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SMR.1573 - 9

SUMMER SCHOOL AND CONFERENCE ON DYNAMICAL SYSTEMS

Polynomial Diffeomorphisms of C^2

(Lecture 1)

John D. Smillie
Department of Mathematics
Cornell University
Ithaca
USA

Polynomial Diffeomorphisms of $\mathbb{C}^2$

① Surface diffeomorphisms in dynamical systems

② Connections between dynamics in one and two variables

③ Potential theory and dynamics

④ Entropy, reality and the boundary of the horseshoe locus.

⑤ A new approach to the horseshoe locus via puzzles

These talks are not surveys of complex dynamics in several complex variables. I will only talk about a small piece of this growing area. For a larger viewpoint you might consult the works of Nessim Sibony and his collaborators and students.

These talks are not a history of dynamical systems.

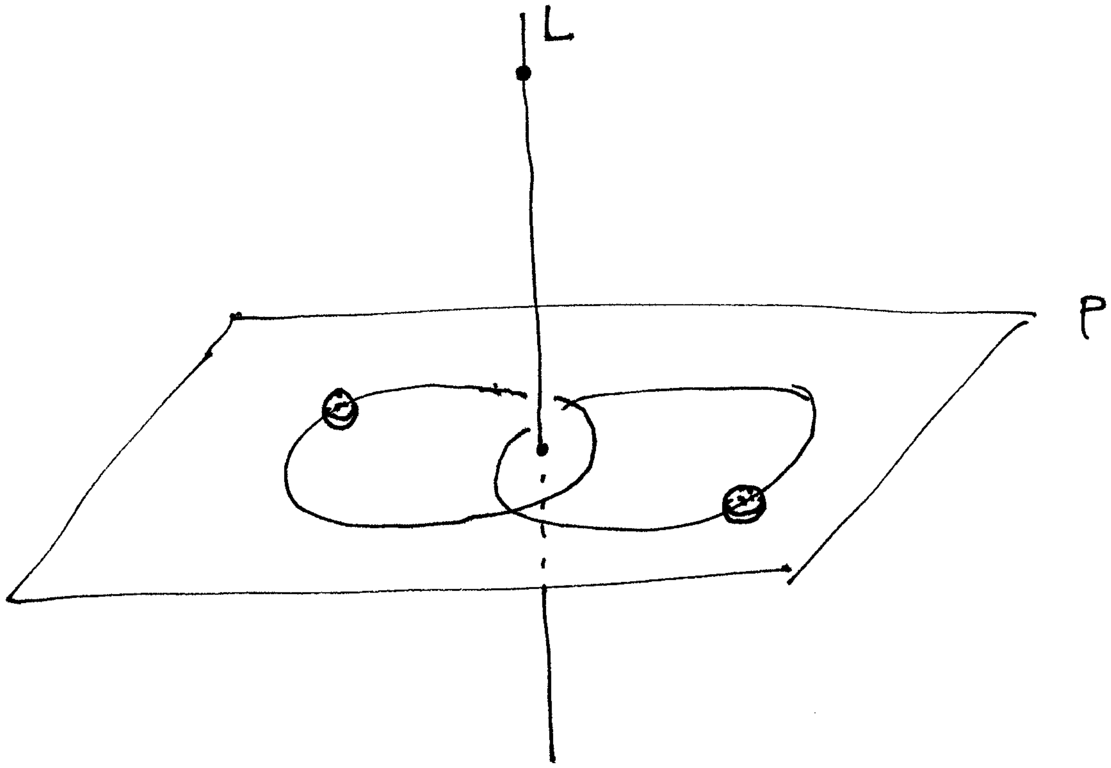
I will talk about the history of dynamical systems as a way of motivating and justifying certain problems but my history is skewed towards my particular interests.

What is a dynamical systems and how do you study it?

Instead of answering the question I will describe what the field looked like in 1973.

"Stable and Random Motions in Dynamical Systems"
by
Jurgen Moser
Princeton University Press 1973

Restricted 3-Body Problem



Two planets of equal mass move in symmetric elliptical orbits with eccentricity ε.

A sattelite of zero mass moves on a line L perpindicular to P.

The motion of the planets is periodic. Normalize the time scale so that the period is 2π.

Let $z(t)$ be the position of the sattelite.

Consider a solution $z(t)$ with infinitely many zeros

$$\ldots t_{-2}, t_{-1}, t_0, t_1, t_2 \ldots$$

Let

$$s_k = \left[\frac{t_{k+1} - t_k}{2\pi} \right]$$

Theorem. (Sitnikov) Given $\varepsilon > 0$ there exists an $m = m(\varepsilon)$ such that any sequence s with $s_k \geq m$ corresponds to a solution of the differential equation.

