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SUMMER SCHOOL AND CONFERENCE ON DYNAMICAL SYSTEMS

Equilibrium States

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Equilibrium States and Pressure

Let $f : M \to M$ be a continuous transformation on a compact space M, and $\phi : M \to \mathbb{R}$ be a continuous function.

Definition An f-invariant measure is an *equilibrium state* of f for the potential ϕ, if it maximizes the functional

$$\eta \mapsto h_\eta(f) + \int \phi \, d\eta$$

among all f-invariant probabilities η.

The variational principle tell us that the supremum of this functional over the invariant probabilities coincides with the *pressure* $P(f, \phi)$ of f for ϕ.

Sinai, Ruelle, Bowen, Parry, Walters (70's): existence and uniqueness of equilibrium states for Expanding/Axiom A systems and Hölder potentials.

Can one extend this theory beyond the uniformly hyperbolic context? To non-uniformly expanding systems?

Our goal: Construct equilibrium measures and (if is possible) prove its uniqueness for large class of maps exibiting expanding and contracting directions.

Main focus: The set H of points $x \in M$ with assimptotical expanding behaviour. More precisely, we consider the points such

$$\limsup \frac{1}{n} \sum_{i=0}^{n-1} \log \|Df(f^i(x))^{-1}\| < -c < 0$$

Warning:

- Maybe the set above is a very small set from the topological point of view.

- There are invariant measures ν such $\nu(H) = 0$ or the map is expanding (Alves-Luzatto-Saussol).

Non-uniformly Expanding Maps

$f : M \to M$ a C^1 local diffeomorphism and M compact Riemannian manifold, satisfying

(H1) There exist pairwise disjoint connected open sets $\mathcal{R} = \{R_1, \ldots, R_{q+p}\}$ such that $\cup_i \bar{R}_i = M$ and every $f|\bar{R}_i$ is injective, and there exist constants $\delta_0 > 0$ and $\sigma_1 > q$ such that

1. f is expanding at every $x \in \bar{R}_{q+1} \cup \cdots \cup \bar{R}_{q+p}$ $\|Df(x)^{-1}\| \leq \sigma_1^{-1}$.

2. f is never too contracting: $\|Df(x)^{-1}\| \leq 1 + \delta_0$ for every $x \in M$.

3. the image $f(\bar{R}_i)$ of every atom is a union of atoms $\bar{R}_j$, and there exists N such for every $i = 1, \ldots, p+q$ we have $f^N(R_i) = M$.

Low Variation Potentials

We associate to each such a transformation a certain positive number $c_0(f)$ that depends only on δ_0, σ_1, and p, q and which goes to $\log q$ if δ_0 goes to zero *or* σ_1 goes to infinity. We assume that the potential $\phi : M \to \mathbb{R}$ satisfies

(H2) ϕ is Hölder continuous and

$$\max \phi - \min \phi < h_{top}(f) - c_0(f)$$

The inequality holds, for instance, if the oscillation of ϕ is less than $h_{top}(f)$ and δ_0 is small enough.

Some notation

For each $n \geq 1$, let $[i_0, \ldots, i_{n-1}]$ be the *cylinder* of all points whose length-n itinerary relative to the partition $\mathcal{R}$ is given by symbols $i_0, \ldots, i_{n-1}$:

$$[i_0, ..., i_{n-1}] = \{y \in M; f^j(y) \in R_{i_j} \text{ for } 0 \leq j \leq n-1\}.$$

Let $\mathcal{R}^n$ be the partition of M into length-n cylinders, and $R^n(x) \in \mathcal{R}^n$ denote an atom that contains a given point $x \in M$.

We write $S_n\phi(x) = \sum_{j=0}^{n-1} \phi(f^j(x))$ for each $x \in M$ and $n \in \mathbb{N}$. An integer sequence $n_j \in \mathbb{N}$ is called *non-lacunary* if it is increasing and n_{j+1}/n_j converges to 1.

Non-lacunary Gibbs Measures

Definition 1. A probability η (not necessarily invariant) is a *Gibbs measure* of f for ϕ if there exists $P \in \mathbb{R}$ and $K > 0$ such that

$$K^{-1} \le \frac{\eta\Big(R^n(x)\Big)}{\exp\Big(S_n\phi(x) - nP\Big)} \le K \qquad (1)$$

for every $x \in M$ and $n \ge 1$. More generally, η is a *non-lacunary Gibbs measure* of f for ϕ if there exists $K > 0$ such that for η-almost every x there exists a non-lacunary sequence of values of $n \ge 1$ for which (1) is satisfied.

In the proofs, this non-lacunary sequence will correspond to *hyperbolic times* of the system.

