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SUMMER SCHOOL AND CONFERENCE ON DYNAMICAL SYSTEMS

Evolutionary Dynamics
(Lecture 1)

Karl Sigmund
Institute of Mathematics
University of Vienna
Vienna
Austria

These are preliminary lecture notes, intended only for distribution to participants

strada costiera, 11 - 34014 trieste italy - tel. +39 040 2240111 fax +39 040 224163 - sci_info@ictp.trieste.it - www.ictp.trieste.it

Evolutionary Game Dynamics

Minicourse 1

Nash equilibria

player I uses strategy $i \in \{1,\ldots,n\}$

player II uses strategy $j \in \{1,\ldots,m\}$

payoff a_{ij} for I, b_{ij} for II

mixed strategies:

player I uses $x = (x_1,\ldots,x_n) \in S_n$ (unit simplex)

player II uses $y \in S_m$

payoff $x^T Ay = \sum a_{ij} x_i y_j$ for I, $x^T By$ for II

Nash equilibria

x best reply to y $(x \in BR(y))$ if

$$z^T Ay \leq x^T Ay$$

for all $z \in S_n$

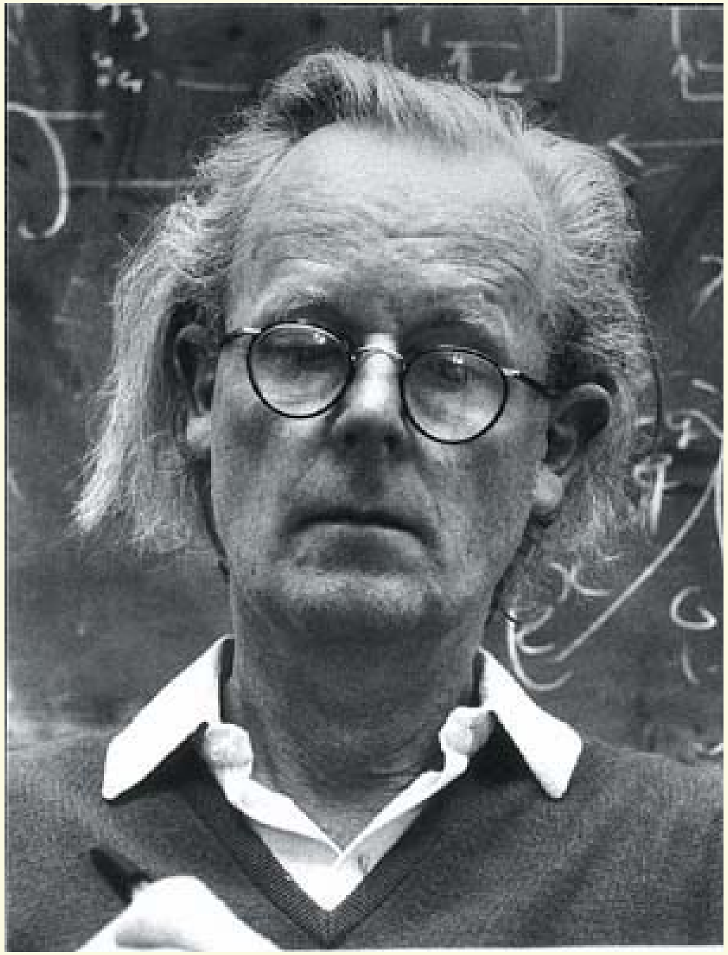
$(x, y) \in S_n \times S_m$ Nash equilibrium if

$$x \in BR(y) \text{ and } y \in BR(x)$$

The greatest conceptual revolution in biology...the replacement of typological thinking by population thinking.

Ernest Mayer

John Maynard Smith



Symmetric Games

players I and II interchangeable in population

$a_{ij} = b_{ji}$ $\qquad A = B^T$

consider only symmetric pairs (x, x)

$x \in S_n$ Nash equilibrium if $x \in BR(x)$

$$z^T A x \le x^T A x$$

for all $z \in S_n$

Chicken Game

to swerve or not to swerve?