The statement can be sharpened to allow half-infinite sequences that start with ∞ (capture orbits) or that end with ∞ (escape orbits).

Note that the theorem reverses the typical analysis of an ordinary differential equation.

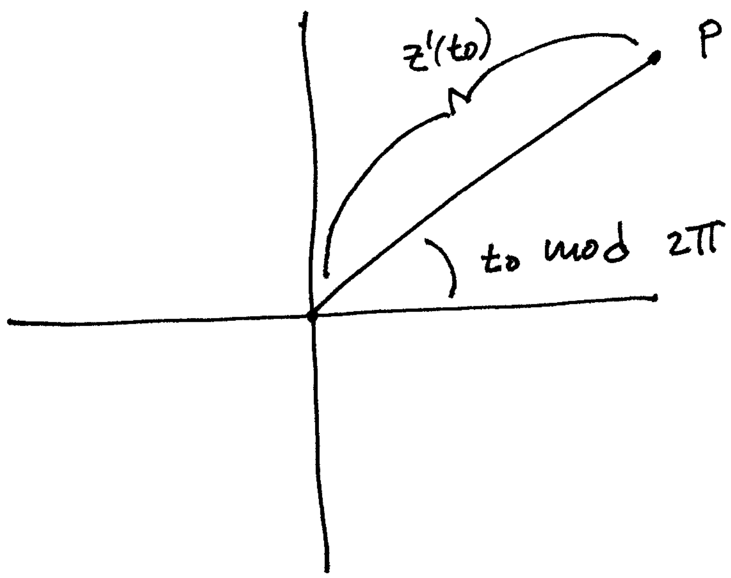
Rather than starting with an initial condition and giving the behavior of the corresponding solution it starts with a behavior and asserts the existence of the corresponding initial condition

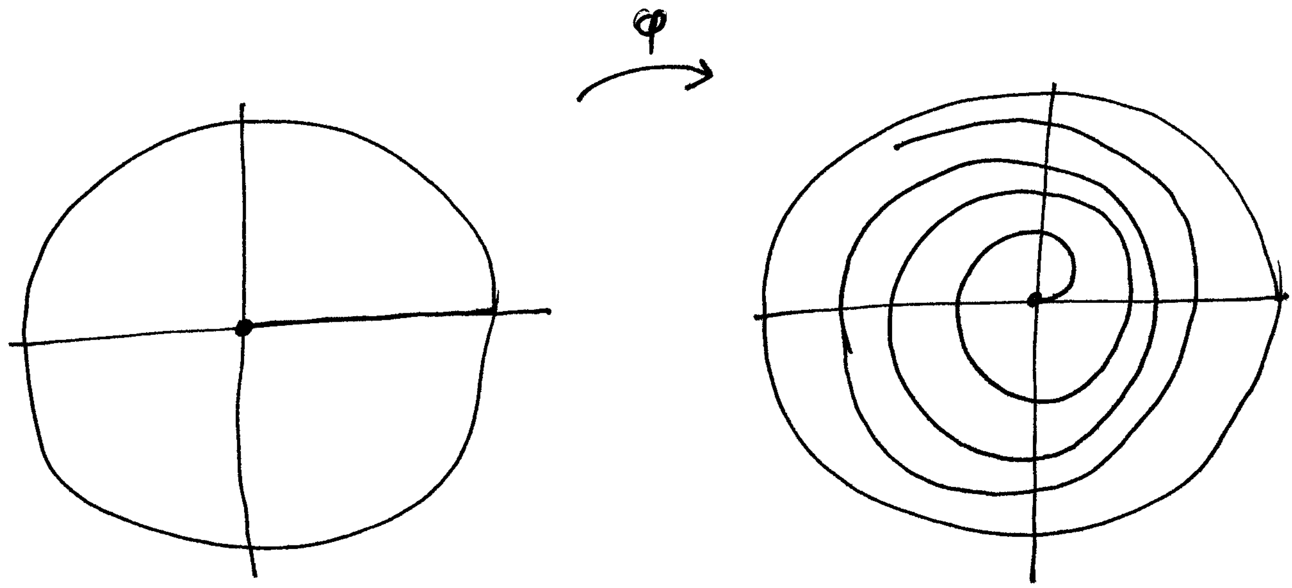
The first step of the proof is the reduction of the analysis of the differential equation to the analysis of a partially defined diffeomorphism of $\mathbb{R}^2$.

