{n \geq 0} \sum_{y \in D_0(n) \sqcup D_1(n)} 2^{-n} \delta_y$$

and the increasing maps $i^+, i^- : I \rightarrow \mathbb{R}$ by

$$\begin{aligned} i^-(y) &= y + \mu([0, y)), \\ i^+(y) &= y + \mu([0, y]). \end{aligned}$$

We therefore have

$$\begin{aligned} i^+(y) &< i^-(y') \quad \text{for } y < y', \\ i^+(y) &= i^-(y) \quad \text{for } y \notin \sqcup_{n \geq 0} (D_0(n) \sqcup D_1(n)), \\ i^+(y) &= i^-(y) + 2^{-n} \quad \text{for } y \in D_0(n) \sqcup D_1(n). \end{aligned}$$

We also define

$$i^-(\lambda^*) = \lambda^* + 4(d-1) = \lim_{y \nearrow \lambda^*} i^\pm(y),$$

and

$$K = i^-(I) \cup i^+(I) \cup \{i^-(\lambda^*)\} = \overline{i^-(I)} = \overline{i^+(I)}.$$

As T is minimal, K is a Cantor set whose gaps are the intervals $(i^-(y), i^+(y))$, $y \in \cup_{n \geq 0} \cup_{\varepsilon \in \{0,1\}} D_\varepsilon(n)$.

Proposition. *There is a unique continuous map $\hat{T} : K \rightarrow K$ such that $\hat{T} \circ i^+ = i^+ \circ T$ on I . Moreover, \hat{T} is a minimal homeomorphism.*

The elementary proof is left to the reader.

Let $\psi : I \rightarrow \mathbb{R}$ be a function which is continuous on each $j_0(I_\alpha)$, with finite limits at the right endpoints of each $j_0(I_\alpha)$. There is a unique continuous function $\hat{\psi} : K \rightarrow \mathbb{R}$ such that $\psi(y) = \hat{\psi} \circ i^+(y)$ for all $y \in I$. Assume that, for some $x_0 \in I$ the Birkhoff sums of ψ for T are bounded. Then the same is true for the Birkhoff sums of $\hat{\psi}$ for \hat{T} at the point $\hat{x}_0 = i^+(x_0)$. By the theorem of Gottschalk and Hedlund, there is a continuous function $\hat{\varphi} : K \rightarrow \mathbb{R}$ satisfying $\hat{\psi} = \hat{\varphi} \circ \hat{T} - \hat{\varphi}$. Define, for $y \in I$

$$\varphi(y) = \hat{\varphi} \circ i^+(y).$$

In general, φ is not continuous. However it is bounded and satisfies $\varphi \circ T - \varphi = \psi$. In the following, we will show that under appropriate circumstances certain Birkhoff sums are bounded.

2.2 Special Birkhoff sums

2.2.1

Let T be an i.e.m. satisfying Keane's condition. Denote by $T^{(k)}$ the i.e.m. obtained by the accelerated Zorich algorithm (with $D = d - 1$ in 1.2.4).

Let $\varphi : I^{(k)} \rightarrow \mathbb{R}$ be a function defined on the domain $I^{(k)}$ of $T^{(k)}$. Let also $l \geq k$. For $\beta \in \mathcal{A}$, $x \in j_0(I_\beta^{(l)})$, the return time of x into $I^{(l)}$ under iteration of $T^{(k)}$ is $Q_\beta(k, l)$. Define a function

$$S(k, l)\varphi : I^{(l)} \rightarrow \mathbb{R}$$

by the formula

$$S(k, l)(\varphi)(x) = \sum_{0 \leq i < Q_\beta(k, l)} \varphi((T^{(k)})^i(x)),$$

for $x \in j_0(I_\beta^{(l)})$. Observe that when φ is constant on each $j_0(I_\beta^{(k)})$, the same is true of $S(k, l)(\varphi)$ in $I^{(l)}$ and the corresponding linear operator has ${}^tQ(k, l)$ as matrix in the canonical basis, as anticipated in 1.3.2.

We just write $S(k)$ for $S(0, k)$.

2.2.2 Some elementary properties of the operators $S(k, l)$

2.2.2.1 For $m \geq l \geq k$ one has

$$S(k, m) = S(l, m) \circ S(k, l) .$$

2.2.2.2 The operators $S(m, n)$ preserve all regularity classes which are invariant by restriction, sum and translation.

2.2.2.3 If φ is an integrable function on $I^{(k)}$,

$$\int_{I^{(k)}} \varphi(x) dx = \int_{I^{(l)}} (S(k, l)\varphi)(x) dx .$$

2.2.2.4 The operators $S(k, l)$ commute with taking derivatives.

2.2.2.5 If the restriction of φ to each $j_0(I_\alpha^{(k)})$ is a polynomial of degree $\leq \mu$, the restriction of $S(k, l)\varphi$ to each $j_0(I_\beta^{(l)})$ is also a polynomial of degree $\leq \mu$. The case $\mu = 0$ has already been considered.

2.2.2.6 Denote by $\text{BV}(\sqcup I_\alpha^{(k)})$ the space of functions φ on $I^{(k)}$ whose restriction to each $j_0(I_\alpha^{(k)})$ has bounded variation and define

$$\text{Var } \varphi = \sum_{\alpha} \text{Var } \varphi|_{j_0(I_\alpha^{(k)})} .$$

(We do not take into account the discontinuities of φ at the discontinuity points of $T^{(k)}$). Then $S(k, l)$ sends $\text{BV}(\sqcup I_\alpha^{(k)})$ into $\text{BV}(\sqcup I_\alpha^{(l)})$ and we have

$$\text{Var } S(k, l)\varphi \leq \text{Var } \varphi .$$

2.2.3 Reduction of Birkhoff sums to special Birkhoff sums

For diffeomorphisms of the circle with irrational rotation number, when trying to estimate the Birkhoff sums of some function, it is a standard trick to consider first the ones associated to the denominators of the convergents of the rotation number. We will do the same here.

Let $\varphi : I \rightarrow \mathbb{R}$ be a function, $x \in I$, and $N > 0$. We want to compute the Birkhoff sums

$$S_N \varphi(x) = \sum_{i=0}^{N-1} \varphi(T^i(x))$$

(with $T = T^{(0)}$).

We first replace x by the point in the orbit $\{x, T(x), \dots, T^{N-1}(x)\}$ which is closest to 0 and cut the Birkhoff sum into two parts (one for T and the other for T^{-1}). Let us assume to keep notations simple that x is actually closest to the origin.

Let $k \geq 0$ be the largest integer such that at least one of the points $T(x), \dots, T^{N-1}(x)$ belongs to $I^{(k)}$; because $T^{(k)}$ is the first return map into $I^{(k)}$, these points are precisely $T^{(k)}(x), (T^{(k)})^2(x), \dots, (T^{(k)})^{b(k)}(x)$ for some integer $b(k) > 0$. Moreover, as none of these points belongs to $I^{(k+1)}$ we must have

$$b(k) < \text{Max}_{\beta \in \mathcal{A}} \sum_{\alpha \in \mathcal{A}} Z_{\alpha\beta}(k+1),$$

the right hand term being the largest return time of $T^{(k)}$ into $I^{(k+1)}$.

We set $x_k = x$, $x_{k-1} = (T^{(k)})^{b(k)}(x)$ and define inductively $b(l)$ and x_l for $0 \leq l \leq k$.

The point x_l has the property that it belongs to $I^{(l)}$ and none of the points $T(x_l), \dots, T^N(x)$ belongs to $I^{(l+1)}$. Those who belong to $I^{(l)}$ are $T^{(l)}(x_l), \dots, (T^{(l)})^{b(l)}(x_l)$ for some integer $b(l) \geq 0$. We have

$$b(l) < \text{Max}_{\beta \in \mathcal{A}} \sum_{\alpha \in \mathcal{A}} Z_{\alpha\beta}(l+1),$$

We define $x_{l-1} = (T^{(l)})^{b(l)}(x_l)$. The process stops when $x_l = T^N(x)$ (or $l = 0$). From this construction it is obvious that we have

$$S_N \varphi(x) = \sum_{l=0}^k \sum_{0 \leq i < b(l)} S(l) \varphi((T^{(l)})^i(x_l))$$

which in particular implies, if φ is bounded:

$$|S_N \varphi(x)| \leq \sum_{l=0}^k \|Z(l+1)\| \|S(l)\varphi\|_{L^\infty},$$

where $\|Z(l+1)\| = \text{Max}_{\beta \in \mathcal{A}} \sum_{\alpha \in \mathcal{A}} Z_{\alpha\beta}(l+1)$.

In particular, if we are able to show that for some $\omega > 0$ we have

$$\|S(l)\varphi\|_{L^\infty} \leq C \|Q(l)\|^{-\omega} \|\varphi\|,$$

and condition (a) in 1.3.1 is satisfied, then the Birkhoff sums of φ will be bounded.

2.3 Estimates for functions of bounded variation

2.3.1

Denote by $BV_*(\sqcup I_\alpha^{(k)})$ the subspace of $BV(\sqcup I_\alpha^{(k)})$ formed by the functions of mean 0. The operator $S(k, l)$ sends this subspace into $BV_*(\sqcup I_\alpha^{(l)})$.

Let $\varphi \in BV_*(\sqcup I_\alpha^{(k)})$. We write

$$S(k, k+1)(\varphi) = \varphi_{k+1} + \chi_{k+1} ,$$

with $\chi_{k+1} \in \Gamma_*^{(k+1)}$ and φ_{k+1} of mean zero on *each* $j_0(I_\alpha^{(k+1)})$. Then we go on with:

$$S(j, j+1)(\varphi_j) = \varphi_{j+1} + \chi_{j+1}$$

with $\chi_{j+1} \in \Gamma_*^{(j+1)}$ and φ_{j+1} of mean zero on *each* $j_0(I_\alpha^{(j+1)})$. We obtain, for $l > k$

$$S(k, l)\varphi = S(l-1, l)\varphi_{l-1} + \sum_{k < j < l} S(j, l)\chi_j .$$

2.3.2

As φ_j differs from $S(k, j)\varphi$ by a function in $\Gamma_*^{(j)}$ and has mean zero on each $j_0(I_\alpha^{(j)})$ we have (see 2.2.2.6)

$$\|\varphi_j\|_{L^\infty} \leq \text{Var } \varphi_j \leq \text{Var } \varphi .$$

On the other hand, we have

$$\begin{aligned} \|\chi_j\|_{L^\infty} &\leq \|S(j-1, j)\varphi_{j-1}\|_{L^\infty} \\ &\leq \|Z(j)\| \|\varphi_{j-1}\|_{L^\infty} \end{aligned}$$

(with $\varphi_{j-1} = \varphi$ when $j = k+1$). We obtain therefore

$$\begin{aligned} \|S(k, l)\varphi\|_{L^\infty} &\leq \|Z(k+1)\| \|S(k+1, l)|_{\Gamma_*^{(k+1)}}\| \|\varphi_0\| \\ &\quad + \sum_{k < j < l} \|Z(j+1)\| \|S(j+1, l)|_{\Gamma_*^{(j+1)}}\| \text{Var } \varphi . \end{aligned}$$

2.3.3

We now take $k = 0$ and estimate the sum

$$\sum_{0 \leq j < l} \|Z(j+1)\| \|S(j+1, l)|_{\Gamma_*^{(j+1)}}\| ,$$

assuming that conditions (a), (b) of Section 1.3 are satisfied.

On one side we have, by condition (a), for all $\varepsilon > 0$:

$$\|Z(j+1)\| \leq C_\varepsilon \|Q(j)\|^\varepsilon .$$

To estimate $\|S(j+1, l)|_{\Gamma_\star^{(j+1)}}\|$ we distinguish two cases. We assume condition (b) of 1.3.2, which involves an exponent $1 - \theta$ with $\theta > 0$.

i) Assume first that $\|Q(j+1)\| \leq \|Q(l)\|^{\theta/d}$. As $Q(j+1)$ belongs to $\text{SL}(d, \mathbb{Z})$, we have

$$\|(Q(j+1))^{-1}\| \leq c \|Q(l)\|^{\theta \frac{d-1}{d}} .$$

Next we write

$$S(j+1, l) = S(0, l) \circ (S(0, j+1))^{-1} ,$$

and it follows from condition (b) that we have

$$\|S(j+1, l)|_{\Gamma_\star^{(j+1)}}\| \leq C \|Q(l)\|^{1 - \frac{\theta}{d}} .$$

ii) Assume now that $\|Q(j+1)\| > \|Q(l)\|^{\theta/d}$. If $l \leq j + 2d - 2$ ($l \leq j + 3$ when $d = 2$), we just use

$$\|S(j+1, l)|_{\Gamma_\star^{(j+1)}}\| \leq C_\varepsilon \|Q(l)\|^\varepsilon$$

by condition (a). If $l > j + 2d - 2$ ($l > j + 3$), we write

$$S(0, l) = S(j', l) S(j+1, j') S(0, j+1)$$

with $j' = j + 2d - 2$ ($j' = j + 3$ when $d = 2$). As the entries of $Q(j+1, j')$ are positive integers we have

$$\|Q(0, l)\| \geq C \|Q(j', l)\| \|Q(0, j+1)\| ,$$

which implies

$$\|Q(j', l)\| \leq C \|Q(l)\|^{1 - \frac{\theta}{d}} .$$

As we have also

$$\|Q(j+1, j')\| \leq C_\varepsilon \|Q(j')\|^\varepsilon ,$$

we obtain in this case that

$$\|S(j+1, l)|_{\Gamma^{(j+1)}}\| \leq C_\varepsilon \|Q(l)\|^{1 - \frac{\theta}{d} + \varepsilon} .$$

Putting the two cases together and inserting this in the sum, we obtain

Proposition For $\varphi \in BV_*(\sqcup_{\alpha \in \mathcal{A}} I_\alpha^{(0)})$, $l \geq 0$, one has

$$\|S(l)\varphi\|_{L^\infty} \leq C \|Q(l)\|^{1-\frac{\theta}{2d}} \|\varphi\|_{BV} .$$

Remark In case i), the estimate we got for $Q(j+1)^{-1}$ is far from optimal (it should be of the order of $Q(j+1)$) but sufficient for our purposes.

