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ICTP 40th Anniversary

SMR.1573 - 8

***SUMMER SCHOOL AND CONFERENCE ON DYNAMICAL SYSTEMS***

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# Evolutionary Dynamics

**(Lecture 5)**

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# Evolutionary Games and Public Goods

Theoretical Models and Experimental Economics

# Examples

- Sheltering
- Group defense
- Foraging
- Brood care

# Examples

- Public transportation
- conservation
- health insurance
- fighting crime

TEMPTATION TO FREE-RIDE

EXPLOITATION OF CO-PLAYERS

# Public goods:

- Groups of cooperators do better than groups of defectors
- Defectors outperform cooperators in each group

# Experiment

Six players

- One euro each
- Contribute to Common Pool?
- Experimenter triples amount in pool
  divides it equally among the six players

Return for each player only 50 cents

selfish individual tempted to defect

and exploit co-players

# Social dilemma

- Tragedy of the Commons
- Free Rider Problem
- Many-Person Prisoner´s Dilemma

# Experimental results:

- Many players contribute
- If game repeated for a few rounds, contribution drops to zero

cost of co-operation $= 1$ (contribution)

$r$ multiplication factor of common good

$n_c$ number of co-operators in group

assume $r < N$

$$P_d = r\frac{n_c}{N} \qquad \text{payoff defectors}$$

$$P_c = P_d - 1 \qquad \text{payoff co-operators}$$

$\Rightarrow$ frequency of co-operators decreases to 0

Part 1: punishment of defectors

Part 2: option to drop out

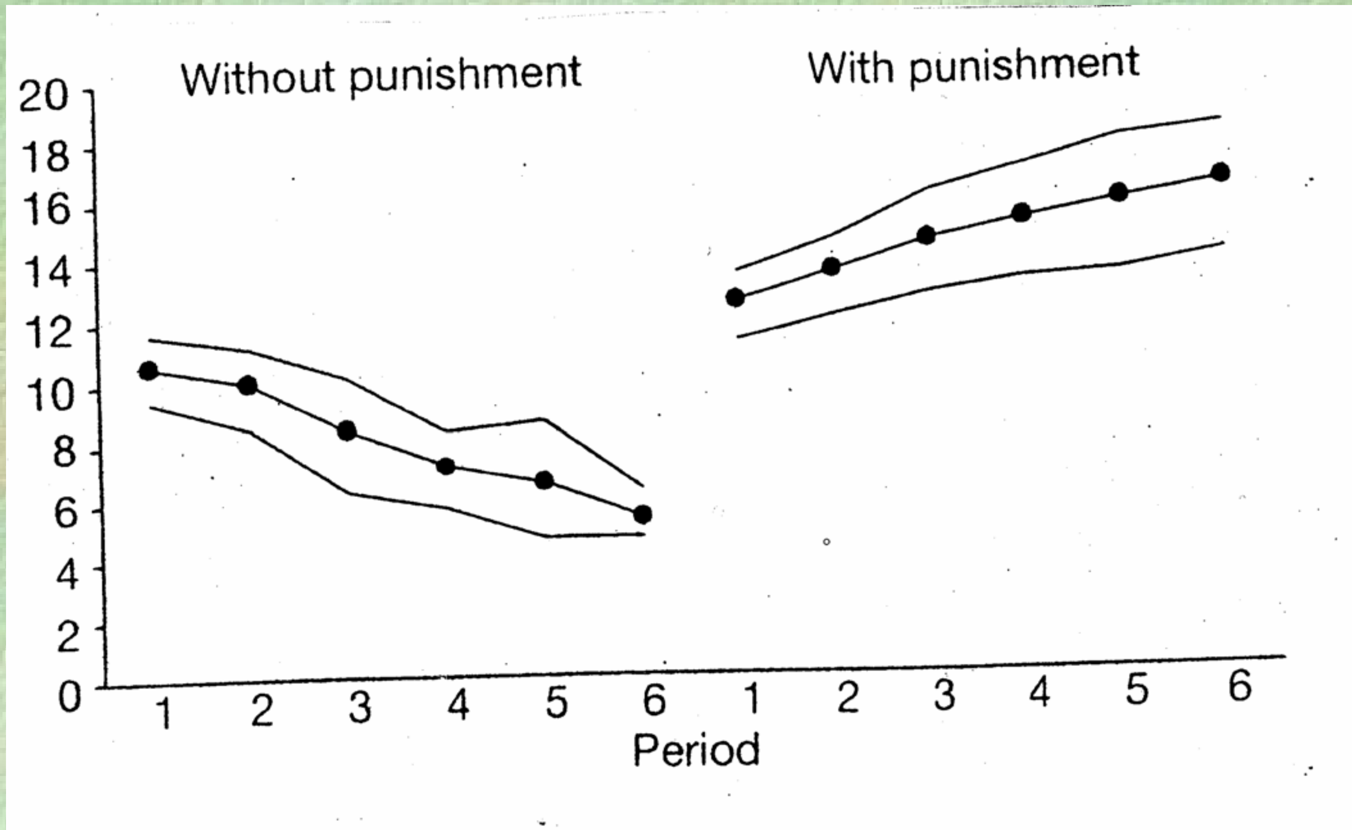
(replicator dynamics with non-linear payoff)

# Punishment

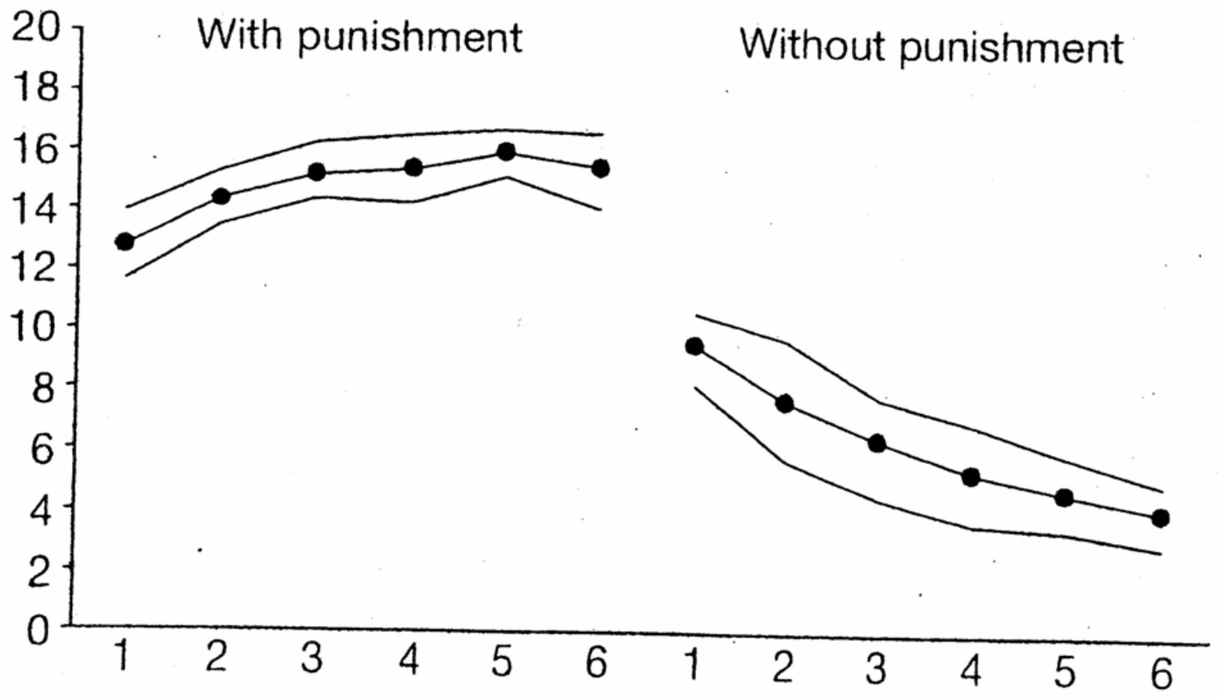
after public goods game, players can fine co-players