The state of the system when $z(t_0)=0$ is completely determined by the velocity of the sattelite $z'(t_0)$ and the position of the planets which is a function of $t_0 \bmod 2\pi$.

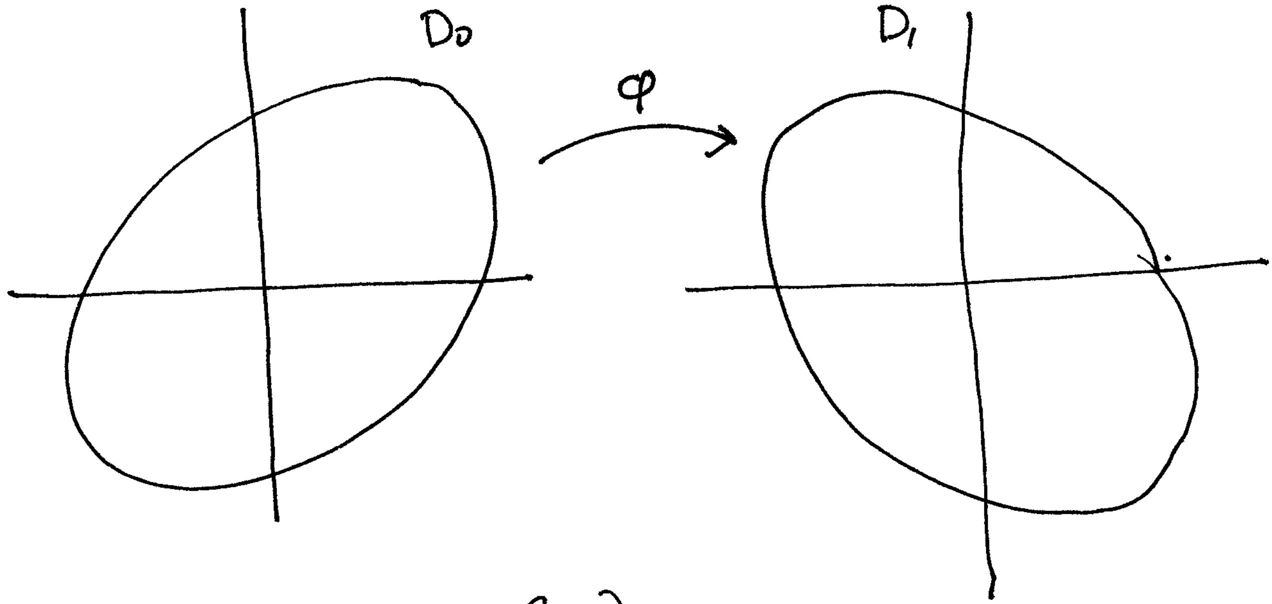
The reflection $z \mapsto -z$ leaves the system unchanged so we may assume that $z'(t_0) \geq 0$.

Interpret $(t_0, z'(t_0))$ as polar coordinates.

