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On Local and Global Existence and Uniqueness of Solutions of the 3D-Navier-Stokes System on R^3

Ya. G. Sinai
Dept. of Mathematics
Princeton University
USA
&
L.D. Landau Inst. for Theoretical Physics
Moscow
Russian Federation

On Local and Global Existence and Uniqueness of Solutions of the 3D – Navier – Stokes System on $\mathbb{R}^3$.

Ya. G. Sinai[*] [†]

To L. Carleson
with great respect.

1 Introduction.

Three – dimensional Navier-Stokes system (NSS) on $\mathbb{R}^3$ without external forcing is written for the velocity vector $u(x,t) = (u_1(x,t), u_2(x,t), u_3(x,t))$ satisfying the incompressibility condition $div\ u = 0$ and for the pressure $p(x,t)$ and has the form

$$\frac{Du}{dt} = \nu\Delta u - \nabla p, \quad x \in \mathbb{R}^3, \quad t \geqslant 0. \tag{1}$$

Here $\left(\frac{Du}{dt}\right)_i = \frac{\partial u_i}{\partial t} + \sum_{k=1}^{3} \frac{\partial u_i}{\partial x_k} u_k$, $\nu > 0$ is the viscosity. In this paper we take $\nu = 1$.

After Fourier transform

$$v(k,t) = \int_{\mathbb{R}^3} e^{-i<k,x>} u(x,t) dx,$$

$$u(x,t) = \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} v(k,t) e^{i<k,x>} dk$$

$v(k,t) \perp k$ for any $k \in \mathbb{R}^3$ in view of incompressibility and NSS takes the form

$$\frac{\partial v(k,t)}{\partial t} = -|k|^2 v(k,t) + i \int_{\mathbb{R}^3} < k, v(k-k',t) > P_k v(k',t) dk' \tag{2}$$

where P_k is the orthogonal projection to the subspace orthogonal to k. Clearly,

$$< k, v(k-k',t) > = < k', v(k-k',t) >.$$

[*]Mathematics Department, Princeton University
[†]Landau Institute of Theoretical Physics, Russian Academy of Sciences

We can reduce the system (2) to the system of non-linear integral equations

$$v(k,t) = e^{-|k|^2 t} v_0(k) + i \int_0^t e^{-|k|^2(t-\tau)} d\tau \times$$

$$\times \int\limits_{\mathbb{R}^3} < k, v(k-k',\tau) > P_k v(k',\tau) dk' \qquad (3)$$

where $v_0(k)$ is the initial condition.

Many problems of hydrodynamics assume the power - like behavior of functions v near $k=0$ and infinity. In this connection it is natural to introduce the spaces $\Phi(\alpha,\omega)$ where $v(k) \in \Phi(\alpha,\omega)$ if

1)$v(k) = \dfrac{c(k)}{|k|^\alpha}$ for $|k| \leqslant 1$, $c(k) \perp k$
and $c(k)$ is bounded and continuous outside $k=0$; put $c = \sup\limits_{|k|\leqslant 1} |c(k)|$;

2)$v(k) = \dfrac{d(k)}{|k|^\omega}$ for $|k| \geqslant 1$
and $d(k)$ is bounded and continuous, $d = \sup\limits_{k \geqslant 1} |d(k)|$.

Certainly 1 plays no essential role in this definition and can be replaced by any other fixed number. We shall use the metric $\|v\| = c + d$.

In some cases v can have infinite energy or enstrophy. Therefore classical existence or uniqueness results like the theorem by T.Kato (see [K]) cannot be applied to the spaces $\Phi(\alpha,\omega)$.

In this paper we consider $0 \leqslant \alpha < 3$, $\omega \geqslant 2$ and in §2 we prove the following theorems.

Theorem 1. Let $\omega > 2$. Then for any $v \in \Phi(\alpha,\omega)$ there exists $t_0 = t_0(\alpha,\omega)$ such that the system (3) has a unique solution on the interval $0 \leqslant t \leqslant t_0$ with the initial condition v.

Theorem 2. Let $\omega > 2$ and $t_0 > 0$ be given. There exists $h > 0$ such that for any $\|v\| < h$ there exists a unique solution of (2) on the interval $0 \leqslant t \leqslant t_0$ which has v as the initial condition.

Our methods do not allow to prove the global existence results in the spaces $\Phi(\alpha,\omega)$ if $\|v\|$ is sufficiently small. There are some reasons to believe that the corresponding statement is even wrong (see §4). Some results in this direction were obtained recently by E. I. Dinaburg

In §3 we consider the case $\alpha = \omega = 2$. A stronger statement is valid.

Theorem 3. If $\|v\|$ is sufficiently small then there exists the unique solution of (3) with this initial condition for all $t \geqslant 0$.

Theorem 3 was proven earlier by Le Jan and Sznitman [LS] and Cannone and Planchon [[CP]] (see also the review by M. Canone [C]). Our methods do not allow to prove for $\alpha = \omega = 2$ the existence of local solutions if $\|v\|$ is large. Probably, this statement is also wrong.