punishment is costly

punishment is efficient (Fehr et al, Nature, 2002)



Without punishment
With punishment
20
18
16
14
12
10
8
6
4
2
0
1
2
3
4
5
6
1
2
3
4
5
6
Period



With punishment
Without punishment
20
18
16
14
12
10
8
6
4
2
0
1 2 3 4 5 6 1 2 3 4 5 6

# Strategies for Public Goods with Punishment

| | | |
|---|---|---|
| $G_1$ | social: | contribute, punish |
| $G_2$ | paradoxical: | don't contribute, punish |
| $G_3$ | asocial: | don't contribute, don't punish |
| $G_4$ | mild: | contribute, don't punish |

$-\beta$ fine (for punished player)

$-\gamma$ cost (for punisher)

# Payoff for Public Goods with Punishment

from contributions of (N - 1) co - players

$$B = (N-1)(x_1 + x_4)\frac{r}{N}$$

and additionally

# Payoff for Public Goods with Punishment

social:

$$P_1 = -(1 - \frac{r}{N}) - (N-1)(x_2 + x_3)\gamma$$

paradoxical:

$$P_2 = -(N-1)(x_1 + x_2)\beta - (N-1)(x_2 + x_3)\gamma$$

asocial:

$$P_3 = -(N-1)(x_1 + x_2)\beta$$

mild:

$$P_4 = -(1 - \frac{r}{N})$$

$$P_1 + P_3 = P_2 + P_4$$

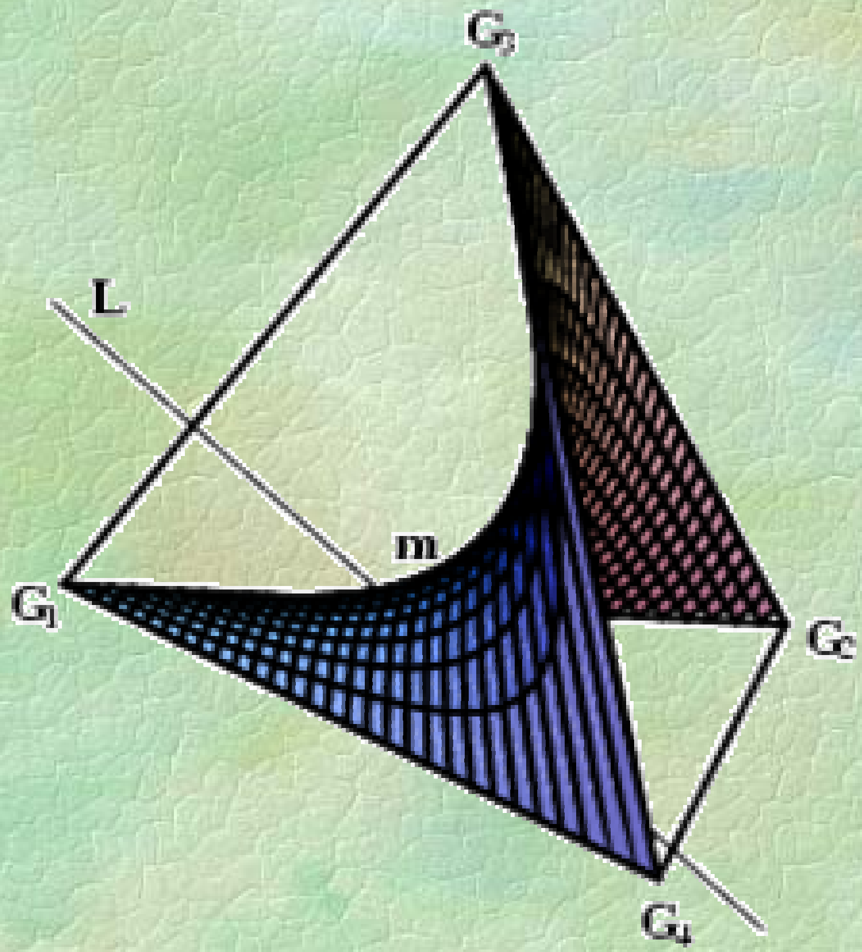
for replicator dynamics

$$\left(\frac{x_1 x_3}{x_2 x_4}\right)^{\bullet} = \left(\frac{x_1 x_3}{x_2 x_4}\right)(P_1 + P_3 - P_2 - P_4) = 0$$

hence

$$W_K = \{x \in S_n : x_1 x_3 = K x_2 x_4\}$$

invariant.