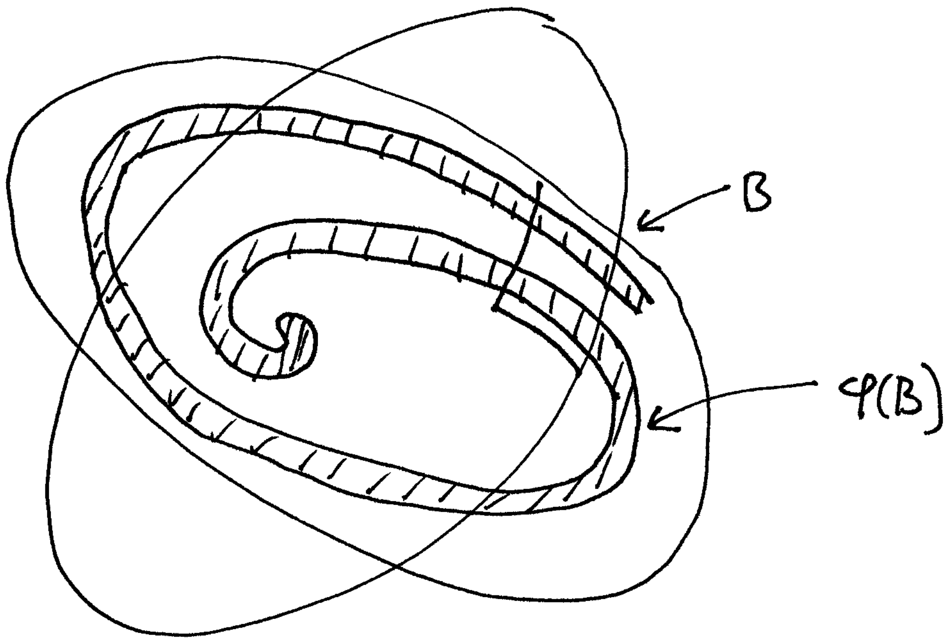




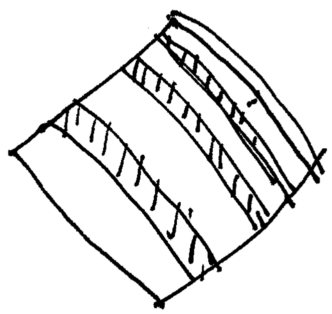
$\varepsilon = 0$



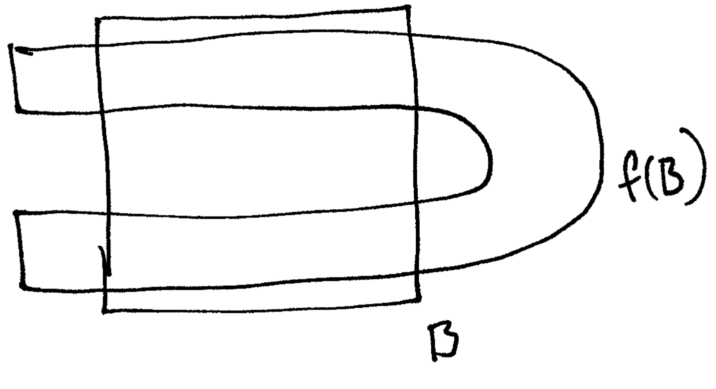
$\varepsilon > 0$

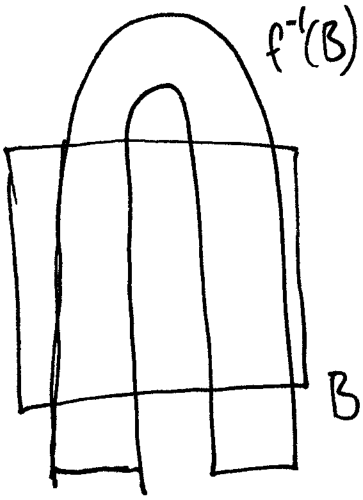


We have a countable collection of strips and a sequence $\underline{s}$ which prescribes the order in which these strips should be visited.



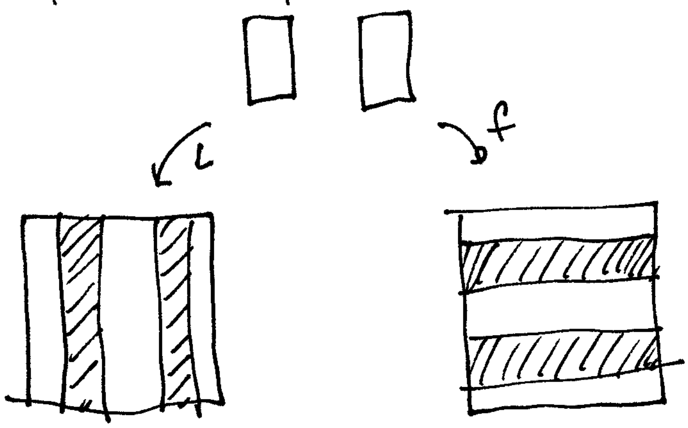
The same phenomenon occurs in a simpler context, the Smale horseshoe where there are just two strips.