2.4 Primitives of functions of bounded variation

2.4.1

For $k \geq 0$, we will denote by $BV^1(\sqcup_{\alpha \in \mathcal{A}} I_\alpha^{(k)})$ the space of functions $\varphi : I^{(k)} \rightarrow \mathbb{R}$ which are absolutely continuous on each $j_0(I_\alpha^{(k)})$ and whose derivative on each $j_0(I_\alpha^{(k)})$ is of bounded variation. The condition that the mean value of the derivative is zero defines an hyperspace $BV_*^1(\sqcup_{\alpha \in \mathcal{A}} I_\alpha^{(k)})$. We recall from 1.3.3 the subspace $\Gamma_s^{(k)}$ of $\Gamma^{(k)}$. We will denote by $\overline{BV}_*^1(\sqcup_{\alpha \in \mathcal{A}} I_\alpha^{(k)})$ the quotient of $BV_*^1(\sqcup_{\alpha \in \mathcal{A}} I_\alpha^{(k)})$ by this finite dimensional subspace.

Given $\varphi \in BV_*(\sqcup_{\alpha \in \mathcal{A}} I_\alpha^{(0)})$, we will find a primitive Φ of φ (given a priori by d constants of integration, one for each $I_\alpha^{(0)}$) for which the special Birkhoff sums are small. The primitive Φ will actually be uniquely determined mod $\Gamma_s^{(0)}$, i.e. in $\overline{BV}_*^1(\sqcup_{\alpha \in \mathcal{A}} I_\alpha^{(0)})$.

2.4.2

For any $\varphi \in BV_*(\sqcup_{\alpha \in \mathcal{A}} I_\alpha^{(k)})$, denote by $P_0^{(k)}\varphi$ the class in $\overline{BV}_*^1(\sqcup_{\alpha \in \mathcal{A}} I_\alpha^{(k)})$ of the primitive of φ which has mean zero on each $j_0(I_\alpha^{(k)})$.

This is the most natural choice of primitive, but unfortunately the special Birkhoff sums $S(k, l)$ do not commute with these primitive operators, i.e. they do not preserve the condition to be of mean value 0 on each $j_0(I_\alpha^{(k)})$.

Therefore, we will modify $P_0^{(k)}$, considering

$$P^{(k)} = P_0^{(k)} + \Delta P^{(k)} ,$$

where

$$\Delta P^{(k)} : BV_*(\sqcup_{\alpha \in \mathcal{A}} I_\alpha^{(k)}) \rightarrow \Gamma^{(k)} / \Gamma_s^{(k)} ,$$

is a bounded linear operator. We want this new choice to be equivariant:

$$S(k, l) \circ P^{(k)} = P^{(l)} \circ S(k, l) .$$

This leads to the following equation for $\Delta P^{(k)}$. Define

$$\Lambda(k, l) = P_0^{(l)} \circ S(k, l) - S(k, l) \circ P_0^{(k)} .$$

This is a bounded linear map from $\text{BV}_* (\sqcup_{\alpha \in \mathcal{A}} I_\alpha^{(k)})$ to $\Gamma^{(l)} / \Gamma_s^{(l)}$. Then we should have

$$S_b(k, l) \circ \Delta P^{(k)} - \Delta P^{(l)} \circ S(k, l) = \Lambda(k, l) , \quad (*)$$

where S_b was defined in 1.3.3.

Equation (*) has the formal solution

$$\Delta P^{(k)} = \sum_{l>k} (S_b(k, l))^{-1} \circ \Lambda(l-1, l) \circ S(k, l-1) , \quad (**)$$

and we will check next that this defines indeed the required primitive.

2.4.3 Estimate for $\Lambda(l-1, l)$.

Let $\varphi \in \text{BV}_* (\sqcup_{\alpha \in \mathcal{A}} I_\alpha^{(l-1)})$. As $P_0^{(l-1)}\varphi$ has mean zero on each $j_0(I_\alpha^{(l-1)})$, we have

$$\|P_0^{(l-1)}\varphi\|_{L^\infty} \leq (\text{Max}_{\alpha \in \mathcal{A}} \lambda_\alpha^{(l-1)}) \|\varphi\|_{L^\infty} .$$

On the other hand we have

$$\begin{aligned} \|S(l-1, l)\varphi\|_{L^\infty} &\leq \|Z(l)\| \|\varphi\|_{L^\infty} , \\ \|S(l-1, l)P_0^{(l-1)}\varphi\|_{L^\infty} &\leq \|Z(l)\| \|P_0^{(l-1)}\varphi\|_{L^\infty} . \end{aligned}$$

Finally, we get

$$\|P_0^{(l)}S(l-1, l)\varphi\|_{L^\infty} \leq (\text{Max}_{\alpha \in \mathcal{A}} \lambda_\alpha^{(l)}) \|Z(l)\| \|\varphi\|_{L^\infty} ,$$

which allows to conclude that

$$\|\Lambda(l-1, l)\varphi\|_{L^\infty} \leq 2\|Z(l)\| (\text{Max}_{\alpha \in \mathcal{A}} \lambda_\alpha^{(l-1)}) \|\varphi\|_{L^\infty} .$$

2.4.4

Assume now the three conditions (a), (b) and (c) of 1.3. From 1.3.1, we get

$$\begin{aligned} \text{Max}_{\alpha \in \mathcal{A}} \lambda_\alpha^{(l-1)} &\leq C_\varepsilon \|Q(l-1)\|^{\varepsilon-1} , \\ \|Z(l)\| &\leq C_\varepsilon \|Q(l-1)\|^\varepsilon , \end{aligned}$$

and from condition (c) that

$$\|(S_b(0, l))^{-1}\| \leq C_\varepsilon \|Q(l)\|^\varepsilon .$$

On the other hand, from the Proposition in 2.3, we obtain

$$\|S(0, l-1)\varphi\|_{L^\infty} \leq C \|Q(l-1)\|^{1-\theta/2d} \|\varphi\|_{BV} .$$

Therefore, for $k = 0$, the series (***) in 2.4.2 is converging and we obtain

$$\|\Delta P^{(0)}\varphi\| \leq \left(\sum_{l>0} C'_\varepsilon \|Q(l)\|^{3\varepsilon-\theta/2d} \right) \|\varphi\|_{BV} .$$

Indeed, we take $\varepsilon < \theta/6d$ and observe that it follows from the Lemma in 1.2.4 that $\|Q(l)\|$ grows at least exponentially fast. In the same way, as $T^{(k)}$ satisfies also conditions (a), (b), (c) (with worse constants but the same exponent θ), the series (***) will converge for all $k \geq 0$. In this case, we prefer to estimate directly $\Delta P^{(k)}S(0, k)\varphi$ for $\varphi \in BV_*(\sqcup_{\alpha \in \mathcal{A}} I_\alpha^{(0)})$. We have

$$\|\Delta P^{(k)}S(0, k)\varphi\| \leq \sum_{l>k} \|(S_b(k, l))^{-1}\| \|\Lambda(l-1, l)\| \|S(0, l-1)\varphi\|_{L^\infty} .$$

The above estimates now give

$$\begin{aligned} \|\Delta P^{(k)}S(0, k)\varphi\| &\leq \left(\sum_{l>k} C'_\varepsilon \|Q(l)\|^{3\varepsilon-\theta/2d} \right) \|\varphi\|_{BV} \\ &\leq C \|Q(k)\|^{-\theta/3d} \|\varphi\|_{BV} . \end{aligned}$$

2.4.5 Special Birkhoff sums for $P^{(0)}\varphi$

Let $\varphi \in BV_*(\sqcup_{\alpha \in \mathcal{A}} I_\alpha^{(0)})$, $\Phi \in BV_*^1(\sqcup_{\alpha \in \mathcal{A}} I_\alpha^{(0)})$ such that the class mod $\Gamma_s^{(0)}$ of Φ is $P^{(0)}\varphi$. The class mod $\Gamma_s^{(k)}$ of $S(k)\Phi$ is $P^{(k)}S(k)\varphi$ by construction.

From the definition of $P_0^{(k)}$ and 2.3, we have

$$\|P_0^{(k)}S(k)\varphi\|_{L^\infty} \leq \left(\text{Max}_{\alpha \in \mathcal{A}} \lambda_\alpha^{(k)} \right) \|Q(k)\|^{1-\theta/d} \|\varphi\|_{BV} ,$$

with $\text{Max}_{\alpha \in \mathcal{A}} \lambda_\alpha^{(k)} \leq C_\varepsilon \|Q(k)\|^{\varepsilon-1}$ by condition (a). Joining this with the estimate for $\Delta P^{(k)}$ above, we obtain

$$\begin{aligned} \|S(k)P^{(0)}\varphi\| &= \|P^{(k)}S(k)\varphi\| \\ &\leq C \|Q(k)\|^{-\theta/3d} \|\varphi\|_{BV} . \end{aligned}$$

By definition of a quotient norm, this means that we may write in $BV_*^1(\sqcup_{\alpha \in \mathcal{A}} I_\alpha^{(k)})$:

$$S(k)\Phi = \Phi_k + \chi_k ,$$

with $\chi_k \in \Gamma_s^{(k)}$ and

$$\|\Phi_k\|_{L^\infty} \leq C\|Q(k)\|^{-\theta/3d}\|\varphi\|_{BV} .$$

we have then

$$\begin{aligned} \chi_{k+1} &= S(k, k+1)\chi_k + S(k, k+1)\Phi_k - \Phi_{k+1} \\ &:= S(k, k+1)\chi_k + \Delta\chi_{k+1} , \end{aligned}$$

with $\|\Delta\chi_{k+1}\| \leq C\|Q(k+1)\|^{-\theta/4d}\|\varphi\|_{BV}$ (using once more condition (a)). Then

$$S(k)\Phi = \Phi_k + \sum_{j \leq k} S(j, k)\Delta\chi_j .$$

In the sum, we separate two cases. Recall that there exists $\sigma > 0$, $C > 0$, such that

$$\left\| S(j)|_{\Gamma_s^{(0)}} \right\| \leq C\|Q(j)\|^{-\sigma} ,$$

for all $j \geq 0$. If $\|Q(j)\| \leq \|Q(k)\|^{\sigma/d}$, we write $S(j, k) = S(k) \circ (S(j))^{-1}$ and get

$$\begin{aligned} \|S(j, k)\Delta\chi_j\| &\leq C\|Q(k)\|^{-\sigma}\|(S(j))^{-1}\Delta\chi_j\| \\ &\leq C\|Q(k)\|^{-\sigma}\|Q(j)\|^{d-1}\|\Delta\chi_j\| \\ &\leq C\|Q(k)\|^{-\sigma/d}\|\varphi\|_{BV} . \end{aligned}$$

In case $\|Q(j)\| > \|Q(k)\|^{\sigma/d}$, we use the second estimate in condition (c) to get

$$\begin{aligned} \|S(j, k)\Delta\chi_j\| &\leq C_\varepsilon\|Q(k)\|^\varepsilon\|\Delta\chi_j\| \\ &\leq C_\varepsilon\|Q(k)\|^{\varepsilon-\theta\sigma/4d^2}\|\varphi\|_{BV} . \end{aligned}$$

We have thus proved the

Theorem. *Let T be an i.e.m. of Roth type. There exists $\omega > 0$, depending only on σ and θ in (b), (c), such that the special Birkhoff sums $S(k)\Phi$ satisfy:*

$$\|S(k)\Phi\|_{L^\infty} \leq C\|Q(k)\|^{-\omega}\|\varphi\|_{BV} .$$

Corollary. *Let T be an i.e.m. of Roth type, $\varphi \in BV_*(\sqcup_{\alpha \in \mathcal{A}} I_\alpha^{(0)})$. For any primitive Φ of φ whose class lie in $P^{(0)}\varphi$, we can solve the cohomological equation*

$$\Psi \circ T - \Psi = \Phi$$

with a bounded solution Ψ .

Proof. This follows from the Theorem, taking into account the remarks at the end of Section 2.1 and 2.2.3.

3. Suspensions of interval exchange maps

We first recall, basically to fix notations, how to suspend i.e.m. 's in order to get a Riemann surface with an holomorphic 1-form. The basic reference is [V1].

3.1 Suspension data

Let $(\mathcal{A}, \pi_0, \pi_1)$ be admissible combinatorial data, and let T be an i.e.m. of this combinatorial type, determined by length data $(\lambda_\alpha)_{\alpha \in \mathcal{A}}$. We will construct a Riemann surface with a flow which can be considered as a suspension of T . In order to do this, we need data which we call *suspension data*. We will identify \mathbb{R}^2 with \mathbb{C} . Consider a family $\tau = (\tau_\alpha)_{\alpha \in \mathcal{A}} \in \mathbb{R}^{\mathcal{A}}$. To this family we associate

$$\begin{aligned}\zeta_\alpha &= \lambda_\alpha + i\tau_\alpha, \quad \alpha \in \mathcal{A} \\ \xi_\alpha^\varepsilon &= \sum_{\pi_\varepsilon \beta \leq \pi_\varepsilon \alpha} \zeta_\beta, \quad \alpha \in \mathcal{A}, \quad \varepsilon \in \{0, 1\}.\end{aligned}$$

We always have $\xi_{\alpha_0}^0 = \xi_{\alpha_1}^1$, where as before $\pi_\varepsilon(\alpha_\varepsilon) = d$. We say that τ defines suspension data if the following inequalities hold:

$$\begin{aligned}\Im \xi_\alpha^0 &> 0 \quad \text{for all } \alpha \in \mathcal{A}, \quad \alpha \neq \alpha_0, \\ \Im \xi_\alpha^1 &< 0 \quad \text{for all } \alpha \in \mathcal{A}, \quad \alpha \neq \alpha_1.\end{aligned}$$

We also set

$$\theta_\alpha = \xi_\alpha^1 - \xi_\alpha^0, \quad \alpha \in \mathcal{A}.$$

We then have

$$\begin{aligned}\theta &= \Omega \zeta, \\ \Re \theta &= \delta,\end{aligned}$$

and define

$$h = -\Im \theta = -\Omega \tau.$$

One has $h_\alpha > 0$ for all $\alpha \in \mathcal{A}$, because of the formula

$$\theta_\alpha = (\xi_\alpha^1 - \zeta_\alpha) - (\xi_\alpha^0 - \zeta_\alpha).$$

One has also

$$\Im \xi_{\alpha_0}^0 = \Im \xi_{\alpha_1}^1 \in [-h_{\alpha_1}, h_{\alpha_0}].$$

3.2 Construction of a Riemann surface

Let $(\mathcal{A}, \pi_0, \pi_1)$ and $(\zeta_\alpha = \lambda_\alpha + i\tau_\alpha)_{\alpha \in \mathcal{A}}$ as above. For $\alpha \in \mathcal{A}$, consider the rectangles in $\mathbb{C} = \mathbb{R}^2$:

$$\begin{aligned} R_\alpha^0 &= (\Re \xi_\alpha^0 - \lambda_\alpha, \Re \xi_\alpha^0) \times [0, h_\alpha] , \\ R_\alpha^1 &= (\Re \xi_\alpha^1 - \lambda_\alpha, \Re \xi_\alpha^1) \times [-h_\alpha, 0] , \end{aligned}$$

and the segments

$$\begin{aligned} S_\alpha^0 &= \{\Re \xi_\alpha^0\} \times [0, \Im \xi_\alpha^0) , \quad \alpha \neq \alpha_0 , \\ S_\alpha^1 &= \{\Re \xi_\alpha^1\} \times (\Im \xi_\alpha^1, 0] , \quad \alpha \neq \alpha_1 . \end{aligned}$$

Let also $S_{\alpha_0}^0 = S_{\alpha_1}^1$ be the half-open vertical segment $[\lambda^*, \xi_{\alpha_0}^0) = [\lambda^*, \xi_{\alpha_1}^1)$.