In §4 we consider $v(k) = \frac{c(k)}{|k|^\alpha}$ for $2 < \alpha < 3$, $\omega = \alpha$ and discuss some possibilities for blow ups in these cases.

During the proofs there appear various constants whose exact values play no role in the arguments. We denote them by capital A with various subindexes.

2 Local Existence and Uniqueness Theorems.

We shall prove theorems 1 and 2 simultaneously pointing out the differences in the arguments at the end of this section. We shall construct solutions of (3) by the method of successive approximations. Put $v^{(0)}(k,t) = e^{-|k|^2 t} v_0(k)$ and define for $n > 0$

$$v^{(n)}(k,t) = v^{(0)}(k,t) + i \int_0^t e^{-|k|^2(t-\tau)} d\tau \int_{\mathbb{R}^3} \left(k, v^{(n-1)}(k-k',\tau)\right) P_k v^{(n-1)}(k',\tau) dk' \tag{4}$$

Clearly, $v^{(n)}(k,t) \perp k$. It follows from (3) that

$$v^{(n+1)}(k,t) - v^{(n)}(k,t) = i \int_0^t e^{-|k|^2(t-\tau)} d\tau \int_{\mathbb{R}^3} \left[\left(k, v^{(n)}(k-k',t) - v^{(n-1)}(k-k',t)\right) P_k v^{(n)}(k',\tau) + \right.$$

$$\left. + \left(k, v^{(n-1)}(k-k',\tau)\right) P_k \left(v^{(n)}(k',\tau) - v^{(n-1)}(k',\tau)\right) \right] dk' \tag{5}$$

From (3) and (4)

$$\left|v^{(n)}(k,t)\right| \leqslant \left|v^{(0)}(k,t)\right| + \int_0^t e^{-|k|^2(t-\tau)} d\tau \cdot |k| \int_{\mathbb{R}^3} \left|v^{(n-1)}(k-k',\tau)\right| \left|v^{(n-1)}(k',\tau)\right| dk' =$$

$$\left|v^{(0)}(k,t)\right| + \frac{1}{|k|} \int_0^{t|k|^2} e^{-\tau} d\tau \int_{\mathbb{R}^3} \left|v^{(n-1)}(k-k', t - \frac{\tau}{|k|^2})\right| \cdot \left|v^{(n-1)}(k', t - \frac{\tau}{|k|^2})\right| dk'. \tag{6}$$

We shall prove that if $v^{(n-1)}(k,\tau) \in \Phi(\alpha,\omega)$ and $\left\|v^{(n-1)}(k,\tau)\right\| \leqslant h$ for all $0 \leqslant \tau \leqslant t$ then $v^{(n)}(k,\tau) \in \Phi(\alpha,\omega)$ and we shall derive the estimate for $\left\|v^{(n)}(k,\tau)\right\|$.

We shall write $v_0^{(n-1)}(k,\tau)$ if $|k| \leqslant 1$ or $v_1^{(n-1)}(k,\tau)$ if $|k| > 1$ instead of $v^{(n-1)}(k,\tau)$ and assume that $v_0^{(n-1)}(k,\tau) = 0$ if $|k| > 1$, $v_1^{(n-1)}(k,\tau) = 0$ if $|k| \leqslant 1$. Consider two cases.

Case 1. $|k| \leqslant 1$. We have from (6)

$$\left|v^{(n)}(k,t)\right| \cdot |k|^\alpha \leqslant \|v_0\| + |k|^{\alpha-1} \int_0^{t|k|^2} e^{-\tau} d\tau \int_{\mathbb{R}^3} \left[\left|v_0^{(n-1)}\left(k-k', t - \frac{\tau}{|k|^2}\right)\right| \cdot \left|v_0^{(n-1)}(k', t - \frac{\tau}{|k|^2})\right| + \right.$$

$$\left. +2 \left|v_0^{(n-1)}\left(k-k', t - \frac{\tau}{|k|^2}\right)\right| \left|v_1^{(n-1)}(k', t - \frac{\tau}{|k|^2})\right| + \left|v_1^{(n-1)}(k-k', t - \frac{\tau}{|k|^2})\right| \left|v_1^{(n-1)}(k', t - \frac{\tau}{|k|^2})\right| \right] dk' \leqslant$$

$$\leqslant \|v_0\| + |k|^{\alpha-1}\left(1-e^{-t|k|^2}\right)h^2\left[\int\limits_{\substack{|k'|\leqslant 1\\|k-k'|\leqslant 1}} \frac{1}{|k-k'|^\alpha}\cdot\frac{1}{|k'|^\alpha}dk' + \int\limits_{\substack{|k'|\geqslant 1\\|k-k'|\leqslant 1}} \frac{1}{|k-k'|^\alpha}\cdot\frac{1}{|k'|^\omega}dk' + \right.$$

$$\left. + \int\limits_{\substack{|k'|\geqslant 1\\|k-k'|\geqslant 1}} \frac{1}{|k-k'|^\omega}\cdot\frac{1}{|k'|^\omega}dk'\right]. \tag{7}$$