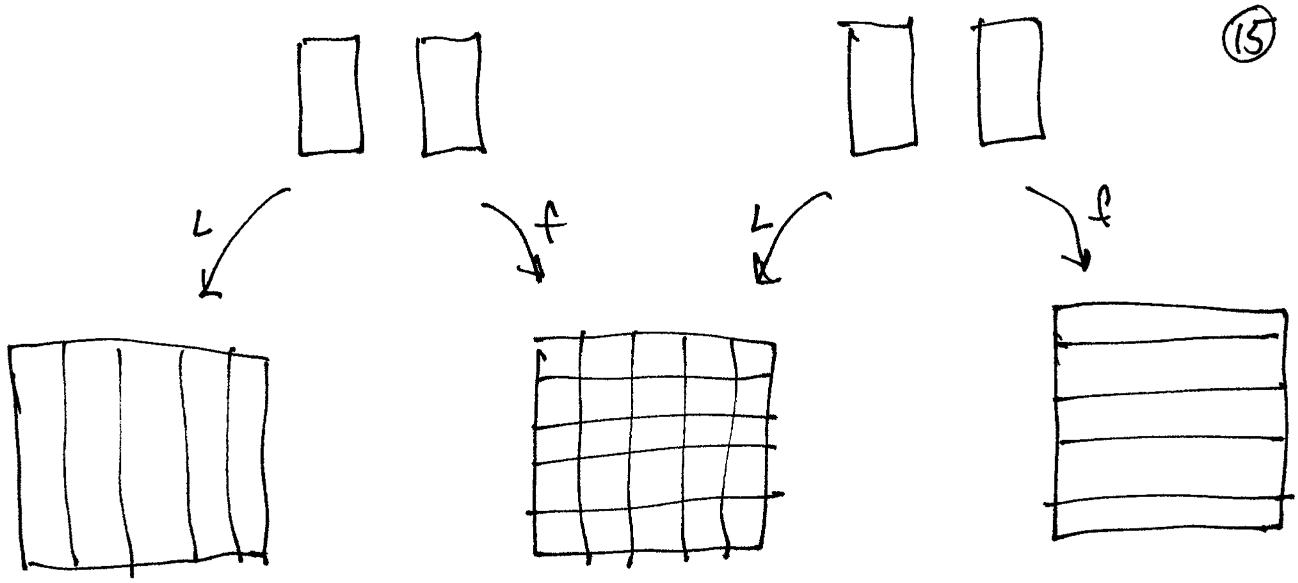


We can construct f so that once a point leaves B in forward time it never returns to B and goes to ∞ in forward time. We can similarly assume that once a point leaves B in backward time it never returns and goes to ∞ in backward time.

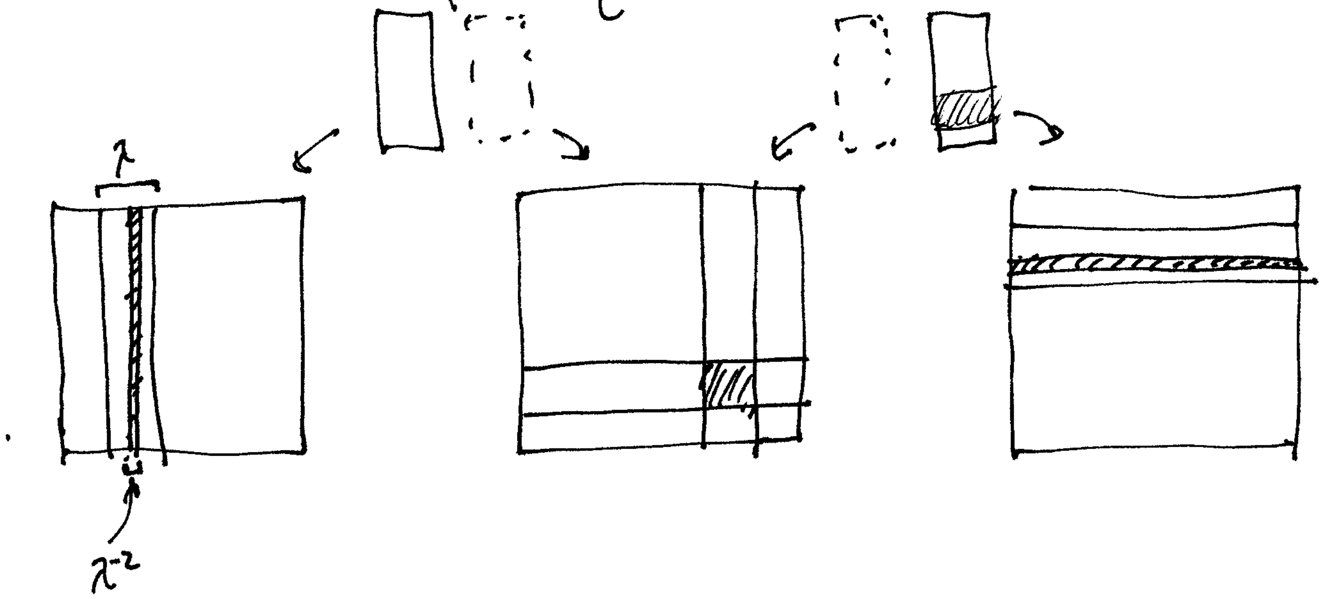
Alternatively we could assume that f is partially defined