my payoff

	if co-player escalates	if co-player swerves
if I escalate	-10	1
if I swerve	-1	0

Nash: swerve with 90 percent

Population setting

large population, n types

x_i frequency of type i

$x = (x_1, \ldots, x_n) \in S_n$ state of population

$(Ax)_i = \sum_j a_{ij} x_j$ average payoff for type i

$x^T Ax$ mean payoff in population

$\dot{x}_i / x_i$ per capita rate of growth

Replicator equation

$$\dot{x}_i = x_i((Ax)_i - x^T Ax)$$

S_n and boundary faces $S_n(J)$ invariant

$S_n(J) = \{x : x_i \geq 0,\ x_i = 0 \text{ if } i \in J,\ \sum x_i = 1\}$

adding constants to colums of A leaves equ. invariant

$$(\frac{x_i}{x_j})^{\cdot} = (\frac{x_i}{x_j})((Ax)_i - (Ax)_j)$$

Replicator dynamics and Nash equilibria

$$\dot{x}_i = x_i((Ax)_i - x^T Ax)$$

rest point : $z_i = 0$ or $(Az)_i = z^T Az$

(generically one or none in intS_n)

z Nash equ. iff rest point and saturated

i.e. $z_i = 0 \Rightarrow (Az)_i \le z^T Az$

Folk theorem of evolutionary game theory

- Nash equilibria are rest points
- strict Nash equilibria are attractors
- stable rest points are Nash equilibria
- limits of interior orbits are Nash equilibria

Existence of Nash equilibria:

$$\dot{x}_i = x_i((Ax)_i - x^T Ax - n\varepsilon) + \varepsilon$$

flow on boundary points into interior of S_n

fixed point $z(\varepsilon)$

$$(Az(\varepsilon))_i - z(\varepsilon)^T Az(\varepsilon) = \varepsilon(n - \frac{1}{z_i(\varepsilon)})$$

accumulation point z saturated rest point

Replicator equation for n=2

$$x_1 = x, \quad x_2 = 1 - x$$

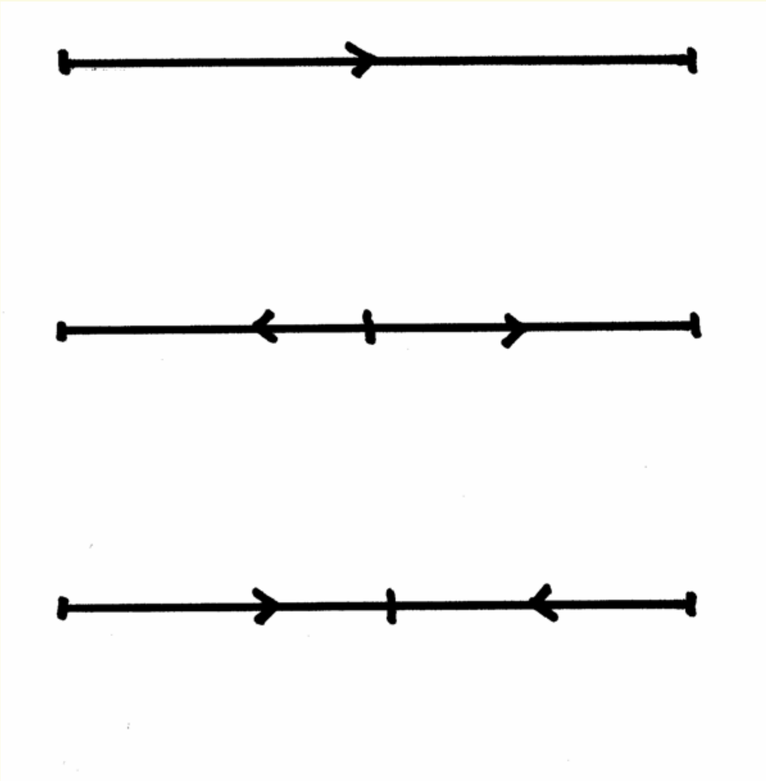
$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \quad \text{or equivalently} \quad \begin{bmatrix} 0 & a \\ b & 0 \end{bmatrix}$$

$$\dot{x} = x(1-x)[a-(a+b)x]$$

rest points for $x = 0, x = 1$ and $x = \frac{a}{a+b}$

(provided $ab > 0$)

Replicator equations for n=2:



- Dominance
- Bistability
- stable coexistence

Example dominance

Prisoner's Dilemma

$$\begin{bmatrix} 10 & -5 \\ 15 & 0 \end{bmatrix}$$

or equivalently

$$\begin{bmatrix} 0 & -5 \\ 5 & 0 \end{bmatrix}$$

$$x \rightarrow 0 \quad \text{(freq. of cooperators)}$$

Example bistability

Iterated Prisoner's Dilemma

$$\begin{bmatrix} 60 & -5 \\ 15 & 0 \end{bmatrix}$$

or equivalently

$$\begin{bmatrix} 0 & -5 \\ -15 & 0 \end{bmatrix}$$

$x \to 0$ or $x \to 1$ (frequ. of Tit For Tat)

Example coexistence

Chicken (or Snowdrift game):

$$\begin{bmatrix} -10 & 1 \\ -1 & 0 \end{bmatrix}$$

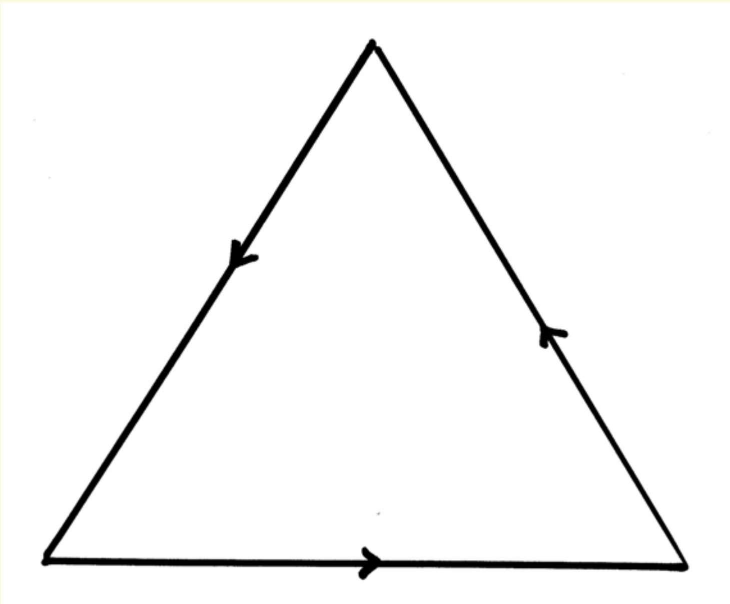
or equivalently

$$\begin{bmatrix} 0 & 1 \\ 9 & 0 \end{bmatrix}$$

$$x \to 1/10 \qquad \text{(frequ. of escalation)}$$

self - regulation!

Replicator equations for n=3:

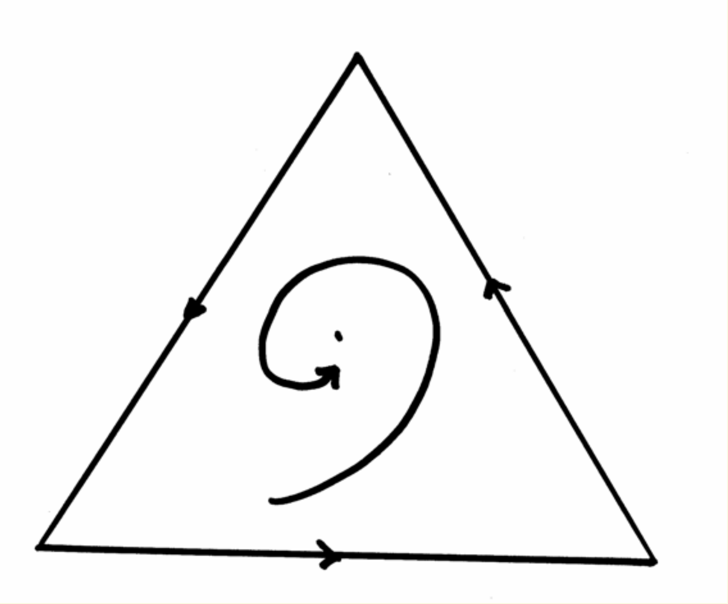


Possibility for

rock-scissors-paper

heteroclinic cycles

Rock-Scissors-Paper



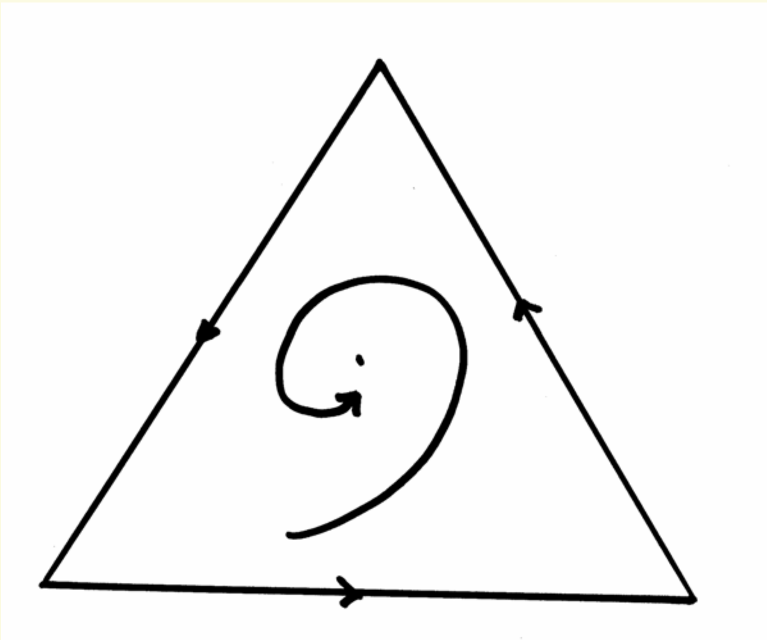
$$A = \begin{bmatrix} 0 & -a_2 & b_3 \\ b_1 & 0 & -a_3 \\ -a_1 & b_2 & 0 \end{bmatrix}$$

$$1 \to 2 \to 3 \to 1$$

corners e_i saddles (not Nash)

eigenvalues b_i and $-a_i$

Rock-Scissors-Paper

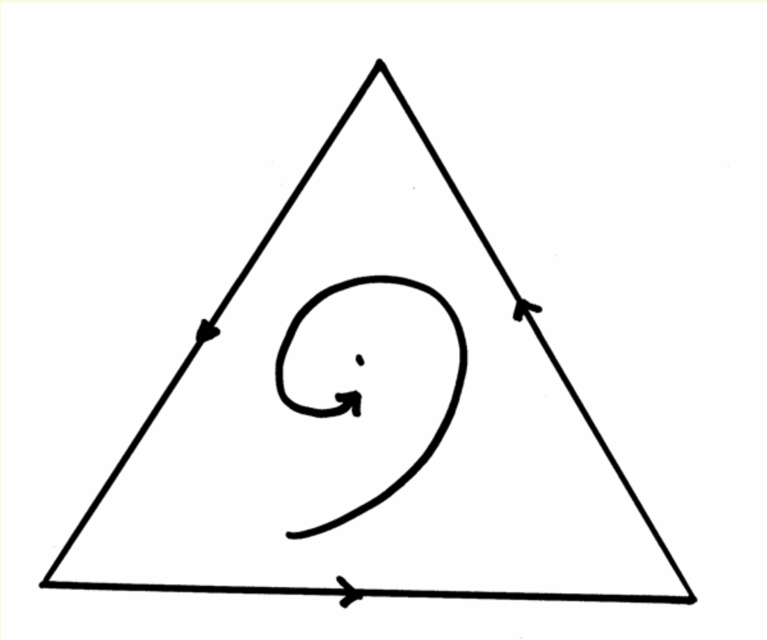


$$A = \begin{bmatrix} 0 & -a_2 & b_3 \\ b_1 & 0 & -a_3 \\ -a_1 & b_2 & 0 \end{bmatrix}$$