Each of these integrals will be estimated separately.

i_1)

$$\int\limits_{\substack{|k'|\leqslant 1\\|k-k'|\leqslant 1}} \frac{1}{|k-k'|^\alpha}\cdot\frac{1}{|k'|^\alpha}dk' = |k|^{3-2\alpha}\int\limits_{\substack{|k'|\leqslant\frac{1}{|k|}\\\left|\frac{k}{|k|}-k'\right|\leqslant\frac{1}{|k|}}} \frac{dk'}{\left|\frac{k}{|k|}-k'\right|^\alpha\cdot|k'|^\alpha}$$

Concerning the last integral we can write

$$\int\limits_{\substack{|k'|\leqslant\frac{1}{|k|}\\\left|\frac{k}{|k|}-k'\right|\leqslant\frac{1}{|k|}}} \frac{dk'}{\left|\frac{k}{|k|}-k'\right|^\alpha\cdot|k'|^\alpha} \leqslant A_1(\alpha) \quad \text{if} \quad \alpha > \frac{3}{2},$$

$$\int\limits_{\substack{|k'|\leqslant\frac{1}{|k|}\\\left|\frac{k}{|k|}-k'\right|\leqslant\frac{1}{|k|}}} \frac{dk'}{\left|\frac{k}{|k|}-k'\right|^\alpha\cdot|k'|^\alpha} \leqslant A_1(\alpha)\ln\frac{1}{|k|} \quad \text{if} \quad \alpha = \frac{3}{2},$$

$$\int\limits_{\substack{|k'|\leqslant\frac{1}{|k|}\\\left|\frac{k}{|k|}-k'\right|\leqslant\frac{1}{|k|}}} \frac{dk'}{\left|\frac{k}{|k|}-k'\right|^\alpha\cdot|k'|^\alpha} \leqslant A_1(\alpha)|k|^{-3+2\alpha} \quad \text{if} \quad \alpha < \frac{3}{2}.$$

From these inequalities it follows that for $|k| \leqslant 1,\ \ t \leqslant t_0$

$$|k|^{\alpha-1}\cdot\left(1-e^{-t|k|^2}\right)\int\limits_{\substack{|k'|\leqslant 1\\|k-k'|\leqslant 1}} \frac{dk'}{|k-k'|^\alpha|k'|^\alpha} \leqslant$$

$$\leqslant |k|^{2-\alpha}\left(1-e^{-t|k|^2}\right)\cdot\int\limits_{|k'|\leqslant\frac{1}{|k|}} \frac{dk'}{\left|\frac{k}{|k|}-k'\right|^\alpha\cdot|k'|^\alpha} \leqslant t_0\cdot A_2(\alpha) \tag{8}$$

i_2) Consider the next integral in (7):

$$\int\limits_{\substack{|k'|\geqslant 1\\|k-k'|\leqslant 1}} \frac{1}{|k-k'|^{\alpha}}\cdot\frac{1}{|k'|^{\omega}}dk' = |k|^{3-\alpha-\omega}\int\limits_{\substack{|k'|\geqslant\frac{1}{|k|}\\ \left|\frac{k}{|k|}-k'\right|\leqslant\frac{1}{|k|}}} \frac{1}{|k'|^{\omega}}\cdot\frac{1}{\left|\frac{k}{|k|}-k'\right|^{\alpha}}dk'$$

It is clear that $\left|\frac{k}{|k|}-k'\right| \geqslant |k'|-1$. Therefore in the domain of integration $|k'|\leqslant\frac{1}{|k|}+1$ and

$$\int\frac{dk'}{|k'|^{\omega}\cdot\left|\frac{k}{|k|}-k'\right|^{\alpha}} \leqslant \int\limits_{\frac{1}{|k|}\leqslant|k'|\leqslant\frac{1}{|k|}+1}\frac{dk'}{|k'|^{\omega}\cdot\left|\frac{k}{|k|}-k'\right|^{\alpha}} \leqslant A_3(\alpha,\omega)\cdot|k|^{\omega}$$

Returning back to (7) we can write

$$|k|^{\alpha-1}\left(1-e^{-t|k|^2}\right)\cdot\int\limits_{\substack{|k'|\geqslant 1\\|k-k'|\leqslant 1}}\frac{1}{|k-k'|^{\alpha}|k|^{\omega}}dk' \leqslant A_4(\alpha,\omega)\cdot t_0.$$

i_3) The estimate of the last integral in (7) is even simpler:

$$\int\limits_{\substack{|k'|>1\\|k-k'|>1}}\frac{dk'}{|k'|^{\omega}|k-k'|^{\omega}}dk' \leqslant A_5(\omega)$$

since $\omega>2$. Therefore

$$|k|^{\alpha-1}\cdot\left(1-e^{-t|k|^2}\right)\int\limits_{\substack{|k'|\geqslant 1\\|k-k'|\geqslant 1}}\frac{dk'}{|k|^{\omega}|k-k'|^{\omega}} \leqslant t\cdot A_6(\alpha,\omega).$$