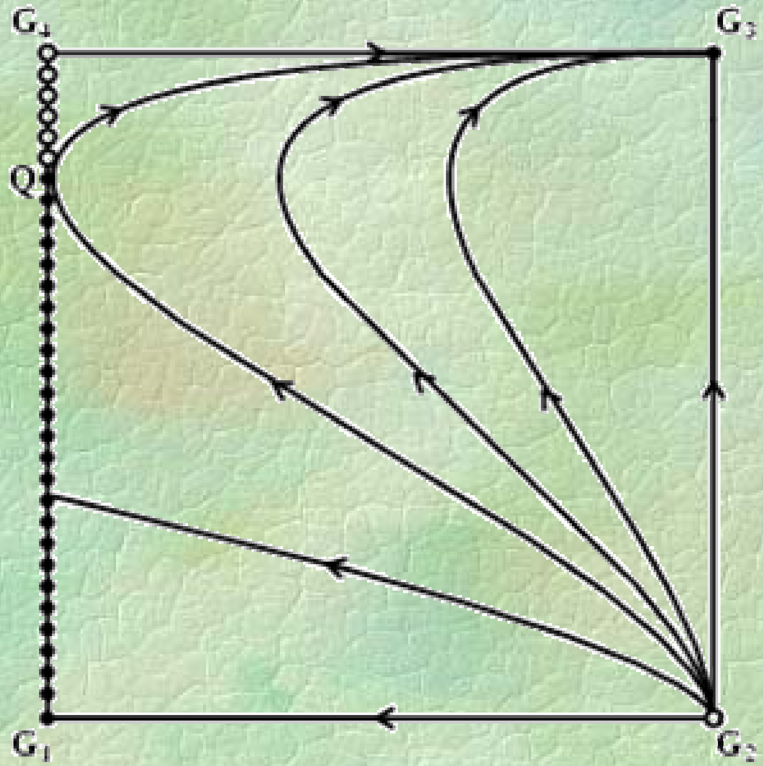

$$x_1 x_3 = K x_2 x_4$$

$$x_1 + x_2 + x_3 + x_4 = 1$$

saddle spanned by

$$G_1 - G_2 - G_3 - G_4 - G_1$$

# Public Goods with Punishment




Nash equilibria : $G_3$ and segment $G_1Q$

saturated fixed points

random shocks lead to $G_3$ (asocial state)

(if $\beta > \frac{N-r}{N(N-1)}$)

# Reputation and Temptation

with small probability $\mu$

'co-operators' $G_1$ or $G_4$ do NOT contribute

IF all other players non-punishers $G_3$ or $G_4$

# additional payoff terms

social:

$$P_1(\mu) = P_1 + \mu(1-\frac{r}{N})(x_3+x_4)^{N-1}$$

paradoxical:

$$P_2(\mu) = P_2$$

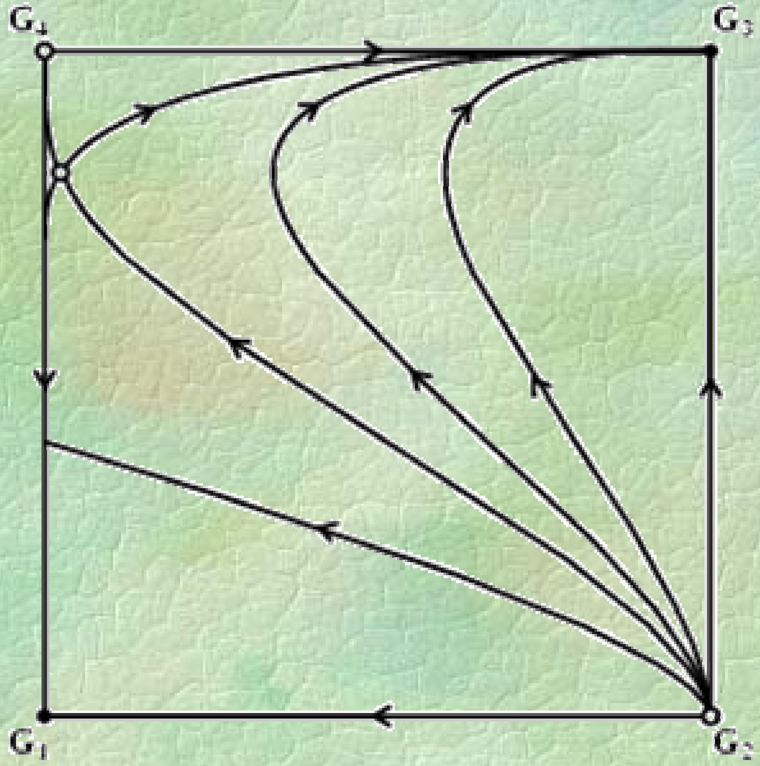
asocial:

$$P_3(\mu) = P_3 - (N-1)\frac{r}{N}\mu(x_1+x_4)(x_3+x_4)^{N-2}$$

mild:

$$P_4(\mu) = P_4 - (N-1)\frac{r}{N}\mu(x_1+x_4)(x_3+x_4)^{N-2} + \mu(1-\frac{r}{N})(x_3+x_4)^{N-1}$$

# Reputation effect




bi - stability

$G_1$ (social) and

$G_3$ (asocial)

both attractors

EXACTLY one rest point in interior of square

with $y := x_3 + x_4$

$P_1(\mu) = P_2(\mu)$ yields

$f(y) = P_1(\mu) - P_2(\mu) =$

$$= \mu \frac{N-r}{N} y^{N-1} - \beta(N-1)y + [\beta(N-1) - \frac{N-r}{N}] = 0$$

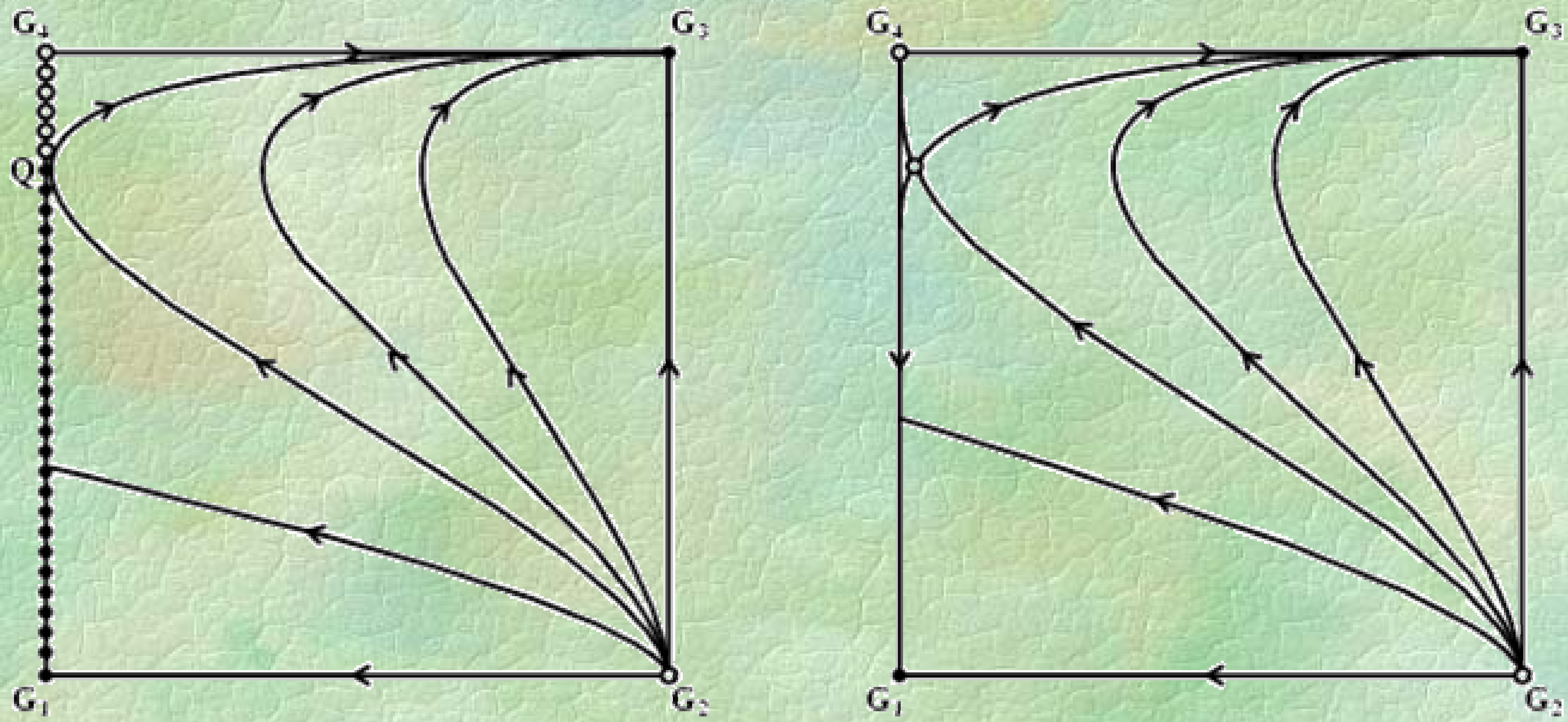
unique solution $y = \hat{y}$ because $f$ convex.