$$1 \rightarrow 2 \rightarrow 3 \rightarrow 1$$

$z \in \operatorname{int} S_n$ unique Nash

Rock-Scissors-Paper



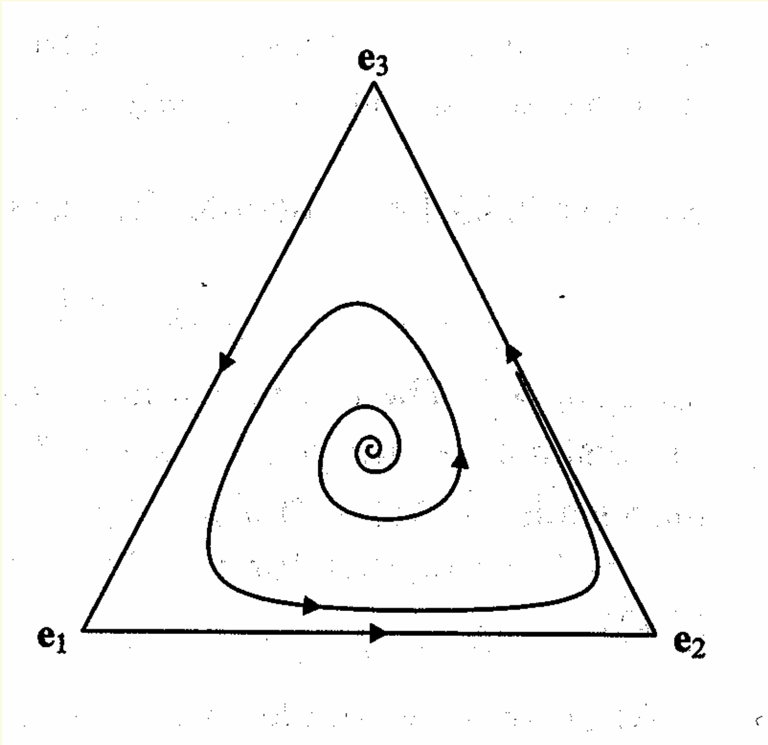
$$A = \begin{bmatrix} 0 & -a_2 & b_3 \\ b_1 & 0 & -a_3 \\ -a_1 & b_2 & 0 \end{bmatrix}$$

z solution of

$$(Az)_1 = (Az)_2 = (Az)_3$$

$$\text{common value} = z^T Az$$

Rock-Scissors-Paper



E.C. Zeeman, Josef Hofbauer :

Equivalent :

(a) z locally stable

(b) z globally stable

(c) $\det A > 0$

(d) $z^T Az > 0$

if not, $\mathrm{bd}S_3$ heteroclinic cycle

Rock-Scissors-Paper

z satisfies $(Az)_1 = (Az)_2 = (Az)_3$

$(Az)_i = -a_{i+1} z_{i+1} + b_{i-1} z_{i-1}$

$z = \sigma (b_2 b_3 + b_2 a_3 + a_2 a_3, \ldots, \ldots)$

with σ s.t. $z \in S_n, \quad \sigma > 0$

$z^T Az = \sigma \det A$

Rock-Scissors-Paper

$P(x) := \prod x_i^{z_i}$

$t \to x(t) \to P(x(t))$

satisies $\dot{P} = P(z^T Ax - x^T Ax) = -P(\xi^T Ax)$

(with $\xi = x - z$)

$= -P(\xi^T Ax - \xi^T Az) = -P(\xi^T A\xi)$

$= -P[\xi_1\xi_2(b_1 - a_2) + \xi_2\xi_3(b_2 - a_3) + \xi_3\xi_1(b_3 - a_1)]$

no restriction of generality to assume $b_i - a_{i+1} = c$

(projective change of coords., c has sign of detA)

Rock-Scissors-Paper

$$\dot{P} = -cP(\xi_1\xi_2 + \xi_2\xi_3 + \xi_3\xi_1) = cP(\xi_1{}^2 + \xi_2{}^2 + \xi_3{}^2)$$

because

$$\xi_1\xi_2 + \xi_2\xi_3 + \xi_3\xi_1 = \frac{1}{2}(\xi_1 + \xi_2 + \xi_3)^2 - (\xi_1{}^2 + \xi_2{}^2 + \xi_3{}^2)$$