Finally for $|k|\leqslant 1$ we get

$$\left|v^{(n)}(k,t)\right|\cdot|k|^{\alpha} \leqslant \|v_0\| + A_7(\alpha,\omega)\cdot t_0\cdot h^2 \tag{9}$$

Case 2. $1\leqslant|k|\leqslant\frac{1}{\sqrt{t_0}}$. From (6)

$$\left|v^{(n)(k,t)}\right|\cdot|k|^{\omega} \leqslant \|v_0\| + h^2|k|^{\omega-1}\left(1-e^{-t|k|^2}\right)\times$$

$$\times\left[\int\limits_{\substack{|k'|\leqslant 1\\|k-k'|\leqslant 1}}\frac{1}{|k-k'|^{\alpha}}\cdot\frac{1}{|k'|^{\alpha}}dk' + \int\limits_{\substack{|k'|\geqslant 1\\|k-k'|\leqslant 1}}\frac{1}{|k-k'|^{\alpha}}\cdot\frac{1}{|k'|^{\omega}}dk' + \int\limits_{\substack{|k'|\geqslant 1\\|k-k'|\geqslant 1}}\frac{1}{|k-k'|^{\omega}}\cdot\frac{1}{|k'|^{\omega}}dk'\right] \tag{10}$$

The first term in (10) can be non - zero for $1 \leqslant |k| \leqslant 2$ and in this case it is not more than

$$\int\limits_{|k'|\leqslant 1} \frac{dk'}{|k'|^\alpha |k-k'|^\alpha} \leqslant A_8(\alpha,\omega)$$

Therefore in view of $1 \leqslant |k| \leqslant 2$

$$|k|^{\omega-1}\left(1-e^{-t|k|^2}\right) \int\limits_{\substack{|k'|\leqslant 1\\ |k-k'|\leqslant 1}} \frac{dk'}{|k-k'|^\alpha |k'|^\alpha} \leqslant A_9(\alpha,\omega)\cdot t_0 \tag{11}$$

For the second term in (10) we can write

$$\int\limits_{\substack{|k'|\geqslant 1\\ |k-k'|\leqslant 1}} \frac{dk'}{|k-k'|^\alpha |k'|^\omega} \leqslant A_{10}(\alpha,\omega)\cdot \frac{1}{|k|^\omega}$$

and this gives

$$|k'|^{\omega-1}\left(1-e^{-t|k|^2}\right) \int\limits_{\substack{|k'|\geqslant 1\\ |k-k'|\leqslant 1}} \frac{dk'}{|k-k'|^\alpha |k'|^\omega} \leqslant A_{11}(\alpha,\omega)\cdot |k|^{\omega+1}\cdot t_0 \cdot \frac{1}{|k|^\omega} =$$

$$= A_{11}(\alpha,\omega)\cdot |k|\cdot t_0 \leqslant A_{11}(\alpha,\omega)\sqrt{t_0} \tag{12}$$

Consider the last term in (10):

$$\int\limits_{\substack{|k'|\geqslant 1\\ |k-k'|\geqslant 1}} \frac{1}{|k-k'|^\omega}\cdot\frac{1}{|k'|^\omega} dk' \leqslant |k|^{3-2\omega}\cdot \int\limits_{\substack{|k'|\geqslant \frac{1}{|k|}\\ \left|\frac{k}{|k|}-k'\right|\geqslant \frac{1}{|k|}}} \frac{dk'}{\left|\frac{k}{|k|}-k'\right|^\omega |k'|^\omega} \tag{13}$$

Again this integral is bounded if $\omega < 3$, diverges as $\ln|k|$ if $\omega = 3$ and behaves as $|k|^{\omega-3}$ if $\omega > 3$. Therefore for $\omega < 3$

$$|k|^{\omega-1}\left(1-e^{-t_0|k|^2}\right) \int\limits_{\substack{|k'|\geqslant 1\\ |k-k'|\geqslant 1}} \frac{1}{|k-k'|^\omega}\frac{1}{|k'|^\omega} dk' \leqslant const\cdot |k|^{\omega+1}\cdot t_0\cdot |k|^{3-2\omega} =$$

$$= const\cdot |k|^{4-\omega}\cdot t_0 \leqslant const\cdot t_0^{1-\frac{4-\omega}{2}} = const\cdot t_0^{\frac{\omega-2}{2}}.$$

For $\omega = 3$ the last expression is not more than

$$const\cdot t_0\cdot |k|^4\cdot |k|^{-3}\ln|k| = const\cdot t_0\cdot |k|\cdot \ln|k| \leqslant const\cdot t_0^{1/3}$$

In this inequality we could take any power less than $\frac{1}{2}$ instead of $\frac{1}{3}$. The value of *const* depends on this number. Collecting all estimates we can write

$$\left|v^{(n)}(k,t)\right| \cdot |k|^{\omega} \leqslant \|v_0\| + A_{12}(\alpha,\omega)\|h\|^2 \cdot t_0^{\delta} \tag{14}$$

for some positive $\delta > 0$. Remark that here we used the fact that $\omega > 2$.