with $z = x_2 + x_3$

$P_3(\mu) = P_2(\mu)$ yields

$$\gamma z = r\mu(1 - \frac{r}{N})(1 - z)\hat{y}^{N-2}$$

# Bifurcation



# Optional Games

Large population

- N players are offered to participate (sample)
- S accept (group)
- N - S decline (loners)
- loners have fallback solution

# Optional Games

three strategies:

- loners
- cooperators
- defectors

  (if only one participates: loner)

Strategies for optional public goods

$$x \text{ freq. of co-operators}$$

$$y \text{ freq. of defectors}$$

$$z \text{ freq. of loners}$$

$$x + y + z = 1$$

# Payoff for optional public goods

loner's payoff

$$P_z = \sigma$$

(assume $0 < \sigma < r - 1$)

payoff for defectors and co-operators as before

$$P_d = r\frac{n_c}{N}$$ payoff defectors

$$P_c = P_d - 1$$ payoff co-operators

# Defector's payoff

$r\frac{m}{S}$ if $m$ co-operators in group with $S$ players

$$\sum_{m=0}^{S-1}\frac{rm}{S}\binom{S-1}{m}\left(\frac{x}{x+y}\right)^{m}\left(\frac{y}{x+y}\right)^{S-1-m}=\frac{r(S-1)}{S}\frac{x}{x+y}$$

$$\sum_{S=1}^{N}\frac{r(S-1)}{S}\frac{x}{x+y}\binom{N-1}{S-1}(1-z)^{S-1}z^{N-S}$$

# Defector‘s payoff

$$P_y = \sigma z^{N-1} + r \frac{x}{1-z}\left(1 - \frac{1-z^N}{N(1-z)}\right)$$

given $S-1$ other players in group,

withholding contribution yields $1-\frac{r}{S}$

$$P_y - P_x = \sum_{S=2}^{N} \binom{N-1}{S-1} (1-z)^{S-1} z^{N-S} (1-\frac{r}{S})$$

$$= 1 + (r-1) z^{N-1} - \frac{r}{N} \frac{1-z^N}{1-z} = F(z)$$

# Payoff for optional public goods

$$P_z = \sigma$$

$$P_y = \sigma z^{N-1} + r\frac{x}{1-z}\left[1 - \frac{1-z^N}{N(1-z)}\right]$$

$$P_y - P_x = 1 + (r-1)z^{N-1} - \frac{r}{N}\frac{1-z^N}{1-z} = F(z)$$

# Rock-Scissors-Paper Cycle

if

$$1 < r < N$$

and

$$0 < \sigma < r - 1$$

# Rock-Scissors-Paper cycle

if most cooperate, best to defect

if most defect, best to abstain

if mostly loners, best to cooperate

(for small groups, Simpson´s Paradox)

# Simpson‘s paradox

in group A, 9 defectors and 1 cooperator

defector earns 1 dollar, cooperator 0

in group B, 9 cooperators and 1 defector

defector earns 11 dollars, cooperator 10

average: defector 2, cooperator 9

# Replicator Dynamics

$$\dot{x} = x(P_x - \overline{P})$$

$$\dot{y} = y(P_y - \overline{P})$$

$$\dot{z} = z(P_z - \overline{P})$$

with $\overline{P} = xP_x + yP_y + zP_z$

$$= \sigma - [(1-z)\sigma - (r-1)x](1 - z^{N-1})$$

# Change in variables

$$(x, y, z) \leftrightarrow (f, z)$$

with $f = \dfrac{x}{x+y}$

$$\dot{f} = \frac{y\dot{x} - x\dot{y}}{(x+y)^2} = \frac{xy}{(x+y)^2}(P_x - P_y)$$