hence P is Lyapunov function

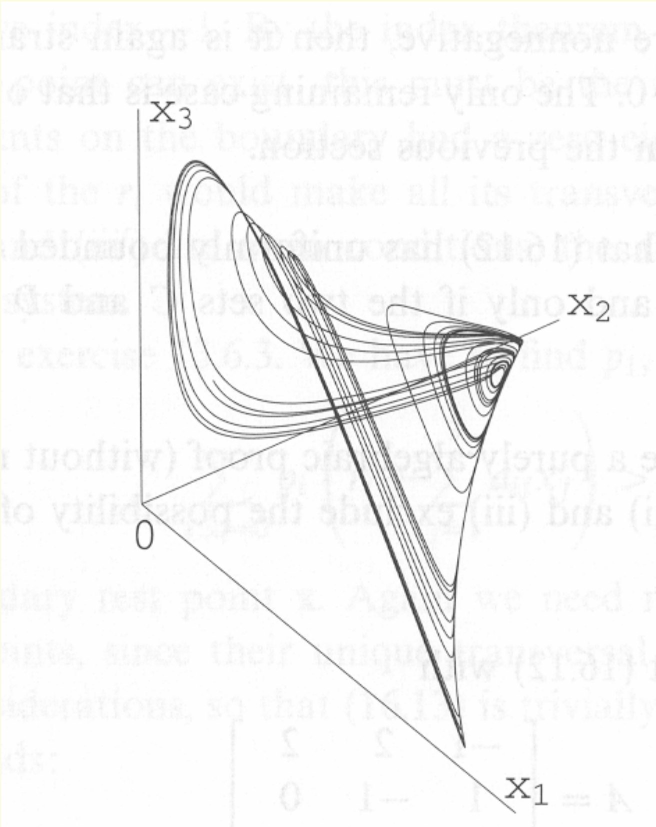
Rock-Scissors-Paper in nature

- Uta stansburiana (lizards)
- males: 3 morphs (inheritable)
- monogamous, guards female
- polygamous, guards harem (less efficiently)
- loose males, sneaky matings

Rock-Scissors-Paper in nature

- Escherichia coli (bacteria)
- can produce colicin (toxic) and immunity protein
- can produce only immunity
- can produce neither nor

Phase portraits of Replicator equations:



for $n \geq 4$ no classif.

equiv. with Lotka - Volterra

$$\dot{y}_i = y_i (r_i + \sum b_{ij} y_j)$$

$i = 1, \ldots, n-1$

one or several limit cycles

chaotic attractors

- Replicator dynamics assumes clonal replication (like begets like)
- other derivation from models of learning and imitation