Case 3. $|k| \geqslant \frac{1}{\sqrt{t_0}}$. Again from (6)

$$\left|v^{(n)}(k,t)\right| \cdot |k|^{\omega} \leqslant \|v_0\| + h^2 \cdot |k|^{\omega-1} \left[\int\limits_{\substack{|k'|\leqslant 1 \\ |k-k'|\leqslant 1}} \frac{1}{|k-k'|^{\alpha}|k'|^{\alpha}} dk' + \right.$$

$$\left. +2 \int\limits_{\substack{|k'|\leqslant 1 \\ |k-k'|\geqslant 1}} \frac{1}{|k-k'|^{\omega}|k'|^{\alpha}} dk' + \int\limits_{\substack{|k'|\geqslant 1 \\ |k-k'|\geqslant 1}} \frac{dk'}{|k-k'|^{\omega}|k'|^{\omega}} \right]$$

The first integral can be non - zero only if $t_0 > 1$. Therefore it is zero if we discuss theorem 1 and consider sufficiently small t_0. If t_0 is not small then h will be small (see below). If $t_0 > 1$ this integral was estimated before (see i_1 in case 1) and it can be non-zero if $|k| \leqslant 2$. In this case it is bounded and

$$|k|^{\omega-1} \left[\int\limits_{\substack{|k'|\leqslant 1 \\ |k-k'|\leqslant 1}} \frac{dk'}{|k-k'|^{\alpha}|k'|^{\alpha}} \right] \leqslant A_{13}(\alpha,\omega).$$

For the second integral we use the estimate

$$\int\limits_{\substack{|k'|\leqslant 1 \\ |k-k'|\geqslant 1}} \frac{1}{|k-k'|^{\omega}|k'|^{\alpha}} dk' \leqslant A_{14}(\alpha,\omega) \cdot \frac{1}{|k|^{\omega}}$$

and therefore

$$2|k|^{\omega-1} \cdot \int\limits_{\substack{|k'|\leqslant 1 \\ |k-k'|\geqslant 1}} \frac{1}{|k-k'|^{\omega}|k'|^{\alpha}} dk' \leqslant A_{15}(\alpha,\omega) \cdot \frac{1}{|k|} \leqslant A_{15}(\alpha,\omega) \cdot t_0^{\frac{1}{2}}.$$

For the third integral $\int\limits_{\substack{|k'|\geqslant 1 \\ |k-k'|\geqslant 1}} \frac{dk'}{|k-k'|^{\omega}|k'|^{\omega}}$ we can write:

if $\omega < 3$ then

$$\int\limits_{\substack{|k'|\geqslant 1 \\ |k-k'|\geqslant 1}} \frac{dk'}{|k-k'|^{\omega}|k'|^{\omega}} \leqslant |k|^{3-2\omega} \cdot const$$

and $|k|^{\omega-1}\cdot|k|^{3-2\omega}=|k|^{2-\omega}\leqslant t_0^{\frac{\omega-2}{2}}$;

if $\omega=3$ then

$$\int\limits_{\substack{|k'|\geqslant 1\\|k-k'|\geqslant 1}}\frac{dk'}{|k-k'|^{\omega}|k'|^{\omega}}\leqslant const\cdot|k|^{-3}\ln|k|;$$

if $\omega>3$ then

$$\int\limits_{\substack{|k'|\geqslant 1\\|k-k'|\geqslant 1}}\frac{dk'}{|k-k'|^{\omega}|k'|^{\omega}}\leqslant\frac{const}{|k|^{\omega}}$$

In all cases

$$|k|^{\omega-1}\cdot\int\frac{dk'}{|k-k'|^{\omega}|k|^{\omega}}\leqslant A_{16}(\alpha,\omega)\cdot\frac{\ln|k|}{|k|}\leqslant A_{17}(\alpha,\omega)\cdot t_0^{\frac{1}{3}}$$

From all presented estimates it follows that

$$\|v^{(n)}\|\leqslant\|v^{(0)}\|+A_{17}(\alpha,\omega)\|v^{(n-1)}\|^2\cdot t_0^{\frac{1}{3}}\tag{15}$$

where $v^{(m)}=\max\limits_{0\leqslant t\leqslant t_0}\|v^{(m)}(k,t)\|$.