hence

$$\dot{f} = -f(1-f)F(z)$$

$$\dot{z} = [\sigma - f(r-1)]z(1-z)(1-z^{N-1})$$

divide by $f(1-f)z(1-z)(1-z^{N-1})$

# Hamiltonian:

$$\dot{f} = -\frac{F(z)}{z(1-z)(1-z^{N-1})}$$

$$\dot{z} = \frac{\sigma - f(r-1)}{f(1-f)}$$

i.e. $$\dot{f} = -\frac{\partial H}{\partial z}$$

$$z = \frac{\partial H}{\partial f}$$

# Rest point in interior:

$$\dot{f} = -\frac{F(z)}{z(1-z)(1-z^{N-1})}$$

$$\dot{z} = \frac{\sigma - f(r-1)}{f(1-f)}$$

in $]0,1[$ $F(z)$ has same zeros as $G(z) = (1-z)F(z)$

$G(0) > 0$ $G(1) = 0$

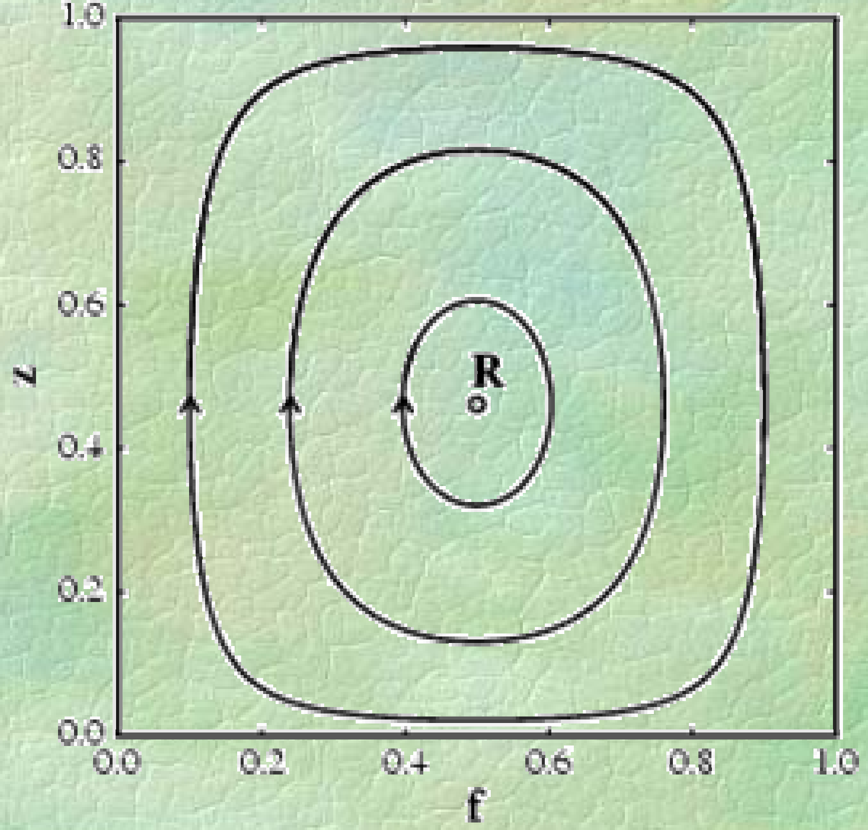
at $z = 1$ local max (min) if $r > 2$ $(r \leq 2)$