Define then

$$R_\zeta = \bigcup_{\varepsilon \in \{0,1\}} \bigcup_{\alpha \in \mathcal{A}} R_\alpha^\varepsilon \bigcup_{\varepsilon \in \{0,1\}} \bigcup_{\alpha \in \mathcal{A}} S_\alpha^\varepsilon .$$

The translation by θ_α sends R_α^0 onto R_α^1 . If $\xi_{\alpha_0}^0 = \xi_{\alpha_1}^1 = 0$, $S_{\alpha_0}^0 = S_{\alpha_1}^1$ is empty, $\xi_{\alpha_1}^0$ is the top right corner of $R_{\alpha_1}^0$ and $\xi_{\alpha_0}^1$ is the bottom right corner of $R_{\alpha_0}^1$. If $\xi_{\alpha_0}^0 = \xi_{\alpha_1}^1 > 0$, the translation by θ_{α_1} sends the top part $\tilde{S}_{\alpha_1}^0 = \{\Re \xi_{\alpha_1}^0\} \times [h_{\alpha_1}, \Im \xi_{\alpha_1}^0)$ of $S_{\alpha_1}^0$ onto $S_{\alpha_1}^1$. If $\xi_{\alpha_0}^0 = \xi_{\alpha_1}^1 < 0$, the translation by θ_{α_0} sends $S_{\alpha_0}^0$ onto the bottom part $\tilde{S}_{\alpha_0}^1 = \{\Re \xi_{\alpha_0}^1\} \times (\Im \xi_{\alpha_0}^1, -h_{\alpha_0}]$ of $S_{\alpha_0}^1$.

We use these translations to identify in R_ζ each R_α^0 to each R_α^1 , and $S_{\alpha_0}^0 = S_{\alpha_1}^1$ (if non empty) to either $\tilde{S}_{\alpha_1}^0$ or $\tilde{S}_{\alpha_0}^1$.

Denote by M_ζ^* the topological space obtained from R_ζ by these identifications.

Observe that M_ζ^* inherits from \mathbb{C} the structure of a Riemann surface, and also a nowhere vanishing holomorphic 1-form ω (given by dz) and a vertical vector field (given by $\frac{\partial}{\partial y}$).

3.3 Compactification of M_ζ^*

Let $\bar{\mathcal{A}}$ be the set with $2d - 2$ elements of pairs (α, L) and (α, R) , except that we identify $(\alpha_0, R) = (\alpha_1, R)$ and $(\alpha'_0, L) = (\alpha'_1, L)$, where $\pi_\varepsilon(\alpha_\varepsilon) = d$, $\pi_\varepsilon(\alpha'_\varepsilon) = 1$.

Let σ be the permutation of $\bar{\mathcal{A}}$ defined by

$$\begin{aligned} \sigma(\alpha, R) &= (\beta_0, L) , \\ \sigma(\alpha, L) &= (\beta_1, R) , \end{aligned}$$

with $\pi_0(\beta_0) = \pi_0(\alpha) + 1$, $\pi_1(\beta_1) = \pi_1(\alpha) - 1$; in particular, we have

$$\begin{aligned} \sigma(\alpha_0, R) &= (\pi_0^{-1}(\pi_0(\alpha_1) + 1), L) , \\ \sigma(\alpha'_1, L) &= (\pi_1^{-1}(\pi_1(\alpha'_0) - 1), R) . \end{aligned}$$

The permutation describes which half planes are met when one winds around an end of M_ζ^* . Denote by Σ the set of cycles of σ . To each $C \in \Sigma$ is associated in a one-to-one correspondance an end q_C of M_ζ^* . From the local structure around q_C , it is clear that the compactification $M_\zeta = M_\zeta^* \cup_{C \in \Sigma} \{q_C\}$ will be a compact Riemann surface, with the set of marked points $\cup_{C \in \Sigma} \{q_C\} = M_\zeta \setminus M_\zeta^*$ in canonical correspondance with Σ . Moreover, the 1-form ω extends to a holomorphic 1-form on M_ζ ; the length of a cycle C is an even number $2n_C$; the corresponding marked point q_C is a zero of ω of order $n_C - 1$.

Let $\nu = \text{card } \Sigma$, and let g be the genus of M_ζ . We have

$$\begin{aligned} d - 1 &= \sum_{C \in \Sigma} n_C , \\ 2g - 2 &= \sum_{C \in \Sigma} (n_C - 1) , \end{aligned}$$

hence

$$d = 2g + \nu - 1 .$$

Example Suppose that π_0, π_1 satisfy

$$\pi_0(\alpha) + \pi_1(\alpha) = d + 1 \quad , \quad \text{for all } \alpha \in \mathcal{A} .$$

If d is even, there is only 1 cycle; we have $d = 2g$ and the only zero of ω has order $2g - 2$. If d is odd, there are two cycles of equal length $d - 1$; we have $d = 2g + 1$, and each of the two zeros of ω has order $g - 1$.

The vertical vector field on M_ζ^* does not extend (continuously) to M_ζ when $g > 1$, unless one slows it near the marked points (which we will not do here). Nevertheless, it can be considered as a suspension of T : starting from a point $(x, 0)$ on the bottom side of R_α^0 , one flows up till reaching the top side where the point (x, h_α) is identified with the point $(x + \delta_\alpha, 0) = (T(x), 0)$ in the top side of R_α^1 . The return time is h_α . The vector field is not complete, as some orbits reach marked points in finite time.

3.4 The cohomological equation for higher smoothness

3.4.1

In this section, we will relate the (discrete) cohomological equation for i.e.m. 's to the (continuous) cohomological equation for the vertical vector field on M_ζ ; this equation is

$$\tilde{\Phi} = \frac{\partial}{\partial y} \tilde{\Psi} ,$$

where now $\tilde{\Phi}, \tilde{\Psi}$ are functions on M_ζ . This allows to compare our results with the pioneering work of Forni ([Fo1]). We will always assume, as he does, that $\tilde{\Phi}$ vanishes in the neighborhood of the marked points of M_ζ .

Considering the cohomological equation on the surface leads naturally to some regularity assumptions on the interval. Because the datum Φ and the solution Ψ are not related to the corresponding functions $\tilde{\Phi}, \tilde{\Psi}$ on the surface in the same way (Ψ is a restriction of $\tilde{\Psi}$ to a segment, while Φ is an integral), the regularity that we introduce for Φ and Ψ are not of the same kind (even taking the loss of derivatives into account).

3.4.2

For each integer $r \geq 1$, we introduce the space $BV_*^r(I)$ of functions $\Phi : I \rightarrow \mathbb{R}$ such that

- for each $\alpha \in \mathcal{A}$, Φ is of class \mathcal{C}^{r-1} on $j_0(I_\alpha)$, $D^{r-1}\Phi$ is absolutely continuous on $j_0(I_\alpha)$ and $D^r\Phi$ is of bounded variation on $j_0(I_\alpha)$;
- each function $D^l\Phi$, for $0 < l \leq r$, has mean value 0 in I .

Remark. *As before, we allow discontinuities at the discontinuities of T . Observe however that the mean value condition implies that the sum of the jumps of $D^l\Phi$ ($0 \leq l < r$) over the discontinuities of T (including the endpoints of I) is zero.*

We will indicate below why the mean value condition is natural.

On the other hand, we will look for solutions in the space $\mathcal{C}^{r-2+Lip}(I)$ of functions Ψ which are \mathcal{C}^{r-2} on all of I , the derivative of order $r-2$ $D^{r-2}\Psi$ being Lipschitz on I . For $r=1$, this is just the space of bounded functions on I . Observe that, as soon as $r \geq 2$, we do not allow discontinuities.

3.4.3

For T an i.e.m. of Roth type, denote by $\Gamma_T = \Gamma_T^{(0)}$ the space of functions $\chi \in \Gamma$ (constant on each $j_0(I_\alpha)$) which can be written as

$$\chi = \psi - \psi \circ T$$

with bounded ψ . This is a linear subspace of Γ which is contained in Γ_* and contains Γ_s . We can rephrase our main theorem by saying that there is a well-defined *obstruction map*

$$BV_*^1(I) \rightarrow \Gamma/\Gamma_T$$

which associates to Φ the function in Γ we must subtract from Φ in order to be able to solve the cohomological equation. We recognize (some of)

Forni distribution conditions, by choosing a basis in the finite-dimensional space Γ/Γ_T . The number of conditions is just the codimension of Γ_T , as the restriction of the obstruction map to Γ is just the quotient map and thus the obstruction map is onto.

3.4.4

Let now $r \geq 1$, $\Phi \in BV_*^r(I)$, and let us try to solve (under finitely many linear conditions on Φ) the cohomological equation

$$\Phi = \Psi - \Psi \circ T ,$$

with $\Psi \in \mathcal{C}^{r-2+Lip}(I)$. We assume that the i.e.m. is of Roth type. Consider the rd -dimensional space $\Gamma(r)$ of functions χ on I whose restrictions to each $j_0(I_\alpha)$ are polynomials of degree $< r$. For $r = 1$, this is our previous space Γ . Consider also

$$\Gamma_*(r) = \Gamma(r) \cap BV_*^r(I)$$

which has codimension $(r - 1)$ in $\Gamma(r)$. We first describe the subspace $\Gamma_T(r)$ of $\Gamma_*(r)$ of functions χ which can be written as

$$\chi = \psi \circ T - \psi ,$$

with $\psi \in \mathcal{C}^{r-2+Lip}(I)$.

Lemma *For $r \geq 1$, the map $\chi \mapsto D\chi$ from $\Gamma(r+1)$ to $\Gamma(r)$ sends $\Gamma_*(r+1)$ to $\Gamma_*(r)$ and $\Gamma_T(r+1)$ to $\Gamma_T(r)$. The kernel, i.e. the intersection $\Gamma \cap \Gamma_T(r+1)$, is equal to $\mathbb{R}\delta$; we have thus*

$$\dim \Gamma_T(r) = \dim \Gamma_T + (r - 1) .$$

Proof. It is clear that $\chi \mapsto D\chi$ sends $\Gamma_*(r+1)$ to $\Gamma_*(r)$ and $\Gamma_T(r+1)$ to $\Gamma_T(r)$. If $\psi_0(x) \equiv x$, then $\psi_0 \circ T(x) - \psi_0(x) = \delta_\alpha$ for $x \in j_0(I_\alpha)$ hence $\mathbb{R}\delta \subset \Gamma \cap \Gamma_T(r)$ for all $r \geq 1$. Conversely, if $\chi \in \Gamma \cap \Gamma_T(r)$, write $\chi = \psi \circ T - \psi$ with $\psi \in Lip(I)$. Taking derivatives, $D\psi$ is T -invariant, hence constant as T is ergodic. Therefore $\chi \in \mathbb{R}\delta$. \square

Theorem *Let $r \geq 1$. For any $\Phi \in BV_*^r(I)$, one can find $\chi \in \Gamma_*(r)$, $\psi \in \mathcal{C}^{r-2+Lip}(I)$ such that*

$$\Phi = \chi + \Psi \circ T - \Psi .$$

In other terms, the map

$$\begin{aligned} \Phi &\mapsto \chi \\ BV_*^r(I) &\rightarrow \Gamma_*(r)/\Gamma_T(r), \end{aligned}$$

is the obstruction map associated with the cohomological equation with the prescribed regularities.

Proof. By induction on r , the case $r = 1$ being our main theorem. Assume $\Phi \in BV_*^{r+1}(I)$. Then $D\Phi \in BV_*^r(I)$. By the induction hypothesis, one can write

$$D\Phi = \chi_1 + \Psi_1 \circ T - \Psi_1,$$

with $\chi_1 \in \Gamma_*(r)$ and $\Psi_1 \in \mathcal{C}^{r-2+Lip}(I)$. Let Ψ be a primitive of Ψ_1 , χ_0 be a primitive of χ_1 . Then $\psi \in \mathcal{C}^{r-1+Lip}(I)$. As $D\Phi$ has mean value 0, χ_1 has also mean value 0 and $\chi_0 \in \Gamma_*(r+1)$. The difference $\chi'_0 = \Phi - \chi_0 - \Psi \circ T + \Psi$ belongs to Γ and we take $\chi = \chi_0 + \chi'_0$. \square

3.4.5

We explain now why the regularities for Φ , Ψ are “natural”.

Let $\zeta = (\zeta_\alpha)_{\alpha \in \mathcal{A}}$ be suspension data, and let M_ζ be the surface constructed from these data as in 3.2.

Let $\tilde{\Phi}$ be a continuous function on M_ζ . With the notations of 3.1, we define, for $\alpha \in \mathcal{A}$:

$$\begin{aligned} \mathcal{I}_\alpha^0 &= \int_0^{\mathfrak{S}^m \xi_\alpha^0} \tilde{\Phi}(\Re \xi_\alpha^0, y) dy, \\ \mathcal{I}_\alpha^1 &= \int_{\mathfrak{S}^m \xi_\alpha^1}^0 \tilde{\Phi}(\Re \xi_\alpha^1, y) dy; \end{aligned}$$

for $\alpha \in \mathcal{A}$, $x \in j_0(I_\alpha)$, we also set

$$\Phi(x) = \int_0^{h_\alpha} \tilde{\Phi}(x, y) dy.$$

Observe that we have

$$\begin{aligned} \Phi((\Re \xi_\alpha^0)^-) &= \mathcal{I}_\alpha^0 + \mathcal{I}_\alpha^1, \\ \Phi(\Re \xi_\alpha^0 - \lambda_\alpha) &= \mathcal{I}_{\beta_0}^0 + \mathcal{I}_{\beta_1}^1, \end{aligned}$$

where $\pi_0(\beta_0) + 1 = \pi_0(\alpha)$, $\pi_1(\beta_1) + 1 = \pi_1(\alpha)$, except if $\pi_0(\alpha) = 1$ (respectively $\pi_1(\alpha) = 1$) when $\mathcal{I}_{\beta_0}^0$ (resp. $\mathcal{I}_{\beta_1}^1$) is declared to be 0.