Take any $\lambda>1$, for example, $\lambda=2$. If $\|v^{(n-1)}\|\leqslant 2\|v_0^{(0)}\|$ then

$$\|v^{(n)}\|\leqslant\|v^{(0)}\|+4A_{17}(\alpha,\omega)\cdot\|v^{(0)}\|^2\cdot t_0^{\frac{1}{3}}=\|v^{(0)}\|\left(1+4A_{17}(\alpha,\omega)\cdot\|v^{(0)}\|\cdot t_0^{\frac{1}{3}}\right)$$

If t_0 is so small that $4A_{17}(\alpha,\omega)\cdot\|v^{(0)}\|\cdot t_0^{\frac{1}{3}}<1$ then $\|v^{(n)}\|\leqslant 2\|v_0^{(0)}\|$. On the other hand, if t_0 is given and $\|v^{(0)}\|$ is so small that $A_{17}(\alpha,\omega)\cdot\|v^{(0)}\|\cdot t_0^{\frac{1}{3}}\leqslant 2$ then $\|v^{(n)}\|\leqslant 2\|v_0^{(0)}\|$. In other words, under conditions of theorems 1 and 2 all $v^{(n)}(k,t)\in\Phi(\alpha,\omega)$ for all $0\leqslant t\leqslant t_0$ and $\|v^{(n)}\|\leqslant 2\|v^{(0)}\|$.

Now we shall prove the convergence of all iterations $v^{(n)}$ to a limit as $n\to\infty$ in the sense of our metric.

Consider $|k|\leqslant 1$. We have from (5)

$$\left|v^{(n+1)}(k,t)-v^{(n)}(k,t)\right|\cdot|k|^{\alpha}\leqslant|k|^{\alpha+1}\int_0^t e^{-|k|^2(t-\tau)}d\tau\int\limits_{\mathbb{R}^3}\Big[\ \left|v^{(n)}(k-k',\tau)-v^{(n-1)}(k-k',\tau)\right|\times$$

$$\times\left(\left|v^{(n)}(k',\tau)\right|+\left|v^{(n-1)}(k-k',\tau)\right|\right)\left|v^{(n)}(k',\tau)-v^{(n-1)}(k',\tau)\right|\Big]\,dk'=$$

$$=|k|^{\alpha+1}\cdot\int_0^t e^{-|k|^2(t-\tau)}d\tau\int\limits_{\mathbb{R}^3}\left(|v^{(n)}(k',\tau)|+|v^{(n-1)}(k',\tau)|\right)\cdot|v^{(n)}(k-k',\tau)-v^{(n-1)}(k-k',\tau)|dk'.$$

The estimates of $|v^{(n)}(k',\tau)|,|v^{(n-1)}(k',\tau)|$ are known because $v^{(n-1)}(k',\tau),v^{(n)}(k',t)$ belong to the space $\Phi(\alpha,\omega)$. Denote $h^{(n)}=\max\limits_{0\leqslant t\leqslant t_0}\left\|v^{(n)}(k,t)-v^{(n-1)}(k,\tau)\right\|$. The same arguments as above show that $h^{(n+1)}\leqslant\varepsilon h^{(n-1)}$ for some $0<\varepsilon<1$. Other cases are considered in a similar way. We omit the details. The last inequality gives the convergence of $v^{(n)}$ to the limit. Theorems 1 and 2 are proven.

3 The case $\alpha = \omega = 2$.

This case is special in several respects. It was considered before by Le Jan and Sznitman [LS] and by Cannone and Planchon [CP]. The authors proved that if the norm $\|v^{(0)}\|$ is sufficiently small then there exists the unique solution of (2) for all $t \geqslant 0$ having this initial condition.

We shall show how this result can be obtained with the help of technique of §2. Thus we assume that $v^{(0)}(k) = \frac{c(k)}{|k|^2}$ and denote $c = \sup\limits_{k \in \mathbb{R}^3} |c(k)|$ We shall show that if $c \leqslant c_0$ then the solution of (2) exists for all $t \geqslant 0$, $v(k,t) \in \Phi(2,2)$ and $\|v(k,t)\| \leqslant c_0$.

We use the same iteration scheme (3). Take any t_0. Assuming that $v^{(n-1)}(k,t) \in \Phi(2,2)$ of all $0 \leqslant t \leqslant t_0$ and $v^{(n-1)} = \max\limits_{0 \leqslant t \leqslant t} \|v^{(n-1)}(k,t)\|$ we can write

$$|k|^2 \cdot |v^{(n)}(k,t)| \leqslant \|v^{(0)}\| + v^{(n-1)^2} |k|^3 \int_0^t e^{-|k|^2(t-\tau)} d\tau \cdot \int\limits_{\mathbb{R}^3} \frac{1}{|k-k'|^2 |k'|^2} dk'. \tag{16}$$

It is easy to see that $\int\limits_{\mathbb{R}^3} \frac{dk'}{|k-k'|^2|k'|^2} \leqslant \frac{A_{18}}{|k|}$. Substitution of the last inequality into (16) gives

$$\|v^{(n)}(k,t)\| \leqslant \|v^{(0)}\| + A_{18} \left(v^{(n-1)}\right)^2 |k|^2 \int_0^t e^{-|k|^2(t-\tau)} d\tau = \|v^{(0)}\| + A_{18} \left(v^{(n-1)}\right)^2 \cdot \left(1 - e^{-|k|^2 t}\right).$$