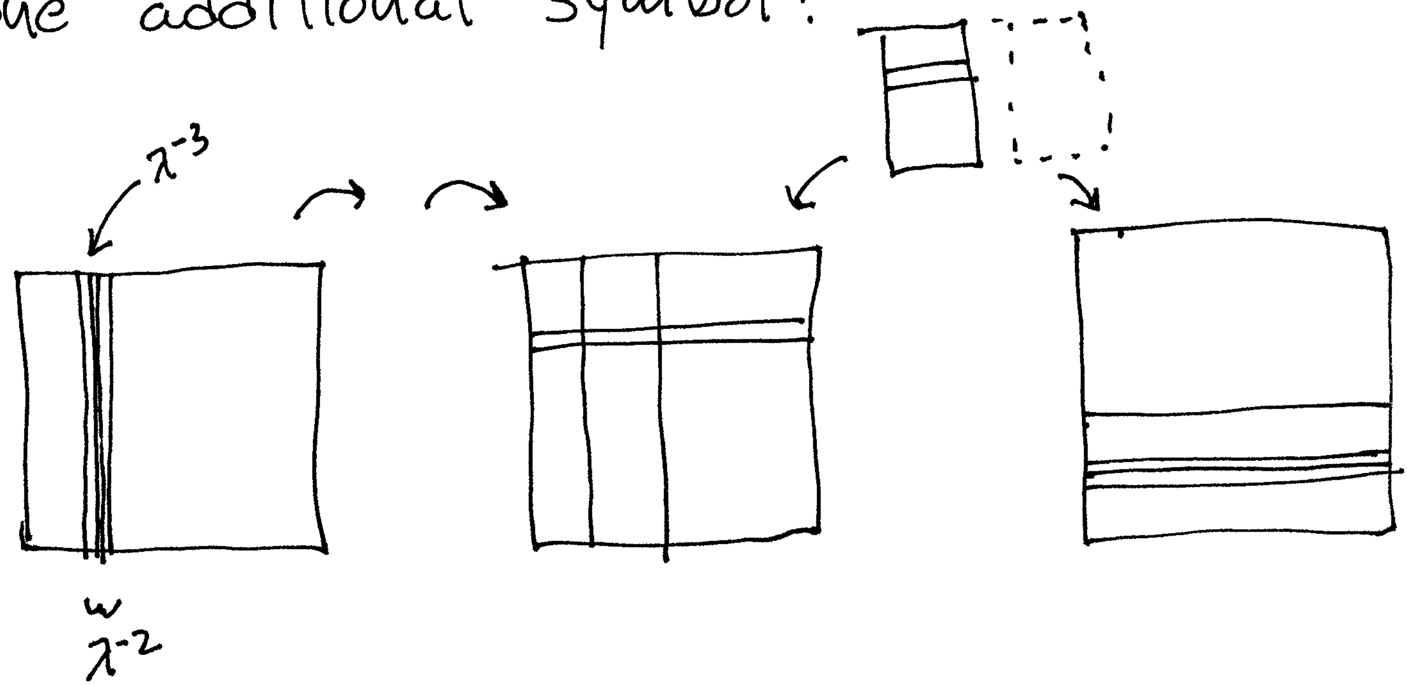
Assume ι is an inclusion and $Df = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{bmatrix}$ with $\lambda > 2$.



Prescribing transitions corresponds to picking one box in each pair. Consider the points that realize a particular 2 step sequence of transitions.



What happens if we prescribe one additional symbol?



Conclusion.

If we prescribe n symbols in the forward direction we get a vertical strip of width λ^{-n}.

If we prescribe ∞ many symbols in the forward direction we get a vertical segment.

If we prescribe n symbols in the backward direction we get a horizontal strip of height λ^{-n}.

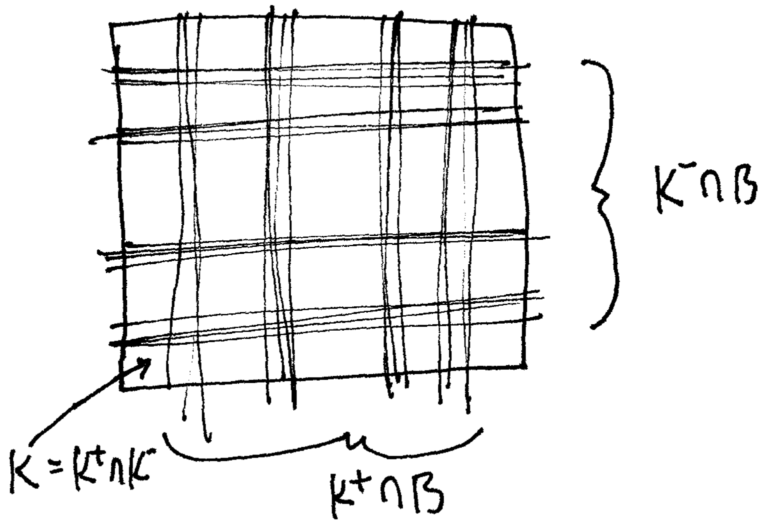
If we prescribe ∞ many symbols in the backward direction we get a horizontal segment.