Imitation Dynamics

input - output

$$\dot{x}_i = x_i \sum [f_{ij}(x) - f_{ji}(x)] x_j$$

where f_{ij} rate for switch $j \to i$

if payoff dependence

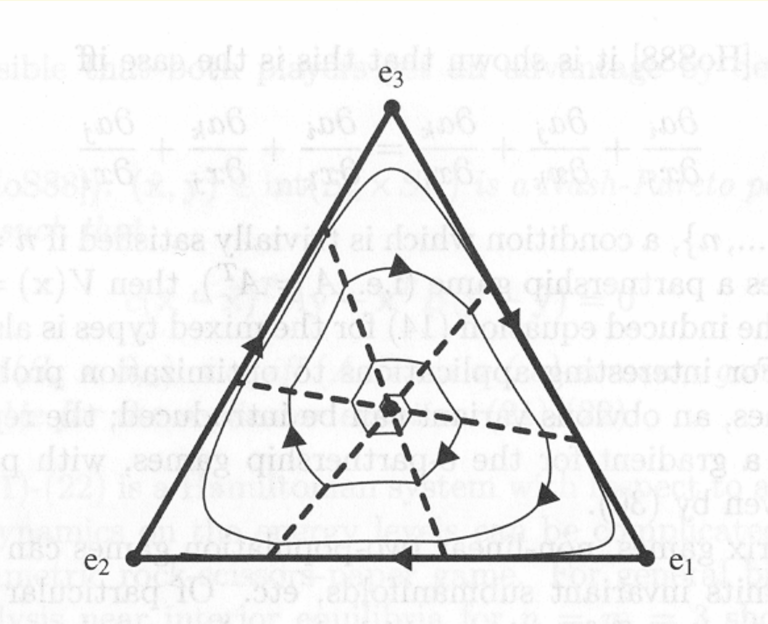
$$f_{ij}(x) = f((Ax)_i, (Ax)_j)$$

imitate the better : $f(u,v) = 0$ if $u \leq v$

$= 1$ if $u > v$

i increases if payoff better than median

Imitation Dynamics



- Piecewise replicator-like

Imitation Dynamics

input - output

$$\dot{x}_i = x_i \sum [f_{ij}(x) - f_{ji}(x)] x_j$$

where f_{ij} rate for switch $j \to i$

if f increasing and $f_{ij} = f([(Ax)_i - (Ax)_j]_+)$

$$\dot{x}_i = x_i [f((Ax)_i) - \bar{f}]$$

similarly if $f_{ij} = f((Ax)_i)$ or $= -f((Ax)_j)$

if f linear $\Rightarrow$ replicator

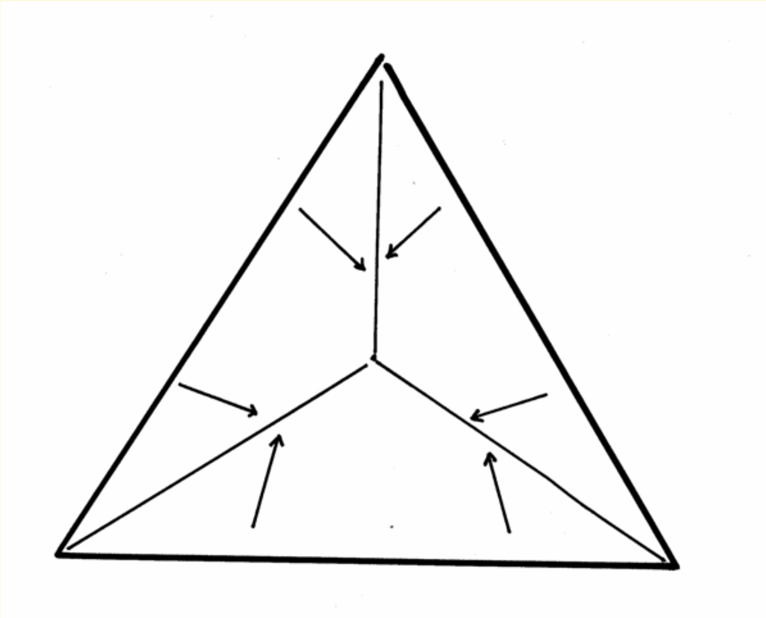
Best Reply Dynamics

$BR(x)$ best reply when in state x

players update by adopting best reply(ies)