From these formulas and $\mathcal{I}_{\alpha_0}^0 + \mathcal{I}_{\alpha_1}^1 = 0$ (with $\pi_\varepsilon(\alpha_\varepsilon) = d$ as usual), we obtain

$$\sum_{\alpha \in \mathcal{A}} \Phi((\Re \xi_\alpha^0)^-) = \sum_{\alpha \in \mathcal{A}} \Phi(\Re \xi_\alpha^0 - \lambda_\alpha),$$

which means that the derivative of Φ (when it exists) has mean value 0. This explains the conditions defining $BV_*^r(I)$. On the other hand, if $\tilde{\Psi}$ is a function on M_ζ satisfying

$$\frac{\partial}{\partial y} \tilde{\Psi} = \tilde{\Phi}$$

and we define

$$\Psi(x) = \tilde{\Psi}(x, 0)$$

then we will have

$$\Psi \circ T - \Psi = \Phi.$$

4. Proof of full measure for Roth type

We will first recall the construction of the finite measure, absolutely continuous w.r.t. Lebesgue measure, which is invariant under the Zorich algorithm (normalized).

4.1 The basic operation of the algorithm for suspensions

Let $(\mathcal{A}, \pi_0, \pi_1)$ and $(\zeta_\alpha = \lambda_\alpha + i\tau_\alpha)_{\alpha \in \mathcal{A}}$ be as above. Construct R_ζ, M_ζ as in 3.2 and 3.3. With $\pi_\varepsilon(\alpha_\varepsilon) = d$ as above, assume that

$$\lambda_{\alpha_0} \neq \lambda_{\alpha_1} .$$

Then the formula $\lambda_{\alpha_\varepsilon} = \text{Max}(\lambda_{\alpha_0}, \lambda_{\alpha_1})$ defines uniquely $\varepsilon \in \{0, 1\}$ and determines uniquely the basic step of the continued fraction algorithm; this step produces new combinatorial data $(\mathcal{A}, \hat{\pi}_0, \hat{\pi}_1)$ and length data $(\hat{\lambda}_\alpha)_{\alpha \in \mathcal{A}}$ given by

$$\begin{aligned} \hat{\lambda}_\alpha &= \lambda_\alpha , \quad \alpha \neq \alpha_\varepsilon \\ \hat{\lambda}_{\alpha_\varepsilon} &= \lambda_{\alpha_\varepsilon} - \lambda_{\alpha_{1-\varepsilon}} . \end{aligned}$$

For suspension data, we just define in the same way

$$\begin{aligned} \hat{\zeta}_\alpha &= \zeta_\alpha , \quad \alpha \neq \alpha_\varepsilon \\ \hat{\zeta}_{\alpha_\varepsilon} &= \zeta_{\alpha_\varepsilon} - \zeta_{\alpha_{1-\varepsilon}} . \end{aligned}$$

This has a nice representation in terms of the corresponding regions $R_\zeta, R_{\hat{\zeta}}$. One cuts from R_ζ the part where $x \geq \hat{\lambda}^* = \lambda^* - \lambda_{\alpha_\varepsilon}$: it is made of $R_{\alpha_{1-\varepsilon}}^{1-\varepsilon}$ and a right part of $R_{\alpha_\varepsilon}^\varepsilon$. We glue back $R_{\alpha_{1-\varepsilon}}^{1-\varepsilon}$ to the free horizontal side of $R_{\alpha_\varepsilon}^{1-\varepsilon}$, and the right part of $R_{\alpha_\varepsilon}^\varepsilon$ to $R_{\alpha_{1-\varepsilon}}^\varepsilon$: see figure 4 below.

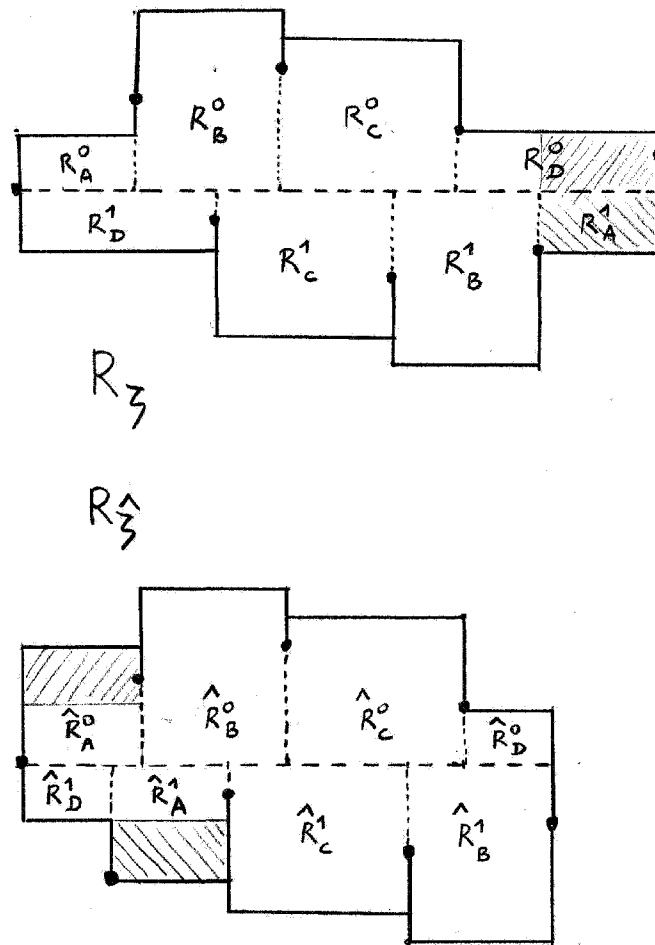


Fig.4 The Rauzy-Veech operation for suspensions.

It is easy to check that the new suspension data satisfy the inequalities required in 3.1; if for instance $\varepsilon = 0$, one has

$$\hat{\xi}_{\alpha}^0 = \xi_{\alpha}^0, \alpha \neq \alpha_0$$

with $\hat{\pi}_0 = \pi_0$ on one hand and

$$\begin{aligned} \hat{\xi}_{\alpha}^1 &= \xi_{\alpha}^1, \alpha \neq \alpha_0, \alpha_1 \\ \hat{\xi}_{\alpha_1}^1 &= \xi_{\alpha_0}^1, \\ \hat{\xi}_{\alpha_0}^1 &= \xi_{\alpha_0}^1 - \zeta_{\alpha_1}. \end{aligned}$$

The last formula gives

$$\begin{aligned}
-\hat{\xi}_{\alpha_0}^1 &= \zeta_{\alpha_1} - \xi_{\alpha_0}^1 \\
&= \zeta_{\alpha_1} - \xi_{\alpha_0}^0 - \theta_{\alpha_0} \\
&= \zeta_{\alpha_1} - \xi_{\alpha_1}^1 - \theta_{\alpha_0} \\
&= -\xi_{\tilde{\alpha}_1}^1 - \theta_{\alpha_0},
\end{aligned}$$

with $\pi_1(\tilde{\alpha}_1) = d - 1$. We therefore have

$$-\mathfrak{S}m \hat{\xi}_{\alpha_0}^1 = -\mathfrak{S}m \xi_{\tilde{\alpha}_1}^1 + h_{\alpha_0} > 0.$$

We also see that (still with $\varepsilon = 0$), if $\hat{\alpha}_1 \in \mathcal{A}$ is such that $\hat{\pi}_1(\hat{\alpha}_1) = d$ (we have $\hat{\alpha}_1 = \tilde{\alpha}_1$ if $\tilde{\alpha}_1 \neq \alpha_0$, $\hat{\alpha}_1 = \alpha_1$ if $\tilde{\alpha}_1 = \alpha_0$), one has

$$\mathfrak{S}m \hat{\xi}_{\hat{\alpha}_1}^1 = \mathfrak{S}m \xi_{\hat{\alpha}_1}^1 < 0.$$

Conversely, given $(\mathcal{A}, \pi_0, \pi_1)$ and $(\zeta_\alpha = \lambda_\alpha + i\tau_\alpha)_{\alpha \in \mathcal{A}}$ as above, assume that

$$\mathfrak{S}m \xi_{\alpha_0}^0 = \mathfrak{S}m \xi_{\alpha_1}^1 \neq 0,$$

and define $\varepsilon = 0$ if $\mathfrak{S}m \xi_{\alpha_1}^1 < 0$, $\varepsilon = 1$ if $\mathfrak{S}m \xi_{\alpha_0}^0 > 0$. Set

$$\begin{aligned}
\hat{\zeta}_\alpha &= \zeta_\alpha, \quad \alpha \neq \alpha_\varepsilon \\
\hat{\zeta}_{\alpha_\varepsilon} &= \zeta_{\alpha_\varepsilon} + \zeta_{\alpha_{1-\varepsilon}},
\end{aligned}$$

and define appropriately new combinatorial data; this operation is the inverse of the one above. Thus the dynamics of the continued fraction algorithm at the level of suspension is invertible (on a full measure set) and can be viewed as the natural extension of the dynamics at the level of i.e.m..

It is clear that the Riemann surfaces $M_\zeta, M_{\hat{\zeta}}$ are canonically isomorphic, and the isomorphism respects the holomorphic 1-form and the vertical vector field.

We can also extend the definition of the Zorich algorithm at the level of suspension data. These accelerated dynamics can actually be thought of as a first return map of the previous dynamics. Indeed, in the polyhedral cone of admissible length and suspension data, consider the polyhedral subcones defined by

$$\begin{aligned}
\mathcal{Z}_0 &= \{\lambda_{\alpha_0} > \lambda_{\alpha_1}, \mathfrak{S}m \xi_{\alpha_0}^0 > 0\}, \\
\mathcal{Z}_1 &= \{\lambda_{\alpha_1} > \lambda_{\alpha_0}, \mathfrak{S}m \xi_{\alpha_1}^1 < 0\}.
\end{aligned}$$

The accelerated dynamics are the first return map to $\mathcal{Z} = \mathcal{Z}_0 \sqcup \mathcal{Z}_1$: this is clear from the description of the basic step above.

4.2 The Teichmüller flow

Fix combinatorial data $(\mathcal{A}, \pi_0, \pi_1)$. Given length data $(\lambda_\alpha)_{\alpha \in \mathcal{A}}$ and suspension data $(\tau_\alpha)_{\alpha \in \mathcal{A}}$, one defines for $t \in \mathbb{R}$

$$U^t(\lambda, \tau) = (e^{t/2}\lambda, e^{-t/2}\tau).$$

This flow is called the *Teichmüller flow*. Observe that the conditions on the length data $\lambda_\alpha > 0$ and on the suspension data (see 3.1) are preserved under the flow.

It is also obvious that the flow commutes with the basic operation of the continued fraction algorithm. In particular, the inequality $\lambda_{\alpha_\varepsilon} > \lambda_{\alpha_{1-\varepsilon}}$ is preserved.

The surface M_ζ is canonically equipped with an area form (coming from \mathbb{C}) for which its area is

$$A := \text{area}(M_\zeta) = \sum_{\alpha \in \mathcal{A}} \lambda_\alpha h_\alpha.$$

The area is preserved by the Teichmüller flow, and also by the basic operation of the continued fraction algorithm.

The Lebesgue measure $d\lambda d\tau$ on the domain $\mathbb{R}^{\mathcal{A}} \times \mathbb{R}^{\mathcal{A}}$ defined by the restrictions on length and suspension data is preserved by the Teichmüller flow, and by the basic operation of the continued fraction algorithm.

One now combines the continued fraction algorithm (in Zorich form) with the Teichmüller flow in order to get a version which is normalized w.r.t. scales.

One could decide to normalize by keeping the total length $\lambda^* = \sum_{\alpha \in \mathcal{A}} \lambda_\alpha$ constant; actually, we prefer in the sequel a slightly different normalization, which leads to simpler formulas.

As in 1.2.1, for $\lambda_{\alpha_\varepsilon} > \lambda_{\alpha_{1-\varepsilon}}$, we set

$$\begin{aligned} \hat{\lambda}_\alpha &= \lambda_\alpha, \quad \alpha \neq \alpha_\varepsilon \\ \hat{\lambda}_{\alpha_\varepsilon} &= \lambda_{\alpha_\varepsilon} - \lambda_{\alpha_{1-\varepsilon}}. \end{aligned}$$

Define now

$$\hat{\lambda}^{**} = \sum_{\alpha \in \mathcal{A}} \hat{\lambda}_\alpha = \lambda^* - \lambda_{\alpha_{1-\varepsilon}}.$$

Let $(\pi_0, \pi_1, \lambda, \tau)$ belong to the domain \mathcal{Z} of the Zorich algorithm, and let $(\bar{\pi}_0, \bar{\pi}_1, \bar{\lambda}, \bar{\tau})$ be the image. Define

$$\begin{aligned} t &= t(\lambda) = 2(\log \hat{\lambda}^{**} - \log \hat{\lambda}^{**}), \\ \bar{G}(\pi_0, \pi_1, \lambda, \tau) &= (\bar{\pi}_0, \bar{\pi}_1, U^{t(\lambda)}(\bar{\lambda}, \bar{\tau})). \end{aligned}$$

The map \bar{G} is called the normalized step for the natural extension of the accelerated algorithm.

4.3 The absolutely continuous invariant measure

We already observed that the restriction of Lebesgue measure $d\lambda d\tau$ to the simplicial cone of admissible length and suspension data is invariant under both the basic step of the algorithm and the Teichmüller flow.

When we further restrict Lebesgue measure to \mathcal{Z} , we obtain a measure m_0 which is still invariant under Teichmüller flow and is now invariant under the accelerated algorithm.

Observe that the function t used in the definition of \bar{G} is constant along the orbits of the Teichmüller flow. It follows that the measure m_0 is also invariant under \bar{G} .

The area function $A = \sum_{\alpha \in \mathcal{A}} \lambda_\alpha h_\alpha$ (where $h = -\Omega\tau$) is also invariant under \bar{G} ; we introduce

$$\mathcal{Z}^{(1)} = \mathcal{Z} \cap \{A \leq 1\},$$

and denote by m_1 the restriction of m_0 to $\mathcal{Z}^{(1)}$; it is invariant under the restriction of \bar{G} to $\mathcal{Z}^{(1)}$.