$G''(z)$ has sign of $(N-2)(r-1) - z(Nr - N - r)$

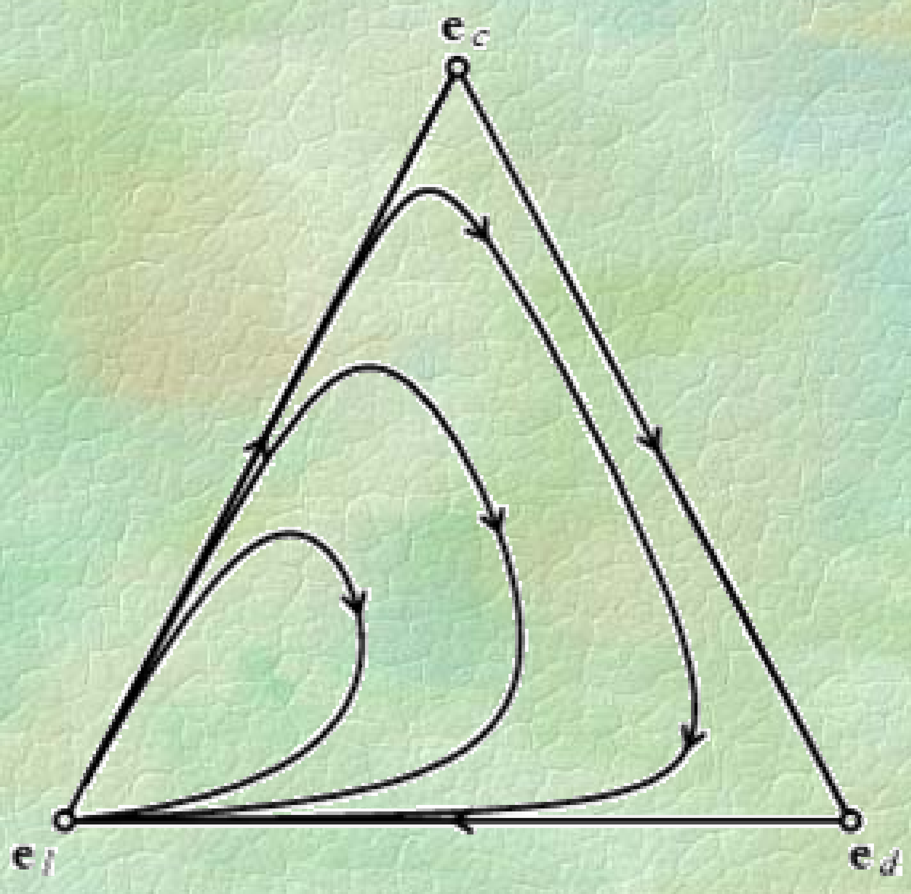
for $1 < r \leq 2$ no zero

for $r > 2$ unique zero at $\hat{z}$

# Hamiltonian dynamics



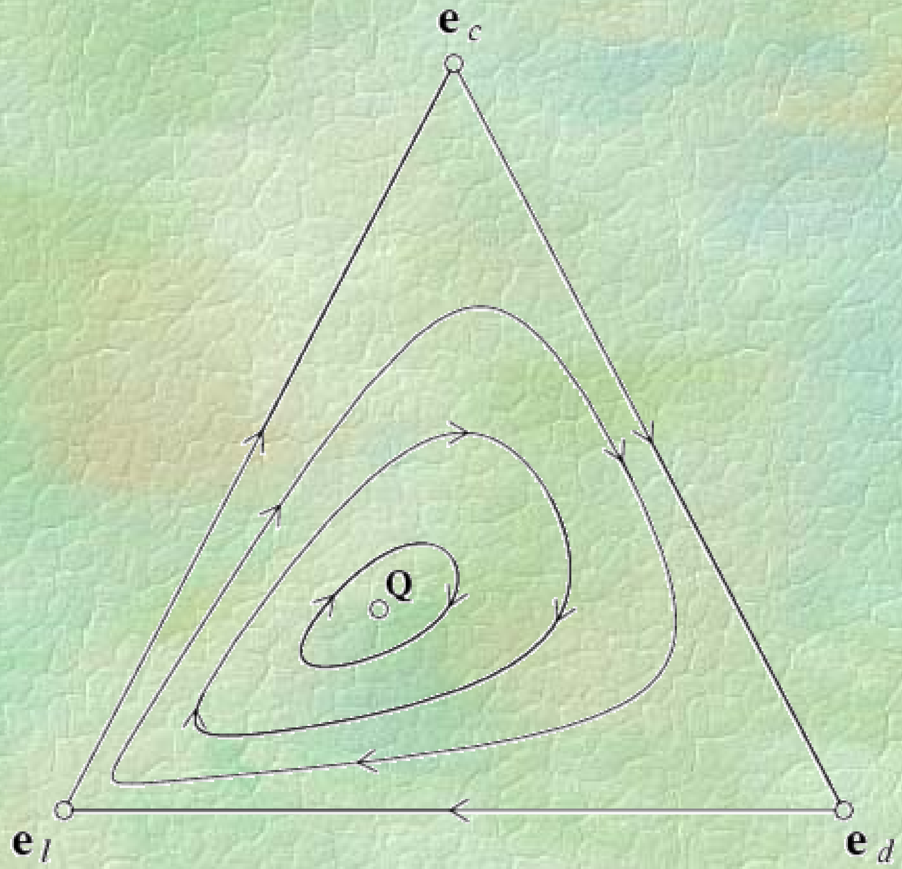
# Replicator dynamics




for $r \leq 2$,

homoclinic orbits

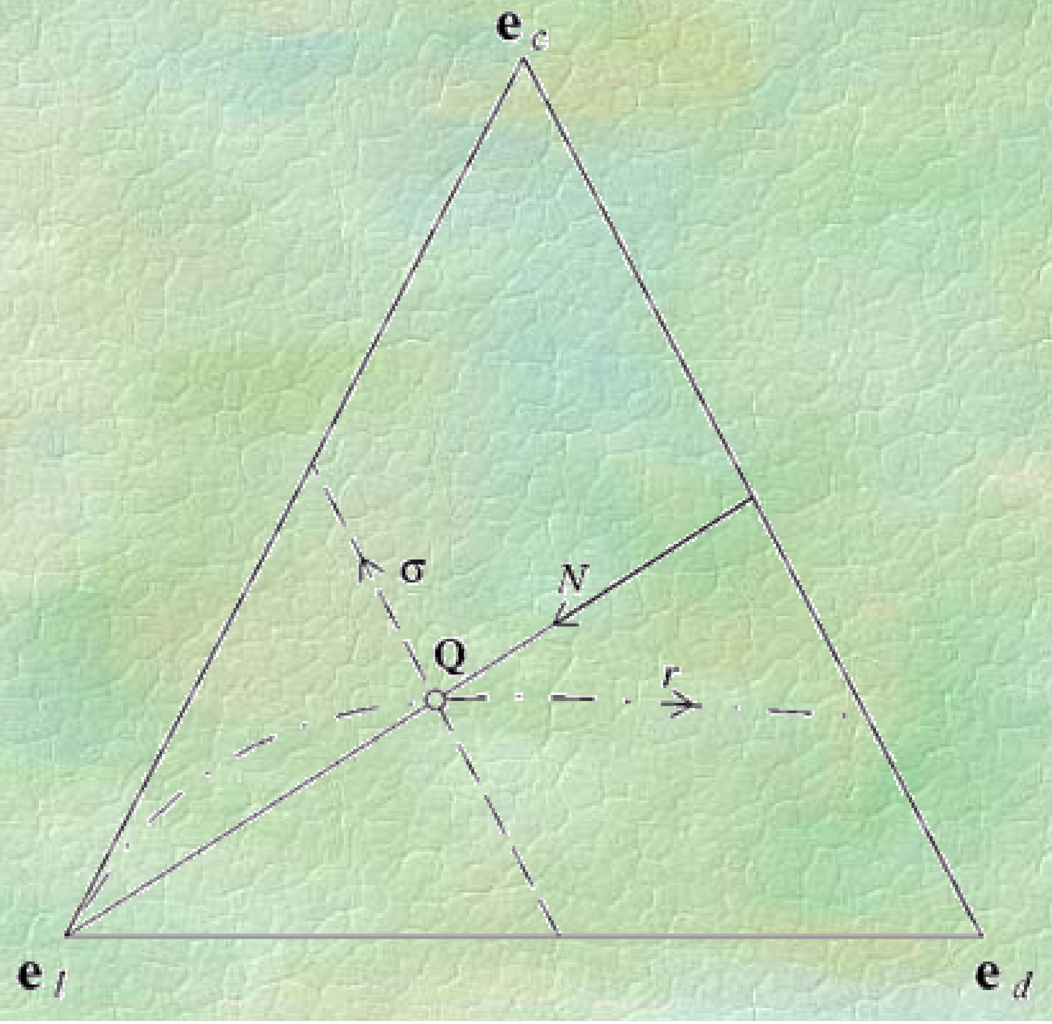
# Replicator dynamics




for $r > 2$,

closed orbits

# Interior equilibrium



# Time averages

time average of $u$ along orbit of period $T$

$$\bar{u} = \frac{1}{T}\int_0^T u\,dt$$

Then

$$\bar{f} = \frac{\sigma}{r-1}$$

and

$$\overline{P_x} = \overline{P_y} = \overline{P_z} = \sigma$$

# Red Queen Dynamics

# Best reply dynamics

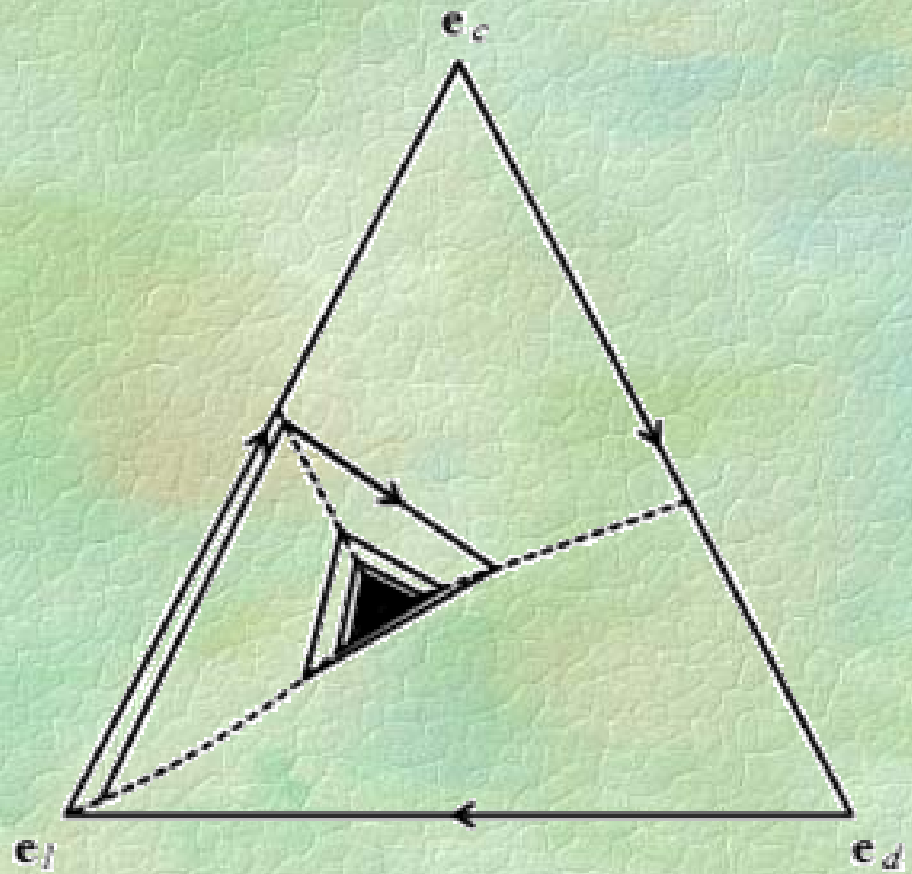
Players occasionally update,

choosing whichever strategy is currently optimal

($\Rightarrow$ rational players)