If for some $\lambda > 1$ we have the inequality $\|v^{(n-1)}\| \leqslant \lambda \|v^{(0)}\|$ then

$$\|v^{(n)}(k,t)\| \leqslant \|v^{(0)}\| + A_{18} \cdot \lambda^2 \cdot \|v^{(0)}\|^2 = \|v^{(0)}\| \left(1 + A_{18} \cdot \lambda \cdot \|v^{(0)}\|\right).$$

Choose λ so that $A_{18} \cdot \lambda \cdot \|v^{(0)}\| \leqslant \lambda - 1$ or $\lambda \geqslant \frac{1}{1 - A_{18} \cdot \|v^{(0)}\|}$. Then $\|v^{(n)}\| \leqslant \lambda \|v^{(0)}\|$. In other words, for any such λ all iterations $v^{(n)}(k,t)$ are defined, $v^{(n)}(k,t) \in \Phi(2,2)$ and $\|v^{(n)}(k,t)\| \leqslant \lambda \|v^{(0)}\|$.

Now we have to prove the convergence of $v^{(n)}$ to a limit as $n \to \infty$. Put

$$h^{(n)} \sup_{0 \leqslant t \leqslant t_0} \sup_{k \in \mathbb{R}^3} \left|v^{(n)}(k,t) - v^{(n-1)}(k,t)\right| |k|^2.$$

We use (5):

$$\left|v^{(n+1)}(k,t) - v^{(n)}(k,t)\right| |k|^2 \leqslant \lambda \|v^{(0)}\| \cdot w^{(n)} \cdot |k|^3 \int_0^t e^{-|k|^2(t-\tau)} d\tau \int\limits_{\mathbb{R}^3} \frac{dk'}{|k-k'||k'|^2}$$

$$\leqslant A_{18} \cdot \lambda \cdot \|v^{(0)}\| \cdot w^{(n)} < 1.$$

Thus if $A_{18} \cdot \lambda \cdot \|v^{(0)}\| < 1$ then $h^{(n)}$ tend to zero exponentially fast and we have the convergence of $v^{(n)}$ to a limit. Remark that we have no restrictions on t_0, i.e. our arguments give the existence of solutions on any interval of time. C.Fefferman explained to one of us [1] how in our case the existence

[1] C.Fefferman, private communication.

of global solutions follows from the existence of local solutions with the help of scaling arguments. It is interesting that presumably for large $|v^{(0)}|$ the local existence theorem is not valid.

4 Some possibilities for blow ups in finite time in the spaces $\Phi(\alpha,\alpha)$.

We shall consider $v^{(0)}(k)=\frac{c^{(0)}(k)}{|k|^\alpha},\ \ k\in\mathbb{R}^3$. It follows from Theorem 1 that for sufficiently small t_0 there exists the unique solution of (2) on the interval $[0,t_0]$. Since it is only a local statement there are all reasons to believe that for α close to 3 there can be blow ups of solutions in finite time. In this section we discuss related possibilities. Take a sequence $\left\{\Delta^{(n)}\right\},\ \ \Delta^{(n)}\to 0$ as $n\to\infty$ so that $\sum\limits_n \Delta^{(n)}<\infty$. We shall consider (3) and neglect for small values of t by the dependence of $v(k,\tau)$ on τ. In this way we get the sequence of recurrent relations:

$$v^{(n+1)}(k)=e^{-|k|^2\Delta^{(n)}}v^{(n)}(k)+i\int\limits_0^{\Delta^{(n)}}e^{-|k|^2(t-\tau)}d\tau\cdot\int\limits_{\mathbb{R}^3}<k,v^{(n)}(k-k')>\cdot P_k v^{(n)}(k')dk'$$

or

$$c^{(n+1)}(k)=e^{-|k|^2\Delta^{(n)}}\cdot c^{(n)}(k)+i|k|^{\alpha-2}\left(1-e^{-|k|^2\Delta^{(n)}}\right)\cdot\int\frac{<k,c^{(n)}(k-k')>\cdot P_k c^{(n)}(k')}{|k-k'|^\alpha\cdot|k'|^\alpha}\qquad(17)$$

The sequence (17) is the main approximation for NSS. We make the following scaling assumptions: $c^{(n)}(k)=V^{(n)}f^{(n)}\left(k\sqrt{\Delta^{(n)}}\right)$ where $f^{(n)}(\varkappa)$ converge to a limit as $n\to\infty$ and the factor $V^{(n)}$ is chosen so that $\sup\limits_{k\in\mathbb{R}^3}\left|f^{(n)}\left(k\sqrt{\Delta^{(n)}}\right)\right|=1$. Then from (17)

$$V^{(n+1)}f^{(n+1)}\left(k\sqrt{\Delta^{(n+1)}}\right)=e^{-|k|^2\cdot\Delta^{(n)}}V^{(n)}f^{(n)}\left(k\sqrt{\Delta^{(n)}}\right)+i\left(V^{(n)}\right)^2|k|^{\alpha-2}\left(1-e^{-|k|^2\Delta^{(n)}}\right)$$

$$\cdot\int\limits_{\mathbb{R}^3}\frac{<k,f^{(n)}\left((k-k')\sqrt{\Delta^{(n)}}\right)>P_k f^{(n)}\left(k'\sqrt{\Delta^{(n)}}\right)}{|k-k'|^\alpha\cdot|k'|^\alpha}dk'\qquad(18)$$