We now project back to the level of i.e.m. , i.e. of length data alone: we obtain a map

$$G(\pi_0, \pi_1, \lambda) = (\bar{\pi}_0, \bar{\pi}_1, e^{t(\lambda)/2} \bar{\lambda})$$

and a measure m_2 image of m_1 under the projection which is invariant under G . As $\hat{\lambda}^*$ is still invariant under G , we can restrict, by homogeneity, the measure m_2 to $\{\hat{\lambda}^* = 1\}$ to obtain a measure m invariant under the restriction of G . This is the measure that we are interested in and that we will now describe.

Let (π_0, π_1, λ) be fixed; assume for instance that $\lambda_{\alpha_0} > \lambda_{\alpha_1}$. Consider in τ -space the polyhedral cone

$$\mathcal{U}_0 = \{\Im m \xi_\alpha^0 > 0, \forall \alpha \in \mathcal{A}, \Im m \xi_\alpha^1 < 0, \forall \alpha \neq \alpha_1\}.$$

The density χ of m_2 at (π_0, π_1, λ) is equal to the volume of $\mathcal{U}_0 \cap \{A \leq 1\}$. Write \mathcal{U}_0 , up to a codimension 1 subset, as a finite union of disjoint simplicial cones \mathcal{U} . For each \mathcal{U} , choose a unimodular basis $\tau^{(1)}, \dots, \tau^{(d)}$ of \mathbb{R}^A generating \mathcal{U} and write $h^{(j)} = -\Omega\tau^{(j)}$. One has

$$\chi_{\pi_0, \pi_1}(\lambda) = (d!)^{-1} \sum_{\mathcal{U}} \prod_1^d \left(\sum_{\alpha \in \mathcal{A}} \lambda_\alpha h_\alpha^{(j)} \right)^{-1}. \quad (*)$$

If we set

$$\begin{aligned}\hat{\lambda}_{\alpha_0} &= \lambda_{\alpha_0} - \lambda_{\alpha_1} \\ \hat{\lambda}_\alpha &= \lambda_\alpha, \alpha \neq \alpha_0 \\ \hat{h}_{\alpha_1} &= h_{\alpha_0} + h_{\alpha_1} \\ \hat{h}_\alpha &= h_\alpha, \alpha \neq \alpha_1\end{aligned}$$

we have

$$\sum_{\alpha \in \mathcal{A}} \lambda_\alpha h_\alpha^{(j)} = \sum_{\alpha \in \mathcal{A}} \hat{\lambda}_\alpha \hat{h}_\alpha^{(j)}.$$

Define

$$W_j = \{\alpha \in \mathcal{A}, \hat{h}_\alpha^{(j)} \neq 0\}.$$

The key property is now the following ([V2],[Z1] see also [Y])

Lemma *For any $X \subset \mathcal{A}$ with $\emptyset \neq X \neq \mathcal{A}$, we have*

$$\text{card}\{j, W_j \cap X = \emptyset\} + \text{card} X < d.$$

When $\lambda_{\alpha_1} > \lambda_{\alpha_0}$, the only difference is that we have to start with

$$\mathcal{U}_1 = \{\Im m \xi_\alpha^0 > 0, \forall \alpha \neq \alpha_0, \Im m \xi_\alpha^1 < 0, \forall \alpha \in \mathcal{A}\}.$$

In the formula (*) above for the density, set

$$\chi_{\mathcal{U}}(\lambda) = \prod_1^d \left(\sum_{\alpha \in \mathcal{A}} \lambda_\alpha h_\alpha^{(j)} \right)^{-1}.$$

Up to a constant factor, the density of m on the simplex

$$\Delta = \{\lambda \in \mathbb{R}^{\mathcal{A}}, \hat{\lambda}_\alpha > 0, \hat{\lambda}^* = 1\}$$

is given by $\sum_{\mathcal{U}} \chi_{\mathcal{U}}$. One has

$$c^{-1} \leq \chi_{\mathcal{U}}(\lambda) \prod_{j=1}^d \left(\sum_{\alpha \in W_j} \hat{\lambda}_\alpha \right) \leq c. \quad (1)$$

To control the size of $\chi_{\mathcal{U}}$, we decompose Δ as follows. Set

$$\mathcal{N} = \{\vec{n} = (n_\alpha)_{\alpha \in \mathcal{A}} \in \mathbb{N}^{\mathcal{A}}, \min_{\alpha} n_\alpha = 0\}.$$

For $\vec{n} \in \mathcal{N}$, $\Delta(\vec{n})$ is the set of $\lambda \in \Delta$ such that

$$\begin{aligned} \hat{\lambda}_\alpha &\geq \frac{1}{2d} \text{ if } n_\alpha = 0, \\ \frac{1}{2d} 2^{1-n_\alpha} &> \hat{\lambda}_\alpha \geq \frac{1}{2d} 2^{-n_\alpha} \text{ if } n_\alpha > 0. \end{aligned}$$

We obtain thus a partition

$$\Delta = \sqcup_{\mathcal{N}} \Delta(\vec{n}),$$

with the estimate

$$c^{-1} \leq 2^{\sum n_\alpha} \text{vol } \Delta(\vec{n}) \leq c. \quad (2)$$

For $\lambda \in \Delta(\vec{n})$, estimate (1) above gives

$$c^{-1} \leq \chi_{\mathcal{U}}(\lambda) 2^{-\sum_j \min_{W_j} n_\alpha} \leq c. \quad (3)$$

With fixed \vec{n} , let $0 = n^0 < n^1 < \dots$ be the values taken by the n_α and $V^i \subset \mathcal{A}$ the set of indices with $n_\alpha \geq n^i$. On one side, one has

$$\begin{aligned} \sum_{\alpha \in \mathcal{A}} n_\alpha &= \sum_{i \geq 0} n^i (\text{card } (V^i \setminus V^{i+1})) \\ &= \sum_{i > 0} (n^i - n^{i-1}) \text{card } V^i. \end{aligned}$$

On the other side, let \tilde{V}^i be the set of j such that $W_j \subset V^i$; one has $\min_{W_j} n_\alpha = n^i$ if and only if $j \in \tilde{V}^i \setminus \tilde{V}^{i+1}$ hence

$$\begin{aligned} \sum_{j=1}^d \min_{W_j} n_\alpha &= \sum_{i \geq 0} n^i (\text{card } (\tilde{V}^i \setminus \tilde{V}^{i+1})) \\ &= \sum_{i > 0} (n^i - n^{i-1}) \text{card } \tilde{V}^i. \end{aligned}$$

By the Lemma above, one has

$$\text{card } \tilde{V}^i < \text{card } V^i$$

as long as $0 < \text{card } V^i < d$. This shows that

$$\sum_{\alpha \in \mathcal{A}} n_\alpha - \sum_{j=1}^d \min_{W_j} n_\alpha \geq |\vec{n}|_\infty := \text{Max}_{\alpha \in \mathcal{A}} n_\alpha.$$

The last estimate, introduced into (2), (3) gives

$$(\text{vol } \Delta(\vec{n})) \text{Max}_{\Delta(\vec{n})} \chi_{\mathcal{U}} \leq c 2^{-|\vec{n}|_{\infty}} . \quad (4)$$

The integrability of $\chi_{\mathcal{U}}$ over Δ now follows from the fact that the number of $\vec{n} \in \mathcal{N}$ with $|\vec{n}|_{\infty} = N$ is of order N^{d-2} .

If we compare (2) and (4), we obtain

$$\text{Max}_{\Delta(\vec{n})} \chi_{\pi_0, \pi_1} \leq c 2^{|\vec{n}|_1 - |\vec{n}|_{\infty}} , \quad (5)$$

with $|\vec{n}|_1 = \sum_{\alpha \in \mathcal{A}} n_{\alpha}$. When $d = 2$, χ_{π_0, π_1} is bounded. Assume now $d > 2$. From (2) and (5), one obtains

$$m(\{\chi_{\pi_0, \pi_1} > 2^N\}) \leq \sum_{|\vec{n}|_1 - |\vec{n}|_{\infty} \geq N - c} c 2^{-|\vec{n}|_{\infty}} ; \quad (6)$$

to have $|\vec{n}|_1 \geq |\vec{n}|_{\infty} + N - c$, one must have $|\vec{n}|_{\infty} \geq \frac{N-c}{d-2}$; an easy computation leads to

$$m(\{\chi_{\pi_0, \pi_1} > 2^N\}) \leq c 2^{-\frac{N}{d-2}} , \quad (7)$$

$$\text{Leb}(\{\chi_{\pi_0, \pi_1} > 2^N\}) \leq c 2^{-N \frac{d-1}{d-2}} . \quad (8)$$

It follows, as χ_{π_0, π_1} is bounded from below that we have, for every Borel set X

$$c^{-1} \text{Leb}(X) \leq m(X) \leq c (\text{Leb}(X))^{\frac{1}{d-1}} .$$

4.4 Integrability of $\log \|Z_{(1)}\|$

Recall the function $Z_{(1)}$, with values in $\text{SL}(\mathbb{Z}^A)$, defined in 1.2.4: the sequence $(\pi_0^{(k)}, \pi_1^{(k)}, \lambda^{(k)})$ given by the Zorich algorithm satisfies

$$\lambda^{(k)} = Z_{(1)}(\pi_0^{(k)}, \pi_1^{(k)}, \lambda^{(k)}) \lambda^{(k+1)} .$$

Following Zorich ([Z1]) we estimate $\|Z_{(1)}\|$ w.r.t. the absolutely continuous invariant measure m . This will be used in two ways:

- applying Oseledets multiplicative ergodic theorem in order to prove that conditions (b) and (c) in 1.3 have full measure;
- as a first step in an induction to prove that condition (a) in 1.3.1 has full measure.

We use as norm the supremum of the coefficients. For $k \geq 0$, $\lambda_{\alpha_\varepsilon} > \lambda_{\alpha_{1-\varepsilon}}$, we have

$$\|Z_{(1)}\| > k \iff \hat{\lambda}_{\alpha_\varepsilon} > k \sum_{\pi_{1-\varepsilon}\alpha > \pi_{1-\varepsilon}\alpha_\varepsilon} \hat{\lambda}_\alpha ;$$

it follows that

$$\|Z_{(1)}\| > (2d)2^{N-1} \Rightarrow \lambda \in \cup_{|\vec{n}|_\infty \geq N} \Delta(\vec{n}) ,$$

which in turn implies that

$$m(\{\|Z_{(1)}\| > 2^N\}) \leq cN^{d-2}2^{-N} .$$

This is the required estimate; it shows that $\|Z_{(1)}\|^\rho$ is m -integrable for all $\rho < 1$ and a fortiori that $\log \|Z_{(1)}\|$ is m -integrable.

4.5 Conditions (b) and (c) have full measure

As $\log \|Z_{(1)}\|$ is m -integrable, we can apply Oseledets theorem and obtain the existence almost everywhere of Lyapunov exponents for the corresponding cocycle.

The space Γ_s is then associated to the negative Lyapunov exponents. The two estimates in condition (c) are immediate consequences of the properties of Oseledets decomposition.

For property (b), we recall the result of Veech ([V3]): the largest Lyapunov exponent is almost everywhere simple. The existence of a spectral gap follows.

In the end of the section, we will prove that property (a) in 1.3.1 has full measure.

4.6 The main step

Let $(\mathcal{A}, \pi_0, \pi_1)$ be combinatorial data, \mathcal{D} the associated Rauzy diagram. For an i.e.m. T satisfying Keane's condition with these data, the Rauzy-Veech algorithm defines an infinite path $(\gamma^{(n)}(T))_{n>0}$ in \mathcal{D} , starting at (π_0, π_1) .

Conversely, if $\gamma = (\gamma^{(n)})_{0 < n \leq N}$ is a finite path in \mathcal{D} starting at (π_0, π_1) , we denote by $\Delta(\gamma)$ the simplex of normalized T in $\Delta(\pi_0, \pi_1)$ such that $\gamma^{(n)}(T) = \gamma^{(n)}$ for $0 < n \leq N$. We use here the old normalization $\{\lambda^* = 1\}$.

To such a path γ is associated a matrix $Q(\gamma) \in \text{SL}(\mathbb{Z}^{\mathcal{A}})$:

$$Q(\gamma) = V(\gamma^{(1)}) \cdots V(\gamma^{(n)}) ,$$

and we write as before

$$Q_\beta(\gamma) = \sum_{\alpha \in \mathcal{A}} Q_{\alpha\beta}(\gamma) .$$

We have

$$1 = \sum_{\alpha \in \mathcal{A}} \lambda_{\alpha}^{(0)} = \sum_{\beta \in \mathcal{A}} Q_{\beta}(\gamma) \lambda_{\beta}$$

(where $\lambda^{(0)} = Q(\gamma)\lambda^{(N)}$), and it follows that

$$\text{vol}_{d-1}(\Delta(\gamma)) = \left[\prod_{\beta \in \mathcal{A}} Q_{\beta}(\gamma) \right]^{-1} \text{vol}_{d-1}(\Delta(\pi_0, \pi_1)).$$

Denote by $(\pi_0^{(N)}, \pi_1^{(N)})$ the endpoint of γ , by $\alpha_0^{(N)}, \alpha_1^{(N)}$ the indices such that $\pi_{\varepsilon}^{(N)}(\alpha_{\varepsilon}^{(N)}) = d$. They are the names of the two arrows going out of $(\pi_0^{(N)}, \pi_1^{(N)})$. The conditional probability, for an i.e.m. T in $\Delta(\gamma)$, that the name of $\gamma^{(N+1)}(T)$ is $\alpha_{\varepsilon}^{(N)}$ is equal to $Q_{\alpha_{1-\varepsilon}^{(N)}}(Q_{\alpha_0^{(N)}} + Q_{\alpha_1^{(N)}})^{-1}$.

Let $1 \leq D < d$. A segment $(\gamma^{(n)}(T))_{k \leq n < l}$ is called a D -segment if the arrows of the segment take no more than D distinct names. It is called *maximal* if $(\gamma^{(n)}(T))_{k \leq n \leq l}$ is not a D -segment.

The following proposition is the main step in proving that condition (a) has full measure.

Proposition *There exist an integer $l = l(d)$ and a constant $\eta = \eta(d) > 0$ with the following properties. Let $\gamma = (\gamma^{(n)})_{0 < n \leq N}$ be a finite path in \mathcal{D} such that the set \mathcal{A}' of names of arrows of γ is distinct from \mathcal{A} . Assume that $D = \text{card } \mathcal{A}' > 1$. There is a subset $\Delta'(\gamma)$ of $\Delta(\gamma)$ with*

$$\text{vol}_{d-1}(\Delta'(\gamma)) \geq \eta \text{vol}_{d-1}(\Delta(\gamma))$$

such that, for every $T \in \Delta'(\gamma)$, there exists $M > N$ with

- the name of $\gamma^{(M)}(T)$ does not belong to \mathcal{A}' ;
- no more than $l(D-1)$ -segments are needed to cover $(\gamma^{(n)})_{N \leq n < M}$.