$$\dot{x} = BR(x) - x$$

# Best reply dynamics




adopt whatever is currently best strategy

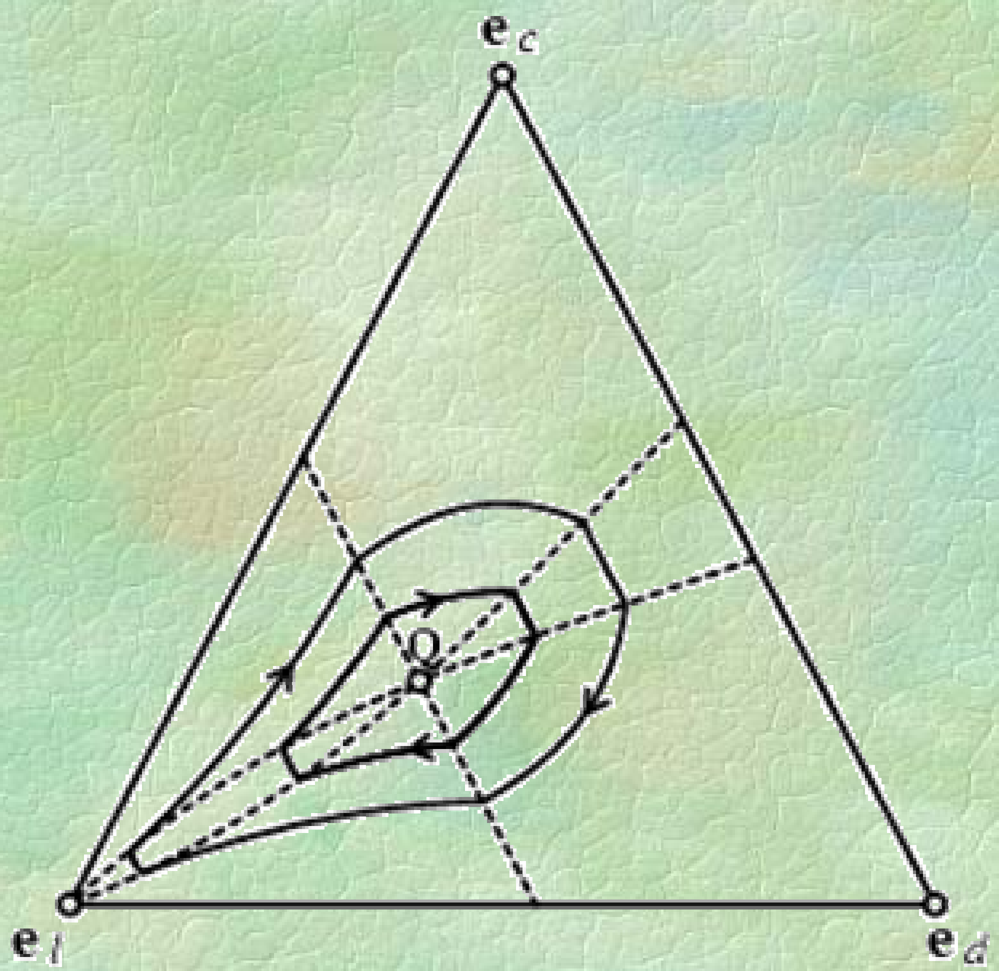
# Imitate the better

choose co-player randomly

adopt strategy whenever payoff higher

(discontinuous vector field)