Put $\beta^{(n+1)}=\frac{V^{(n+1)}}{V^{(n)}}$ and $\varkappa=k\sqrt{\Delta^{(n)}}$. From (18)

$$\beta^{(n+1)}f^{(n+1)}\left(\varkappa\sqrt{\frac{\Delta^{(n+1)}}{\Delta^{(n)}}}\right)=e^{-\varkappa^2}f^{(n)}(\varkappa)+iV^{(n)}\cdot\left(\Delta^{(n)}\right)^{\frac{\alpha-2}{2}}\cdot\left(1-e^{-\varkappa^2}\right)\cdot$$

$$\cdot|\varkappa^{\alpha-2}|\int\limits_{\mathbb{R}^3}\frac{<\varkappa,f^{(n)}(\varkappa-\varkappa')>P_\varkappa f^{(n)}(\varkappa')}{|\varkappa-\varkappa'|^\alpha\cdot|\varkappa'|^\alpha}d\varkappa'$$

Choose $\Delta^{(n)}$ so that $V^{(n)} \cdot \left(\Delta^{(n)}\right)^{\frac{\alpha-2}{2}} = a$ where a is an arbitrary constant, for example, $a = 1$. This gives the expression of $\Delta^{(n)}$ through $V^{(n)}$. From the definitions

$$\beta^{(n+1)} = \max_{\varkappa \in \mathbb{R}^3} \left| e^{-\varkappa^2} f^{(n)}(\varkappa) + i\left(1 - e^{-\varkappa^2}\right) \cdot |\varkappa|^{\alpha-2} \cdot \int\limits_{\mathbb{R}^3} \frac{< \varkappa, f^{(n)}(\varkappa - \varkappa') > P_\varkappa f^{(n)}(\varkappa')}{|\varkappa - \varkappa'|^\alpha \cdot |\varkappa'|^\alpha} d\varkappa' \right|$$

If in the limit $n \to \infty$ the functions $f^{(n)}$ converge in the uniform metric to a limit then the limiting function f satisfies the following equation:

$$\beta f(\varkappa \cdot h) = e^{-\varkappa^2} f(\varkappa) + i\left(1 - e^{-\varkappa^2}\right) \cdot |\varkappa|^{\alpha-2} \int\limits_{\mathbb{R}^3} \frac{< \varkappa, f^{(n)}(\varkappa - \varkappa') > P_\varkappa f^{(n)}(\varkappa')}{|\varkappa - \varkappa'|^\alpha \cdot |\varkappa'|^\alpha} d\varkappa' \qquad (19)$$

where

$$\beta = \max \left| e^{-\varkappa^2} f(\varkappa) + i\left(1 - e^{-\varkappa^2}\right) \cdot |\varkappa|^{\alpha-2} \int \frac{< \varkappa, f^{(n)}(\varkappa - \varkappa') > P_\varkappa f^{(n)}(\varkappa')}{|\varkappa - \varkappa'|^\alpha \cdot |\varkappa'|^\alpha} d\varkappa' \right|$$

The value of h is found from the relation:

$$\sqrt{\frac{\Delta^{(n+1)}}{\Delta^{(n)}}} = \left(\frac{V^{(n)}}{V^{(n+1)}}\right)^{\frac{1}{\alpha-2}} = \beta_n^{-\frac{1}{\alpha-2}} \to \beta^{-\frac{1}{\alpha-2}}$$

The equation (19) can be considered as the equation for the fixed point of a renormalization group. We are interested only in solutions for which $\beta > 1$. Also the space of odd functions is invariant under the non-linear transformation in the rhs of (19). Therefore the simplest fixed point can be from this subspace. The integral

$$\int \frac{< \varkappa, f^{(n)}(\varkappa - \varkappa') > P_\varkappa f^{(n)}(\varkappa')}{|\varkappa - \varkappa'|^\alpha \cdot |\varkappa'|^\alpha} d\varkappa'$$

behaves as an homogenous function of degree $4 - 2\alpha$. Therefore for $\varkappa \to 0$ the non-linear term in (19) decays as $|k|^{4-\alpha}$. For $\varkappa \to \infty$ it decays as $\varkappa^{2-\alpha}$ and it is natural to consider (19) in the space of functions with this type of decay. This scenario of blow up assumes strong connection between the behavior of initial conditions near $k = 0$ and $k = \infty$. We hope to discuss the main equation, its solution, the role of parameter a and the relation to blow ups in another paper.

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