We will first explain how the full measure estimate for condition (a) follows from the proposition, and then prove the proposition.

4.7 Condition (a) has full measure

For $T \in \Delta(\pi_0, \pi_1)$, satisfying Keane's condition, and $1 \leq D < d$, denote by $Z_{(D)}(T)$ the matrix in $\text{SL}(\mathbb{Z}^A)$ associated to the initial maximal D -segment in $(\gamma^{(n)}(T))_{n > 0}$. Denote by $M_{(D)}(T)$ (resp. $M_{(D)}^1(T)$) the minimal number of $(D-1)$ -segments (resp. 1-segments) needed to cover this initial maximal D -segment.

Corollary Let $N > 0$. Except on a set of measure $\leq c2^{-cN^{1/D}}$, one has

$$\begin{aligned} \|Z_{(D)}(T)\| &\leq 2^N, \\ M_{(D)}(T) &\leq N^{1/D}, \\ M_{(D)}^1(T) &\leq N^{\frac{D-1}{D}}. \end{aligned}$$

Remark The measure referred to can be either Lebesgue or the invariant measure m : in view of the last formula of 4.3, it changes only the values of the constants.

Proof. The estimate for $Z_{(1)}$ has been shown in 4.5. Let us show the estimate for $M_{(D)}$.

Let $\gamma = (\gamma^{(n)})_{0 < n \leq N}$ be any finite path such that $(\gamma^{(n)})_{0 < n < N}$ is a $(D-1)$ -segment but γ is not. Apply a first time the proposition in each $\Delta(\gamma)$. One obtains that

$$\text{Leb}(\{M_{(D)} > l + 1\}) < 1 - \eta.$$

We next subdivide the set $\{M_{(D)} > l + 1\}$ into simplices $\Delta(\gamma_1)$, where $\gamma_1 = (\gamma_1^{(n)})_{0 < n \leq N_1}$ is a D -segment and $(\gamma_1^{(n)})_{0 < n < N_1}$ is the concatenation of $(l+1)$ maximal $(D-1)$ -segments. Applying once again the proposition in each $\Delta(\gamma_1)$ gives

$$\text{Leb}(\{M_{(D)} > 2l + 1\}) < (1 - \eta)^2.$$

Iterating this process leads to the required estimate for $M_{(D)}$.

We next show by induction on D that

$$m(M_{(D)}^1 > N^{\frac{D-1}{D}}) \leq c2^{-cN^{1/D}}.$$

For $D = 2$, one has $M_{(D)} = M_{(D)}^1$; the comparison between m and the Lebesgue measure gives the estimate. Assume $D > 2$ and write

$$Z_{(D)}(T) = Z_{(D-1)}(T_0)Z_{(D-1)}(T_1) \cdots Z_{(D-1)}^*(T_{M-1})$$

with $T_0 = T$, $M = M_{(D)}(T)$, and T_i is obtained from T_0 by n_i iterations of the Zorich algorithm (we have $0 = n_0 < n_1 < n_2 < \dots$); $Z_{(D-1)}^*(T_{M-1})$ denotes some initial part in the product giving $Z_{(D-1)}(T_{M-1})$.

Neglecting a set of measure $\leq c2^{-cN^{1/D}}$, we can assume that $M \leq N^{1/D}$.

By the induction hypothesis, applied with $N' = N^{\frac{D-1}{D}}$, we have

$$m(M_{(D-1)}^1(T_0) > N^{\frac{D-2}{D}}) \leq c2^{-cN^{1/D}}.$$

As the measure m is invariant under the Zorich algorithm, the same estimate holds when we put instead of T_0 any given iterate $T^{(k)}$ of T_0 under the algorithm. Thus we have

$$\begin{aligned} m(\text{Max}_{0 \leq k < N} M_{(D-1)}^1(T^{(k)}) > N^{\frac{D-2}{D}}) &\leq cN2^{-cN^{1/D}} \\ &\leq c'2^{-c'N^{1/D}}. \end{aligned}$$

On the other side, when

$$\text{Max}_{0 \leq k < N} M_{(D-1)}^1(T^{(k)}) \leq N^{\frac{D-2}{D}},$$

we have

$$n_i \leq iN^{\frac{D-2}{D}}$$

for

$$0 \leq i \leq M - 1 < N^{1/D}$$

and

$$M_{(D)}^1(T) \leq N^{\frac{D-1}{D}}.$$

This proves the estimate for $M_{(D)}^1$.

The estimate on $Z_{(D)}$ is again proven by induction on D , the case $D = 1$ having been done in 4.4. Neglecting a set of measure $c2^{-cN^{1/D}}$, we may assume $M_{(D)}(T) \leq N^{1/D}$ and $M_{(D)}^1(T) \leq N^{\frac{D-1}{D}}$. Write $Z_{(D)}(T)$ as above.

If $\|Z_{(D)}(T)\| > 2^N$, one can find $i \in \{0, 1, \dots, M - 1\}$ such that

$$\|Z_{(D-1)}(T_i)\| > 2^{N/M} \geq 2^{N \frac{D-1}{D}}.$$

By the induction hypothesis, we have

$$m\left(\{\|Z_{(D-1)}(T_0)\| > 2^{N \frac{D-1}{D}}\}\right) \leq c2^{-cN^{1/D}}$$

and the same estimate holds if we replace T_0 by any given $T^{(k)}$. It is sufficient to consider $k \leq N^{\frac{D-1}{D}}$. Again, one has

$$cN^{\frac{D-1}{D}}2^{-cN^{1/D}} \leq c'2^{-c'N^{1/D}},$$

and this concludes the proof of the corollary. \square

The proof that condition (a) has full measure follows now from a usual Borel–Cantelli argument. Take $D = d - 1$ and write $N = (\kappa \log k)^{d-1}$ with fixed large $\kappa > 0$ and an integer $k \geq 0$. One has

$$m(\{\|Z_{(d-1)}(T)\| > 2^N\}) \leq ck^{-c\kappa}.$$

If κ is large enough, the right hand term form a converging series. As m is invariant under the Zorich algorithm, we conclude that almost surely, the iterates $T^{(k)}$ of T under the Zorich algorithm satisfy

$$\|Z_{(d-1)}(T^{(k)})\| \leq 2^{\kappa(\log k)^{d-1}},$$

for all large k .

On the other hand, the exponential rate of growth of the $Q(k)$ (in the Zorich algorithm) is given by the largest Lyapunov exponent of the Teichmüller flow, which is positive.

We conclude that there exists κ_1 such that almost all i.e.m. T satisfy

$$\log \|Z_{(d-1)}(k)\| \leq \kappa_1 [\log \log \|Q(k)\|]^{d-1}$$

for all large enough k .

Question. *Does one have almost surely*

$$\|Z_{(d-1)}(k)\| = O([\log \|Q(k)\|]^C)$$

for some $C > 0$?

4.8 Proof of the Proposition

Let γ, \mathcal{A}', D be as in the proposition. Let T be an i.e.m. in $\Delta(\gamma)$ satisfying Keane’s condition. Define, for $n \geq 0$

$$Q'(n, T) = \sum_{\alpha \in \mathcal{A}'} Q_\alpha(n, T),$$

$$Q_{ext}(n, T) = \sum_{\alpha \in \mathcal{A} \setminus \mathcal{A}'} Q_\alpha(n, T),$$

where $Q_\alpha(n, T)$ is the shorthand for $Q_\alpha((\gamma_j(T))_{0 \leq j \leq n})$ (see the beginning of 4.6).

Lemma 1. *If the names of the arrows $\gamma^{(m)}(T)$ belong to \mathcal{A}' for $m \leq n$, we have*

$$Q_{ext}(n, T) \leq (d - 2)Q'(n, T).$$

(Recall that, as $1 < D < d$, we have $d \geq 3$).

Proof. We start with $Q_\alpha(0, T) = 1$ for all $\alpha \in \mathcal{A}$. Divide the segment $[1, n]$ into maximal 1-segments into which the name of the arrows is the same; let $[n_i, n_{i+1})$ be such a segment, with arrows of name $\alpha_i \in \mathcal{A}'$. The secondary names of these arrows appear with some periodicity $d_i < d$; moreover, if $n_i > 1$, the secondary name of $\gamma^{(n_i)}$ is $\alpha_{i-1} \in \mathcal{A}'$; if $n_i = 1$, $i = 0$, $n_1 > d_0$, the secondary name of $\gamma^{(m)}$ is α_1 for each $m = n_1 - kd_0$, $k > d_0$. For $n_i \leq m < n_{i+1}$ we have

$$\begin{aligned} Q_{ext}(m, T) &= Q_{ext}(m-1, T), \\ Q'(m, T) &= Q'(m-1, T) + Q_{\alpha_i}(n_i-1, T), \end{aligned}$$

if the secondary name of $\gamma^{(m)}$ is in \mathcal{A}' and

$$\begin{aligned} Q_{ext}(m, T) &= Q_{ext}(m-1, T) + Q_{\alpha_i}(n_i-1, T), \\ Q'(m, T) &= Q'(m-1, T), \end{aligned}$$

otherwise. In each segment except perhaps the first one, the number of secondary names in $\mathcal{A} \setminus \mathcal{A}'$ does not exceed $(d-2)$ times the number of secondary names in \mathcal{A}' . In the first segment, we write $n_1 = kd_0 + n'_1$, $0 < n'_1 \leq d_0$; again the number of secondary names in $\mathcal{A} \setminus \mathcal{A}'$ does not exceed $(d-2)$ times the number of secondary names in \mathcal{A}' in the subsegment $[n'_1, n_1)$. Finally we have for $0 \leq m < n'_1$ that $Q'(m, T) \geq D \geq 2$, $Q_{ext}(m, T) \leq Q_{ext}(0, T) + m \leq d - D + d_0 - 1 \leq 2d - 4$ and the estimate of the lemma follows. \square

Let $1 \leq D_1 \leq D$, $n \geq 0$, $C_1 > 0$. We say that $T \in \Delta(\gamma)$ is (D_1, n, C_1) -balanced if we have

$$Q_\alpha(n, T) \geq C_1^{-1} Q'(n, T)$$

for at least D_1 indices $\alpha \in \mathcal{A}'$. The property only depends on the path $(\gamma^{(m)}(T))_{0 < m \leq n}$ and we will also say that this path is (D_1, n, C_1) -balanced. Clearly, any T is $(1, n, D)$ -balanced (for all $n \geq 0$).

Lemma 2. *Assume that γ is (D, n, C_0) -balanced, for some constant $C_0 > 0$. Then we can find $\Delta'(\gamma) \subset \Delta(\gamma)$ satisfying the conclusions of the proposition, with $l = l(d)$ and $\eta = \eta(d, C_0)$.*

Proof. Let $\gamma' = (\gamma^{(n)})_{0 < n \leq M}$ be an extension of γ with minimal length such that the name α of $\gamma^{(M)}$ is not in \mathcal{A}' . Then $M - N$ is bounded by the diameter of \mathcal{D} , i.e. in terms of d only. Therefore there exists $C_* = C_*(d)$ such that γ' is $(D, M-1, C_*C_0)$ -balanced; moreover, the path $\gamma'' = (\gamma^{(n)})_{0 < n < M}$ satisfies

$$\text{Vol}_{d-1} \Delta(\gamma'') \geq \eta'' \text{Vol}_{d-1} \Delta(\gamma),$$

with $\eta'' = \eta''(C_0, d)$. Then, for all $\beta \in \mathcal{A}'$, we have

$$\begin{aligned} Q_\alpha(M-1) &\leq Q_{ext}(M-1) \\ &\leq (d-2)Q'(M-1) \\ &\leq (d-2)C_*C_0Q_\beta(M-1), \end{aligned}$$

and therefore

$$\text{Vol}_{d-1} \Delta(\gamma') \geq \eta' \text{Vol}_{d-1} \Delta(\gamma''),$$

with $\eta' = (1 + (d-2)C_*C_0)^{-1}$. We take $\eta = \eta'\eta''$, $\Delta'(\gamma) = \Delta(\gamma')$. Finally l is bounded because $M - N$ is bounded. \square

When γ is only $(\tilde{D}, N, \tilde{C})$ -balanced for some $\tilde{D} < D$, the strategy will be to extend γ without losing volume in order to obtain a more balanced path; at the end we should be able to apply Lemma 2 (unless we have already found $\Delta'(\gamma)$).

We therefore assume that γ is $(\tilde{D}, N, \tilde{C})$ -balanced. This is certainly satisfied with $\tilde{D} = 1$, $\tilde{C} = D$. Denote by $\tilde{\mathcal{A}}$ the set of α such that

$$Q_\alpha(N) \geq \tilde{C}^{-1}Q'(N).$$

The first step is to extend γ to a path $\gamma' = (\gamma^{(n)}(T))_{0 < n \leq N'}$ of minimal length such that the name of $\gamma^{(N')}$ is not in $\tilde{\mathcal{A}}$. When $N' = N + 1$, there might be two choices for $\gamma^{(N')}$ and we choose the one which gives the largest volume to $\Delta(\gamma')$.

In any case, an argument completely similar to the one in the proof of Lemma 2 leads to the estimate

$$\text{Vol}_{d-1} \Delta(\gamma') \geq \eta' \text{Vol}_{d-1} \Delta(\gamma),$$

with a constant $\eta' = \eta'(d, \tilde{C})$.

If the name of $\gamma^{(N')}$ does not belong to \mathcal{A}' , we can take as in Lemma 2 $\Delta'(\gamma) = \Delta(\gamma')$ and the proof of the proposition is over. We now assume that the name of $\gamma^{(N')}$ belongs to $\mathcal{A}' \setminus \tilde{\mathcal{A}}$.

The subset $\Delta'(\gamma)$ of $\Delta(\gamma)$ we are looking for will be contained in $\Delta(\gamma')$. Observe that there exists $C_* = C_*(d)$ such that γ' is $(\tilde{D}, N', C_*\tilde{C})$ -balanced.

Case A: In the loop of arrows of the same name which starts with $\gamma^{(N')}$, no secondary name belongs to $\mathcal{A}' \setminus \tilde{\mathcal{A}}$.

Let α be the name of $\gamma^{(N')}$, $\beta_0, \dots, \beta_{r-1}$ being the successive secondary names in the loop. Let $k > 0$, that we write $k = rl + m$, $0 \leq m < r$. Let $\gamma_1(k)$ be the path extending γ' such that

- the name of $\gamma_1(k)^{(n)}$ is α for $N' < n < N' + k := N_1(k)$;
- the name of $\gamma_1(k)^{(N_1(k))}$ is β_m .