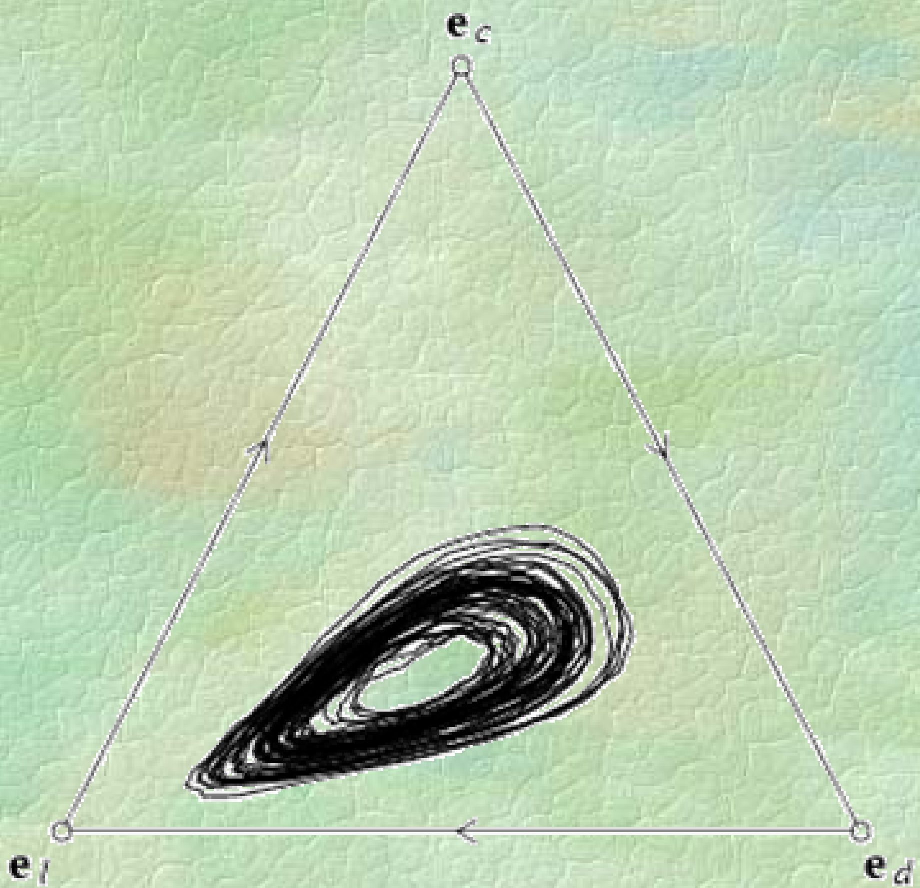
# Imitate the better




rank ordering of payoffs

six regions

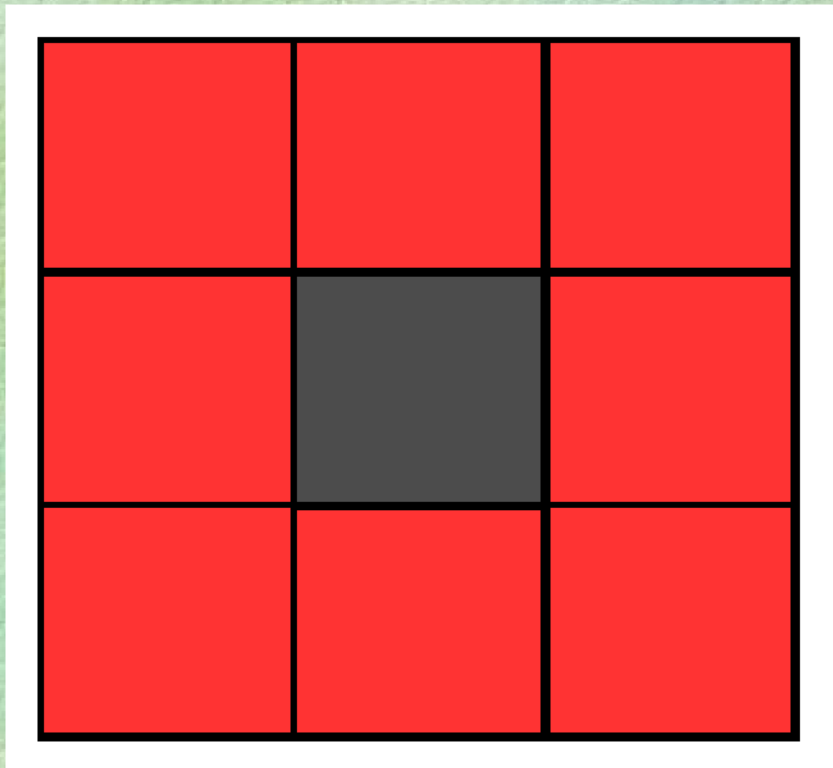
# Imitate the better




individual based simulations

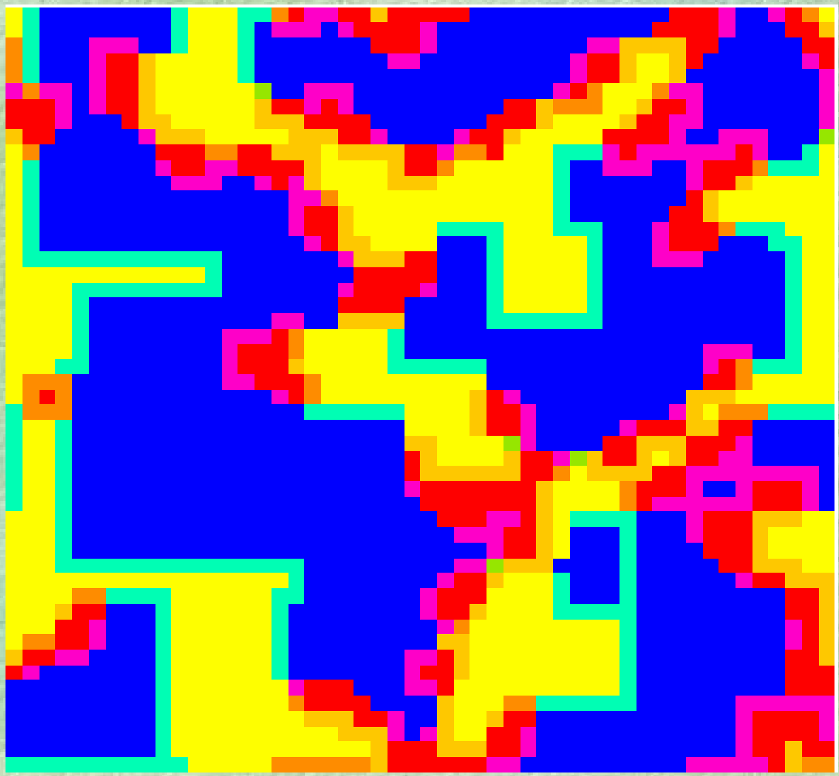
(population size 5000)

# Neighborhood structure



- Interaction with nearest neighbors
- best takes over

# best takes over



Red: defectors

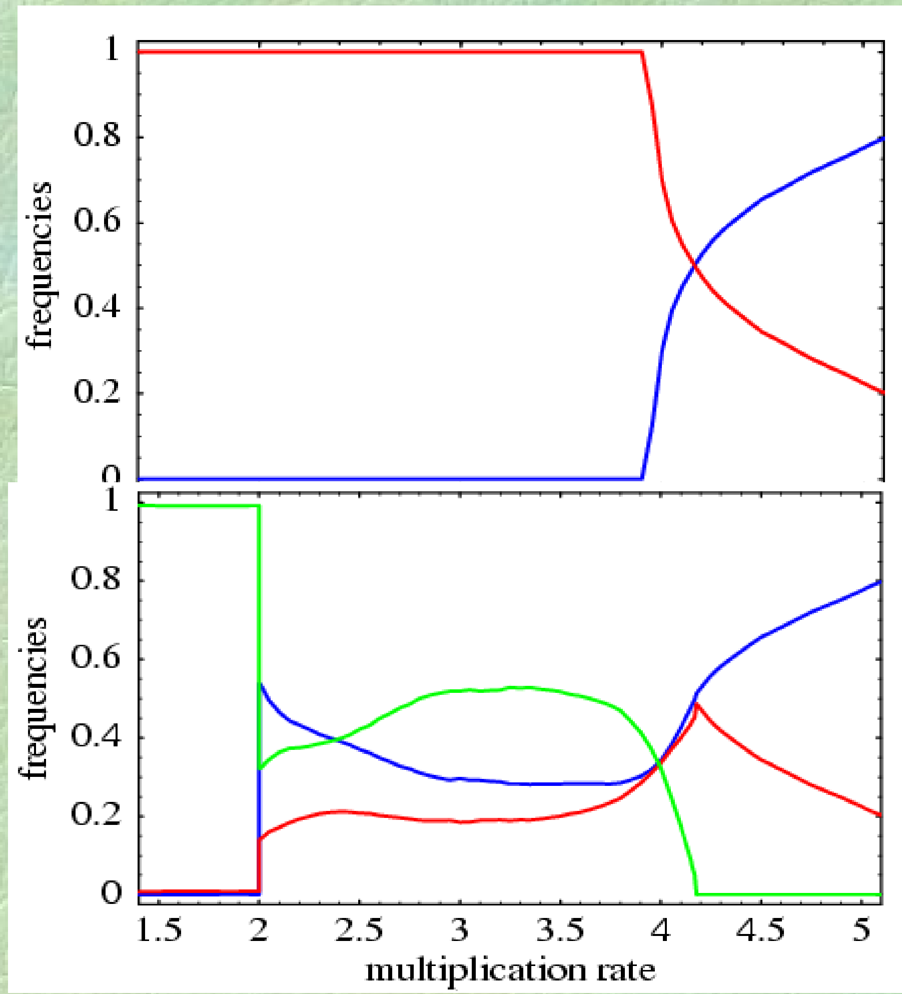
Blue: cooperators

Yellow: loners

# frequencies on the grid

- compulsory

loner´s option



# Morals?

More freedom yields more cooperation

Individuals that are less social
make better societies