$$\dot{x} = BR(x) - x$$

Best reply Dynamics



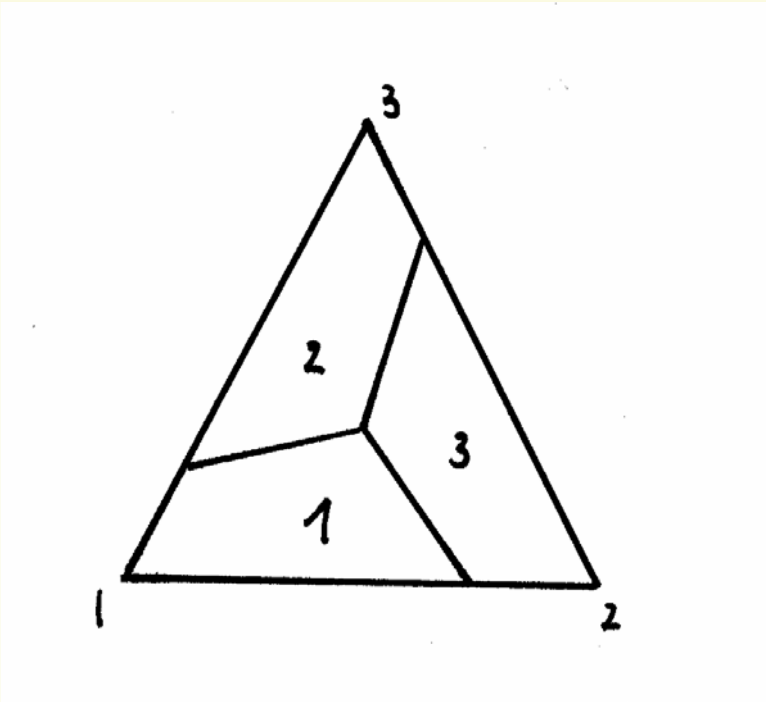
$\dot{x} \in BR(x) - x$

piecewise linear orbits

$$\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

rational but myopic players

Best reply Dynamics



rock - scissors - paper

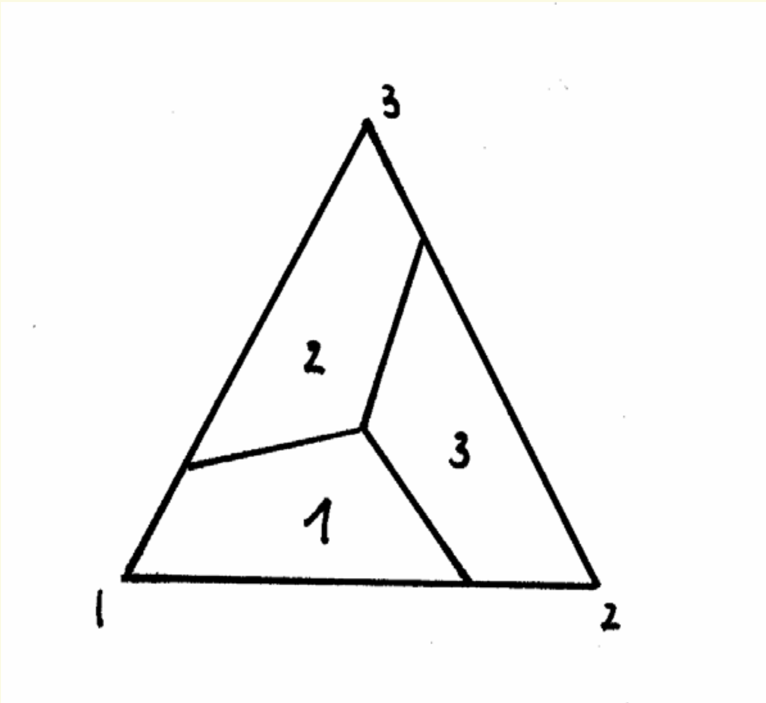
$$\begin{bmatrix} 0 & -a_2 & b_3 \\ b_1 & 0 & -a_3 \\ -a_1 & b_2 & 0 \end{bmatrix}$$

for $\det A < 0$ convergence

to Nash equ. point

(same with rep. dynamics)

Best reply Dynamics



rock - scissors - paper

$V(x) := \max(Ax)_i$

minimal value $z^T Az$ at z

if i best reply,

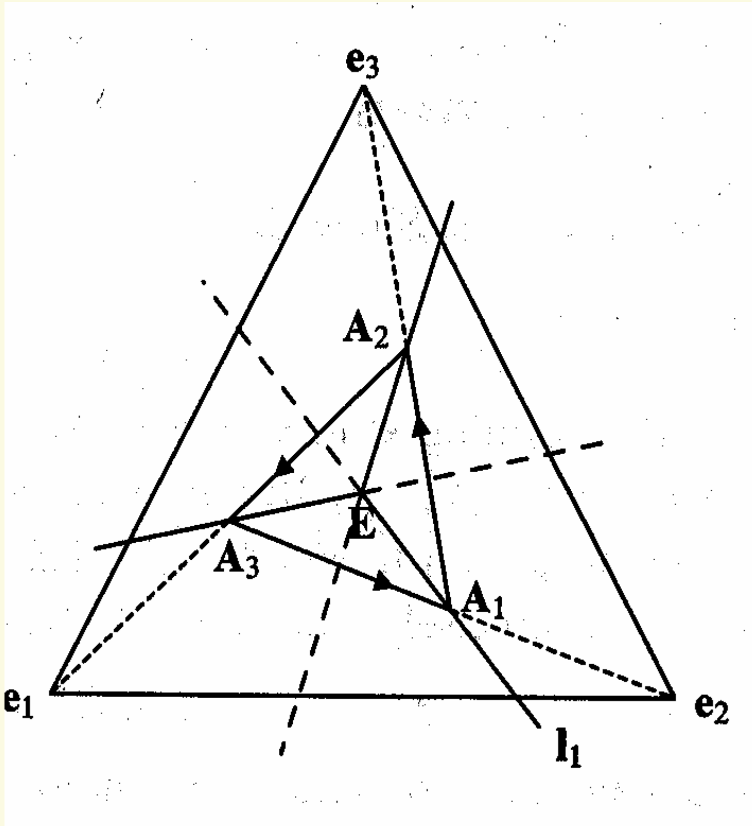
$\dot{V} = (A\dot{x})_i = (Ae_i)_i - (Ax)_i$

(because $\dot{x} = e_i - x$ and

$(Ae_i)_i = a_{ii} = 0)$

$\Rightarrow \dot{V} = -V(x)$

Best reply Dynamics



rock - scissors - paper

$$\begin{bmatrix} 0 & -a & b \\ b & 0 & -a \\ -a & b & 0 \end{bmatrix}$$

for $\det A > 0$

Shapley - triangle

(rep. dynamics $\rightarrow \mathrm{bd} S_n$

but time average

$$z_i(T) = \frac{1}{T}\int_0^T x_i(t)dt \rightarrow \text{Shapley)}$$