Observe that it follows immediately from the definition of R_0, R_1 in 1.2.1 that the indices $\beta_0, \dots, \beta_{r-1}$ are distinct. Therefore, we will have, for $0 \leq k_1 = rl_1 + m_1 < k$:

$$Q_{\beta_j}(N' + k_1) = Q_{\beta_j}(N' - 1) + \begin{cases} l_1 Q_\alpha(N' - 1) & \text{if } m_1 < j, \\ (l_1 + 1) Q_\alpha(N' - 1) & \text{if } m_1 \geq j, \end{cases}$$

$$Q_\alpha(N' + k_1) = Q_\alpha(N' - 1),$$

and also

$$Q_{\beta_j}(N' + k) = Q_{\beta_j}(N' + k - 1),$$

$$Q_\alpha(N' + k) = Q_\alpha(N' - 1) + Q_{\beta_m}(N' + k - 1).$$

For any $k > 0$, the extension from γ to $\gamma_1(k)$ is covered by the same number of $(D - 1)$ segments, which is bounded in terms of d only.

For those k such that $\beta_m \notin \mathcal{A}'$, we include $\Delta(\gamma_1(k))$ in $\Delta'(\gamma)$.

The formulas for the volumes give

$$\text{Vol}_{d-1}([\cup_{0 < k_1 < k} \Delta(\gamma_1(k_1))]^c) = \frac{\prod Q_{\beta_j}(N')}{\prod Q_{\beta_j}(N' + k - 1)} \text{Vol}_{d-1} \Delta(\gamma').$$

We keep for further consideration all $\gamma_1(k)$ with

$$kQ_\alpha(N' - 1) \leq Q'(N' - 1).$$

The formula above shows that together they will fill a definite proportion of $\Delta(\gamma')$.

We also see that when $\beta_m \in \tilde{\mathcal{A}}$, $\gamma_1(k)$ will be $(\tilde{D} + 1, N_1(k), C_1)$ -balanced, with C_1 depending only on d . For each such $\gamma_1(k)$, we either apply Lemma 2 (if $\tilde{D} + 1 = D$) or repeat the discussion, with $\gamma_1(k)$ in the place of γ , from a better starting hypothesis.

Case B: The complement of case A.

For an i.e.m. in $\Delta(\gamma')$ satisfying Keane's condition, we consider the three mutually exclusive possibilities:

- T is of type I if there exists $N_1 \geq N'$ such that all arrows $\gamma^{(n)}(T)$, $N' \leq n \leq N_1$, have names in $\mathcal{A}' \setminus \tilde{\mathcal{A}}$, and we have

$$\sum_{\alpha \in \mathcal{A}' \setminus \tilde{\mathcal{A}}} Q_\alpha(N_1, T) \geq Q'(N', T).$$

We take a minimal such N_1 .

- T is of type II (respectively of type III) if it is not of type I and the first name of an arrow $\gamma^{(n)}(T)$, $n > N'$, which does not belong to $\mathcal{A}' \setminus \tilde{\mathcal{A}}$ belongs to $\tilde{\mathcal{A}}$ (resp. to $\mathcal{A} \setminus \mathcal{A}'$).

We deal separately with the three types.

a) All T of type III will be contained in $\Delta'(\gamma)$; for such a T , M is the first integer $> N'$ for which the name does not belong to $\mathcal{A}' \setminus \tilde{\mathcal{A}}$. Observe that the segment $(\gamma^{(n)}(T))_{N' \leq n < M}$ is a $(D-1)$ -segment because $\text{card}(\mathcal{A}' \setminus \tilde{\mathcal{A}}) < D$. As $N' - N$ is bounded in function of d only, the number of $(D-1)$ -segments needed to cover $(\gamma^{(n)}(T))_{N \leq n < M}$ is bounded in terms of d only.

b) Assume that T is of type II. Let N_1 be the smallest integer $n > N'$ such that the name of $\gamma^{(n)}(T)$ does not belong to $\mathcal{A}' \setminus \tilde{\mathcal{A}}$; this name belongs to $\tilde{\mathcal{A}}$. Let $\gamma_1 = (\gamma^{(n)}(T))_{0 < n \leq N_1}$. When T varies among i.e.m. 's of type II, the γ_1 form an at most countable collection such that the corresponding simplices $\Delta(\gamma_1)$ have disjoint interiors (and are contained in $\Delta(\gamma')$). Every T_1 belonging to some $\Delta(\gamma_1)$ is also of type II. We claim that every γ_1 is (D_1, N_1, C_1) -balanced with $D_1 > \tilde{D}$ and $C_1 = C_1(\tilde{C}, d)$ (see the proof below). As for type III, the number of $(D-1)$ -segments needed to cover $\gamma_1^{(n)}$, $N \leq n < N_1$, is bounded in terms of d only.

c) Assume that T is of type I. With N_1 minimal as in the definition of type I, take $\gamma_1 = (\gamma^{(n)}(T))_{0 < n \leq N_1}$. When T varies among i.e.m. 's of type I, the γ_1 form again an at most countable collection for which the corresponding simplices $\Delta(\gamma_1)$ have disjoint interiors (and are contained in $\Delta(\gamma')$). Every T_1 belonging to some $\Delta(\gamma_1)$ is also of type I. We claim that every γ_1 is (D_1, N_1, C_1) -balanced with $D_1 > \tilde{D}$ and $C_1 = C_1(\tilde{C}, d)$ (see the proof below). The number of $(D-1)$ -segments needed to cover $\gamma_1^{(n)}$, $N \leq n < N_1$, is bounded in terms of d only.

The discussion above leads in case B to a countable partition (up to a codimension one subset) of $\Delta(\gamma')$ into subsimplices of type III which will be included in $\Delta'(\gamma)$ and simplices $\Delta(\gamma_1)$ (of type I or II) which satisfy the same hypotheses than $\Delta(\gamma)$ but are better balanced (i.e. $D_1 > \tilde{D}$); when $D_1 = D$, we can apply Lemma 2 to γ_1 ; when $D_1 < D$, we repeat the discussion with γ_1 instead of γ . The process stops in less than D steps and gives the conclusion of the proposition. \square

Proof of the claim for type II. As T is not of type I, we have

$$\sum_{\alpha \in \mathcal{A}' \setminus \tilde{\mathcal{A}}} Q_\alpha(N_1 - 1, T) < Q'(N', T).$$

Let us consider a maximal 1-segment contained in $(\gamma^{(n)}(T))_{N' \leq n < N_1}$. As

we are not in case A, there is a definite proportion, depending only of d , of secondary names which belong to $\mathcal{A}' \setminus \tilde{\mathcal{A}}$. This implies that we must have

$$Q'(N_1 - 1, T) \leq C'_1 Q'(N', T),$$

with C'_1 depending only on d . On the other hand, if $\alpha \in \mathcal{A}' \setminus \tilde{\mathcal{A}}$ and $\beta \in \tilde{\mathcal{A}}$ are the names of $\gamma^{(N_1-1)}(T)$, $\gamma^{(N_1)}(T)$ respectively, we have

$$\begin{aligned} Q_\alpha(N_1) &= Q_\alpha(N_1 - 1) + Q_\beta(N_1 - 1) \\ &\geq (C_* \tilde{C})^{-1} Q'(N', T). \end{aligned}$$

It follows that γ_1 is $(\tilde{D} + 1, N_1, C_1)$ -balanced with $C_1 = C'_1 C_* \tilde{C}$. \square

Proof of the claim for type I. By definition of N_1 , we have again

$$\sum_{\alpha \in \mathcal{A}' \setminus \tilde{\mathcal{A}}} Q_\alpha(N_1 - 1, T) < Q'(N', T),$$

and it follows again that

$$Q'(N_1 - 1, T) \leq C'_1 Q'(N', T).$$

By definition of N_1 , we have now

$$\sum_{\alpha \in \mathcal{A}' \setminus \tilde{\mathcal{A}}} Q_\alpha(N_1, T) \geq Q'(N', T),$$

and it follows that γ_1 is $(\tilde{D} + 1, N_1, C_1)$ -balanced with C_1 depending only on \tilde{C} and d . \square

The proof of the proposition, and therefore also of the full measure statement, is now complete.

Appendix A. Roth–type conditions in a concrete family of interval exchange maps

A.1 Let $\mathcal{A} = (A, B, C, D)$. The Rauzy diagram of the pair $(\pi_0, \pi_1) = \begin{pmatrix} A & B & C & D \\ D & C & B & A \end{pmatrix}$ is indicated in 1.2.2. The suspension of an i.e.m. with these combinatorial data leads to an holomorphic 1–form with a double zero on a genus two surface.

In this diagram, we define for $n \geq 0$ a loop $\gamma(n)$ based at (π_0, π_1) by asking that the names of the successive arrows should be $D^2CDA^2B^nA$. The product of the V matrices around this loop is

$$M(n) = \begin{pmatrix} 1 & 1 & 1 & 1 \\ n & n+1 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ n+1 & n+2 & 2 & 2 \end{pmatrix}$$

with characteristic polynomial

$$\chi_n(X) = X^4 - (n+6)X^3 + (3n+10)X^2 - (n+6)X + 1$$

Setting $U = X + X^{-1}$ leads to

$$\chi_n(X) = X^2(U^2 - (n+6)U + 3n+8).$$

The eigenvalues of $M(n)$ are thus given by

$$\lambda + \lambda^{-1} = U^\pm := \frac{1}{2}(n+6 \pm \sqrt{n^2+4}).$$

The case $n = 0$ is degenerate, with $U^+ = 4$, $U^- = 2$. When $n > 0$, both U^+ , U^- are > 2 ; we will denote the eigenvalues by $\lambda_u^+ > \lambda_u^- (> 1) > \lambda_s^- > \lambda_s^+$, by $e_u^+, e_u^-, e_s^-, e_s^+$ the corresponding eigenvectors of the *transposed* matrix ${}^tM(n)$.

The eigenvector associated to the eigenvalue λ is proportional to

$$((\lambda-1)(\lambda^2-4\lambda+2), \lambda^3-4\lambda^2+3\lambda-1, \lambda(\lambda-1), (\lambda-1)^2).$$

A.2 As $n \rightarrow +\infty$, one has

$$\begin{aligned} \lim U^+ - (n+3) &= \lim \lambda_u^+ - (n+3) = 0, \\ \lim U^- &= 3, \quad \lim \lambda_u^- = G := \frac{\sqrt{5}+3}{2}. \end{aligned}$$

One can also choose eigenvectors to obtain:

$$\begin{aligned}\lim e_u^+ &= E_u^+ := (1, 1, 0, 0) \\ \lim e_u^- &= E_u^- := (-1, -1, G - 1, 1) \\ \lim e_s^- &= E_s^- := (-1, -1, G^{-1} - 1, 1) \\ \lim e_s^+ &= E_s^+ := (2, 1, 0, -1) .\end{aligned}$$

These four limit vectors form a basis of \mathbb{R}^4 in which we rewrite ${}^tM(n)$:

$$\begin{aligned}{}^tM(n)E_u^+ &= (n + 3)E_u^+ - E_s^+ + \frac{1}{\sqrt{5}}(E_u^- - E_s^-) , \\ {}^tM(n)E_s^+ &= E_u^+ , \\ {}^tM(n)E_u^- &= G(E_u^+ + E_u^-) , \\ {}^tM(n)E_s^- &= G^{-1}(E_u^+ + E_s^-) .\end{aligned}$$

For the corresponding coordinates, this gives

$$\begin{aligned}X_u^+ &= (n + 3)x_u^+ + x_s^+ + Gx_u^- + G^{-1}x_s^- , \\ X_s^+ &= -x_u^+ , \\ X_u^- &= \frac{1}{\sqrt{5}}x_u^+ + Gx_u^- , \\ X_s^- &= -\frac{1}{\sqrt{5}}x_u^+ + G^{-1}x_u^- .\end{aligned}$$

A.3 The following two lemmas express that for $n \geq 4$ certain cone conditions are satisfied.

Lemma 1. For $n \geq 4$, $x_u^+ \geq \text{Max}(|x_s^+|, |x_u^-|, |x_s^-|)$ one has

$$X_u^+ \geq \text{Max} \left((n - 1)|X_s^+|, (n - 1)|X_s^-|, \frac{10 - 3\sqrt{5}}{3}|X_u^-| \right) ,$$

and

$$X_u^+ \geq (n - 1)x_u^+ .$$

Proof. As $G^{-1} + \frac{1}{\sqrt{5}} < 1$ and $G + G^{-1} = 3$, we have

$$\text{Max} \left((n - 1)|X_s^+|, (n - 1)|X_s^-|, (n - 1)|x_u^+| \right) \leq X_u^+ .$$

If $x_u^- \geq 0$, one has

$$X_u^+ \geq (n+1)x_u^+ + Gx_u^- \geq 2X_u^-$$

because $n+1 - 2/\sqrt{5} > G$ for $n \geq 4$.

For $x_u^- < 0$, one has $X_u^+ \geq (n+1)x_u^+ - G|x_u^-|$. On one hand

$$(n+1)x_u^+ - G|x_u^-| \geq (n+1)\sqrt{5}X_u^- ,$$

on the other

$$(n+1)x_u^+ - G|x_u^-| \geq -\gamma X_u^-$$

as soon as $n+1 + \gamma/\sqrt{5} \geq (\gamma+1)G$, which allows to take $\gamma = \frac{10-3\sqrt{5}}{3}$ for $n \geq 4$.

□

Lemma 2. For $n \geq 4$, $\text{Max}(|x_u^+|, |x_u^-|) \geq \text{Max}(|x_s^+|, |x_s^-|)$, one has

$$\begin{aligned} \text{Max}(|X_u^+|, |X_u^-|) &\geq \left(G - \frac{1}{\sqrt{5}}\right) \text{Max}(|X_s^+|, |X_s^-|) , \\ \text{Max}(|X_u^+|, |X_u^-|) &\geq \left(G - \frac{1}{\sqrt{5}}\right) \text{Max}(|x_u^+|, |x_u^-|) . \end{aligned}$$

Proof. When $|x_u^+| \geq |x_u^-|$, this follows from Lemma 1. If $0 \leq |x_u^+| \leq |x_u^-|$, one has

$$X_u^- \geq \left(G - \frac{1}{\sqrt{5}}\right) x_u^- \geq \left(G - \frac{1}{\sqrt{5}}\right) \text{Max}(|X_s^+|, |X_s^-|)$$

because $G^{-1} + \frac{1}{\sqrt{5}} < 1$. □

One should observe in Lemma 1 that $\frac{10-3\sqrt{5}}{3} > 1$ and in Lemma 2 that $G - \frac{1}{\sqrt{5}} > 1$.