If we prescribe ∞ many symbols in both directions we get a point.

Let $K^{\pm} = \{p : f^n(p) \not\to \infty \text{ as } n \to \pm\infty\}$.

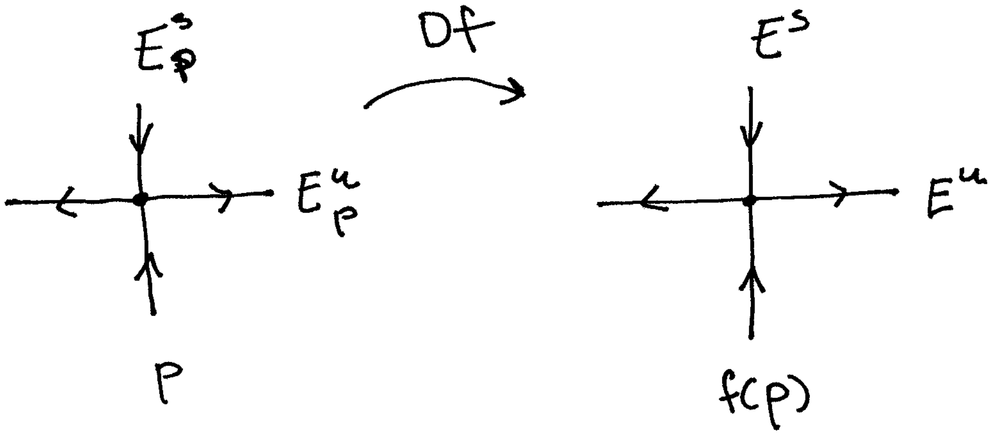
Let $K = K^+ \cap K^-$.

For the horseshoe we have

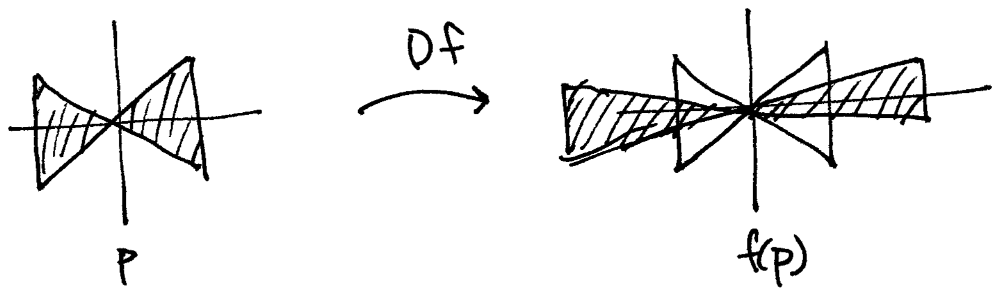


K is an <u>invariant</u> <u>set</u> in that $f(K) = K$.

An invariant set Λ is a <u>hyperbolic set</u> for f if we have a splitting of the tangent bundle $T_p = E^u_p \oplus E^s_p$ at each $p \in \Lambda$ so that $Df(E^u_p) = E^u_{f(p)}$, $Df(E^s_p) = E^s_{f(p)}$ and furthermore there are metrics on E^u and E^s so that $Df|E^u$ expands ~~dist~~ lengths uniformly and $Df|E^s$ contracts lengths uniformly.



In order to show that the restricted 3-body problem contains an invariant hyperbolic set one uses the method of cone fields

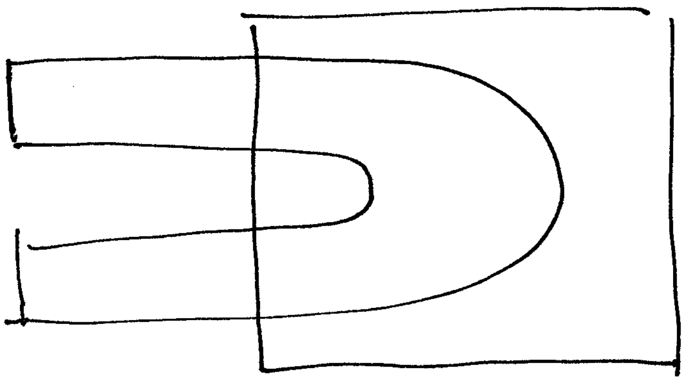


By iterating forward and intersecting the cones we can construct an invariant line field. If we assume that there is a metric on the vectors in the cone which is expanded then the invariant line field is expanded.