Non-lacunary versus weak Gibbs measures

Remark 2. If η is a non-lacunary Gibbs measure then for η-almost every point $x \in M$ there exists a sequence $K_n = K_n(x)$ such that

$$K_n^{-1} \leq \frac{\eta\Big(R^n(x)\Big)}{\exp\Big(S_n\phi(x) - nP\Big)} \leq K_n$$

for $n \geq 1$ and $\lim \frac{1}{n} \log K_n = 0$.

In some examples, the subexponential estimate can not be achieved for all cylinders.

Theorem A. *Assume f and ϕ satisfy hypotheses (H1) and (H2). Then f admits some invariant ergodic probability which is a non-lacunary Gibbs measure for ϕ and some constant P which all of its Lyapunov exponents are positive. Also, this measure give zero measure to the boundary of $\mathcal{R}$.*

Warning: here, $P =$ log spectral radius of $\mathcal{L}_\phi$ is not necessarily the pressure of ϕ!

Theorem B. *Assume f and ϕ satisfy hypotheses (H1) and (H2). Then, there exists a Hölder continuous function h bounded from zero, such that*

$$\mathcal{L}_\phi h = e^P h$$

As a consequence of Theorem A and B, we obtain

Theorem C. *Assume f and ϕ satisfy hypotheses (H1) and (H2). If δ_0 is small enought, then f has some unique equilibrium state μ for ϕ, which is a non-lacunary Gibbs measure and H has full measure for μ.*

Hyperbolic Times

Definition 3. We say that $n \in \mathbb{N}$ is a *hyperbolic time* for $x \in M$ if

$$\prod_{k=0}^{j-1} \|Df(f^{n-k}(x))^{-1}\| \leq e^{-2cj},$$

for every $1 \leq j \leq n$.

We say that n is a hyperbolic time for a cylinder $R^n \in \mathcal{R}^n$ if n is a hyperbolic time for every $x \in R^n$. We denote by $\mathcal{R}^n_h$ the set of the cylinders $R^n \in \mathcal{R}^n$ for which n is a hyperbolic time and H as the set of $x \in M$ that x have infinitely many hyperbolic times. Note that this set is invariant but needs not be compact.

Further Results

In a very related setting with some additional hypothesis we have the following theorem (in E T & D S, 23:6):

Theorem (O) - *Assuming that f is $C^{1+\alpha}$ local diffeomorphism satisfying some hypothesis related to $[H1]$. Then, all* **continuous** *potentials with low variation have equilibrium states. Moreover, all equilibrium states give full measure to H.*

In Arbieto-Matheus-Oliveira (Nonlinearity,2004) the theorem above is extended to the setting of *random* non-uniformly maps.

Proof of Theorem A

- To define a reference measure ν via RPF operator and prove that $\nu(H) = 1$.

- To prove that ν is non-lacunary Gibbs measure and the first hyperbolic map is integrable.

- To define an induced map F via first hyperbolic time and prove that F admits some invariant probability absolutely continuous with respect o ν with bounded density.

- To push this measure to a f-invariant measure.

Proof of Theorem B

- To define a sequence of operators T_i that "forgets" pre-images $x_n \in f^{-n}(x)$ such n is not a hyperbolic time for x_n.

- To prove that the average $n^{-1}\sum_{i=0}^{n-1} e^{-iP}T_i$ is uniformly bounded from above and below and equicontinuous.

- To prove that the some subsequence of the average above converges to some eigein-function of $\mathcal{L}_\phi$.

Proof of Theorem C

- To prove that there exists only one invariant non-lacunary Gibbs measure.

- To prove that this measure is an equilibrium measure.

- To prove that any equilibrium measure is an non-lacunary Gibbs measure

Questions

- What about the hyphotesis of low variation? How to extend it for more potentials?
- If our map admits singularities?
- What one can say about the growth of periodic points? and zeta functions?
- Is really the logarithm of the spectral radius for RPF operator equal to the pressure?
- Decay of correlations? Continuity of these measures? Stochastic Stability?
- And the Diffeomorphism case (Partially hyperbolic, to fix ideas)?

Lemma 4. *Given* $\gamma \in (0,1)$ *define*

$$c_\gamma = \limsup_{n\to\infty} \frac{1}{n} \log \# I_{\gamma,n},$$

where $I_{\gamma,n}$ *is the set of* $(i_0, \ldots, i_{n-1})$ *such*

$$\#\{0 \le j \le n-1; i_j \in \{1, \ldots, q\}\} > \gamma n.$$

Then c_γ *goes to* 0 *when* $\gamma \to 1$.

Note that c_γ depends only on γ and p, q. Fix $0 < \gamma < 1$ such that

$$(1+\delta_0)^\gamma \sigma_1^{-(1-\gamma)} < 1 \tag{2}$$

and then take $c_0(f) = c_\gamma$. As we announced, $c_0(f)$ depends only on δ_0, σ_1, p, and it goes to zero when either $\delta_0 \to 0$ or $\sigma_1 \to \infty$.