Lemma 3. Equip \mathbb{R}^A with the sup norm. Then, for any integers $n_1, \dots, n_k > 0$, we have

$$\prod_{i=1}^k (n_i + 1) \leq \| {}^t M(n_k) \cdots {}^t M(n_1) \|_\infty \leq \prod_{i=1}^k (2n_i + 4) .$$

Proof. The upper bound follows from $\| {}^t M(n) \| \leq 2n + 4$ for $n > 0$, the lower bound from the fact that

$${}^t M(n)(1, 1, 0, 0) - (n+1)(1, 1, 0, 0)$$

is a non negative vector. □

A.4 Let Σ be the set of sequences $(n_i)_{i>0}$ of integers ≥ 4 . To each sequence in Σ we associate the infinite path $\gamma(n_1)\gamma(n_2)\cdots$ starting at (π_0, π_1) . The cone property of Lemma 1 guarantees that there is exactly one i.e.m. satisfying Keane's condition associated with this path. On the space Γ of functions constant on each $j_0(I_\alpha)$, we have a complete filtration: the space Γ_s has dimension 2 according to Lemma 2, contains the line $\mathbb{R}\delta$ (where δ is the displacement vector) and is contained in the hyperplane Γ_* of zero mean.

Therefore conditions (b) and (c) in 1.3.2, 1.3.3 are automatically satisfied. Condition (a) is equivalent, in view of Lemma 3, to

$$\log n_k = o\left(\sum_{i<k} \log n_i\right).$$

Appendix B. A non-uniquely ergodic interval exchange map satisfying condition (a)

B.1 Let m, n, p be non negative integers. In the Rauzy diagram of the pair $(\pi_0, \pi_1) = \begin{pmatrix} A & B & C & D \\ D & C & B & A \end{pmatrix}$ (cf. 1.2.2), consider the loop $\gamma_0(m, n, p)$ based at (π_0, π_1) such that the names of the successive arrows are

$$D^{3m+1}BC^nBDC^pD.$$

We also consider the dual loop $\gamma_1(m, n, p)$ which is deduced from $\gamma_0(m, n, p)$ by means of the canonical involution and whose arrows have names

$$A^{3m+1}CB^nCAB^pA.$$

Given three sequences $(m_k)_{k \geq 0}$, $(n_k)_{k \geq 0}$ and $(p_k)_{k \geq 0}$ we also consider the infinite path Γ , based at $\begin{pmatrix} A & B & C & D \\ D & C & B & A \end{pmatrix}$ which is obtained composing

$$\gamma_0(m_0, n_0, p_0)\gamma_1(m_1, n_1, p_1) \cdots \gamma_0(m_{2k}, n_{2k}, p_{2k})\gamma_1(m_{2k+1}, n_{2k+1}, p_{2k+1}) \cdots$$

The matrix $Z_0(m, n, p)$ associated to $\gamma_0(m, n, p)$ is

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & p+2 & p+1 \\ 0 & n & (n+1)(p+1) & p(n+1) \\ m+1 & m(n+2)+1 & m(n+2)(p+1)+m+1 & pm(n+2)+m+1 \end{pmatrix}$$

where the vectors of the canonical basis of \mathbb{R}^4 are ordered alphabetically. Analogously the matrix $Z_1(m, n, p)$ associated to $\gamma_1(m, n, p)$ is

$$\begin{pmatrix} pm(n+2)+m+1 & m(n+2)(p+1)+m+1 & m(n+2)+1 & m+1 \\ p(n+1) & (n+1)(p+1) & n & 0 \\ p+1 & p+2 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

We set

$$Q(k) = Z_0(m_0, n_0, p_0)Z_1(m_1, n_1, p_1) \cdots Z_\varepsilon(m_{k-1}, n_{k-1}, p_{k-1}),$$

with $k-1 \equiv \varepsilon \pmod{2}$. We denote $e_A(k), e_B(k), e_C(k), e_D(k)$ the column vectors of $Q(k)$.

B.2 Let $m_0 = 0$ and choose $n_0 \gg 1$. The integer p_0 will be chosen later but it will be such that $p_0 \geq n_0$. One has

$$\begin{aligned} e_A(1) &= {}^t(1 \ 0 \ 0 \ 1), \\ e_B(1) &= n_0 [{}^t(0 \ 0 \ 1 \ 0) + O(n_0^{-1})], \\ e_C(1) &= n_0 p_0 [{}^t(0 \ 0 \ 1 \ 0) + O(n_0^{-1})], \\ e_D(1) &= n_0 p_0 [{}^t(0 \ 0 \ 1 \ 0) + O(n_0^{-1})]. \end{aligned}$$

We determine then $m_1, p_0, n_1, m_2, \dots, m_k, p_{k-1}, n_k, m_{k+1}, p_k, \dots$ through the following formulas:

$$\begin{aligned} \Pi_0 &:= n_0, & m_1 &:= n_0^3, \\ \Pi_1 &:= m_1 \Pi_0^{-1} = n_0^2, & p_0 &:= (n_0 + 1)^2 \Pi_1 = n_0^2 (n_0 + 1)^2, \\ \Pi_2 &:= p_0 \Pi_1^{-1} = (n_0 + 1)^2, & n_1 &:= (n_0 + 2)^2 \Pi_2 = (n_0 + 1)^2 (n_0 + 2)^2, \\ &\vdots & &\vdots \\ \Pi_{3l} &:= n_l \Pi_{3l-1}^{-1} = (n_0 + 3l - 1)^2, & m_{l+1} &:= (n_0 + 3l)^2 \Pi_{3l}, \\ \Pi_{3l+1} &:= m_{l+1} \Pi_{3l}^{-1} = (n_0 + 3l)^2, & p_l &:= (n_0 + 3l + 1)^2 \Pi_{3l+1}, \\ \Pi_{3l+2} &:= p_l \Pi_{3l+1}^{-1} = (n_0 + 3l + 1)^2, & n_{l+1} &:= (n_0 + 3l + 2)^2 \Pi_{3l+2}, \\ &\vdots & &\vdots \end{aligned}$$

Thus one has, for $l \geq 0$

$$\begin{aligned} p_l &= (n_0 + 3l)^2 (n_0 + 3l + 1)^2, \\ n_{l+1} &= (n_0 + 3l + 1)^2 (n_0 + 3l + 2)^2, \\ m_{l+2} &= (n_0 + 3l + 2)^2 (n_0 + 3l + 3)^2, \end{aligned}$$

and also $m_1 = n_0^3$. For all $k \geq -1$ we set

$$c_k = n_0^3 \left[\frac{(n_0 + k)!}{n_0!} \right]^2$$

so that one has $n_0 = c_{-1}$ and

$$\begin{aligned}
c_1 &= n_0 p_0, \\
c_2 &= m_1 n_1, \\
c_4 &= m_1 n_1 p_1, \\
c_5 &= m_2 n_2 n_0 p_0, \\
c_7 &= m_2 n_2 p_2 n_0 p_0, \\
c_8 &= m_3 n_3 m_1 n_1 p_1, \\
&\vdots \\
c_{10} &= m_3 n_3 p_3 m_1 n_1 p_1, \\
&\vdots
\end{aligned}$$

Let us check by induction that, setting $c_{-2} = 1$, one has for $l \geq 0$:

$$\begin{aligned}
e_A(2l-1) &= c_{6l-8} \left[\begin{smallmatrix} t & 1 & 0 & 0 & 1 \end{smallmatrix} + O(n_0^{-1}) \right], \\
e_B(2l-1) &= c_{6l-7} \left[\begin{smallmatrix} t & 0 & 0 & 1 & 0 \end{smallmatrix} + O(n_0^{-1}) \right], \\
e_C(2l-1) &= c_{6l-5} \left[\begin{smallmatrix} t & 0 & 0 & 1 & 0 \end{smallmatrix} + O(n_0^{-1}) \right], \\
e_D(2l-1) &= c_{6l-5} \left[\begin{smallmatrix} t & 0 & 0 & 1 & 0 \end{smallmatrix} + O(n_0^{-1}) \right], \\
e_D(2l) &= c_{6l-5} \left[\begin{smallmatrix} t & 0 & 0 & 1 & 0 \end{smallmatrix} + O(n_0^{-1}) \right], \\
e_C(2l) &= c_{6l-4} \left[\begin{smallmatrix} t & 1 & 0 & 0 & 1 \end{smallmatrix} + O(n_0^{-1}) \right], \\
e_B(2l) &= c_{6l-2} \left[\begin{smallmatrix} t & 1 & 0 & 0 & 1 \end{smallmatrix} + O(n_0^{-1}) \right], \\
e_A(2l) &= c_{6l-2} \left[\begin{smallmatrix} t & 1 & 0 & 0 & 1 \end{smallmatrix} + O(n_0^{-1}) \right],
\end{aligned}$$

We have already checked the first four relations for $l = 1$. Assume that the first four relations are verified for a given value of l . Then

$$e_D(2l) = e_D(2l-1) + (m_{2l-1} + 1)e_A(2l-1),$$

with

$$m_{2l-1}c_{6l-8} = c_{6l-6} = c_{6l-5}(n_0 + 6l - 5)^{-2}.$$

Moreover

$$e_C(2l) = 2e_C(2l-1) + n_{2l-1}e_B(2l-1) + [m_{2l-1}(n_{2l-1} + 2) + 1]e_A(2l-1),$$

with

$$\begin{aligned}
m_{2l-1}n_{2l-1}c_{6l-8} &= c_{6l-4}, \\
n_{2l-1}c_{6l-7} &= (n_0 + 6l - 6)^{-2}c_{6l-4}, \\
2c_{6l-5} &= 2(n_0 + 6l - 4)^{-2}c_{6l-4}, \\
(2m_{2l-1} + 1)c_{6l-8} &= O((n_0 + 6l)^{-2})c_{6l-4},
\end{aligned}$$

and

$$e_B(2l) = (p_{2l-1} + 2)e_C(2l - 1) + (n_{2l-1} + 1)(p_{2l-1} + 1)e_B(2l - 1) \\ + (m_{2l-1}(n_{2l-1} + 2)(p_{2l-1} + 1) + m_{2l-1} + 1)e_A(2l - 1)$$

with

$$(p_{2l-1} + 2)c_{6l-5} = O((n_0 + 6l)^{-2}c_{6l-2}), \\ (n_{2l-1} + 1)(p_{2l-1} + 1)c_{6l-7} = O((n_0 + 6l)^{-2}c_{6l-2}), \\ [1 + m_{2l-1}(n_{2l-1} + 2p_{2l-1} + 1)]c_{6l-8} = O((n_0 + 6l)^{-4}c_{6l-2}), \\ m_{2l-1}n_{2l-1}p_{2l-1}c_{6l-8} = c_{6l-2}.$$

The formula for $e_A(2l)$ is completely similar.

Since one has

$$\prod_{k \geq 0} [1 + O((n_0 + k)^{-2})] - 1 = O(n_0^{-1}),$$

one gets the four last relations. Taking into account the canonical involution one can analogously obtain the first four relations.

B.3 The decomposition of the infinite path Γ into loops $\gamma_0(m_{2k}, n_{2k}, p_{2k})$ and $\gamma_1(m_{2k+1}, n_{2k+1}, p_{2k+1})$ is nothing else than the decomposition for the accelerated Zorich algorithm. One has

$$\|Z_\varepsilon(m_k, n_k, p_k)\| \sim (n_0 + 3k)^{12},$$

(with $\varepsilon \equiv k \pmod{2}$), and

$$\|Q(k)\| \sim c_{3k-2} = n_0^3 \left[\frac{(n_0 + 3k - 2)!}{n_0!} \right]^2.$$

Thus one obtains

$$\|Z_\varepsilon(m_k, n_k, p_k)\| = o\left([\log \|Q(k)\|]^{12}\right)$$

and the first condition in the definition of Roth type interval exchange map is (by far) satisfied.

B.4 From the formula and estimates of Section A.2.2 one gets

$$\begin{aligned}\frac{e_D(2l)}{\|e_D(2l)\|_1} &= \frac{e_D(2l-1)}{\|e_D(2l-1)\|_1} + O((n_0 + 6l)^{-2}), \\ \frac{e_C(2l)}{\|e_C(2l)\|_1} &= \frac{e_A(2l-1)}{\|e_A(2l-1)\|_1} + O((n_0 + 6l)^{-2}), \\ \frac{e_B(2l)}{\|e_B(2l)\|_1} &= \frac{e_A(2l-1)}{\|e_A(2l-1)\|_1} + O((n_0 + 6l)^{-2}), \\ \frac{e_A(2l)}{\|e_A(2l)\|_1} &= \frac{e_A(2l-1)}{\|e_A(2l-1)\|_1} + O((n_0 + 6l)^{-2}),\end{aligned}$$

and by applying the canonical involution one obtains similar formulas at the order $2l+1$. Therefore one can conclude that if there exist two vectors u_A and u_D in \mathbb{R}^A such that

$$\begin{aligned}\|u_A\| &= \|u_D\| = 1, \\ u_A &= \begin{pmatrix} 1/2 \\ 0 \\ 0 \\ 1/2 \end{pmatrix} + O(n_0^{-1}), \\ u_D &= \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} + O(n_0^{-1}),\end{aligned}$$

$$\begin{aligned}\lim_{l \rightarrow +\infty} \frac{e_A(l)}{\|e_A(l)\|_1} &= \lim_{l \rightarrow +\infty} \frac{e_B(2l)}{\|e_B(2l)\|_1} = \lim_{l \rightarrow +\infty} \frac{e_C(2l)}{\|e_C(2l)\|_1} = u_A, \\ \lim_{l \rightarrow +\infty} \frac{e_D(l)}{\|e_D(l)\|_1} &= \lim_{l \rightarrow +\infty} \frac{e_C(2l+1)}{\|e_C(2l+1)\|_1} = \lim_{l \rightarrow +\infty} \frac{e_B(2l+1)}{\|e_B(2l+1)\|_1} = u_D.\end{aligned}$$

It is now easy to see that each point u of the segment $[u_A, u_D] \subset (\mathbb{R}^+)^A$ is the lengths datum for an interval exchange map with combinatorial datum $\begin{pmatrix} A & B & C & D \\ D & C & B & A \end{pmatrix}$, verifying Keane's condition and which is *not* uniquely ergodic: the interval exchange maps of this one parameter family are topologically conjugate.

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