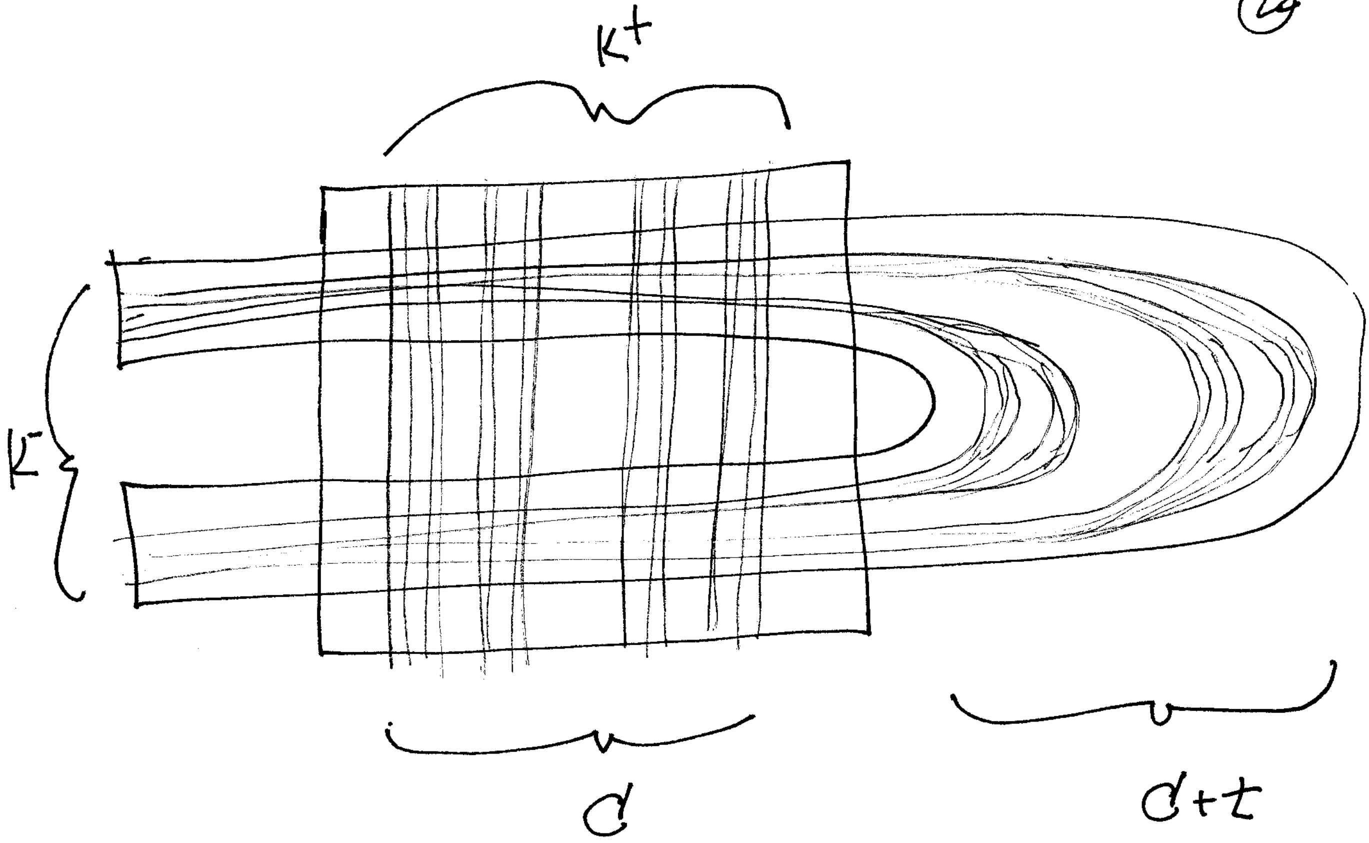
How often can diffeomorphisms of $\mathbb{R}^2$ be analyzed by these techniques?

We say a diffeomorphism is Axiom A if the "recurrent set" is a hyperbolic set. Axiom A diffeomorphisms are an open set in the space of diffeomorphisms. Are they dense?

Cartoon model of the degeneration of the horseshoe.



f_t folds the square and then translates by a distance t.
In real life the effect on K^+ and K^- can be dramatic.
In our cartoon model assume that K^+ is fixed and K^- is translated by t.



We observe tangencies of stable and unstable manifolds when

$$C \cap (C+t) \neq \emptyset$$

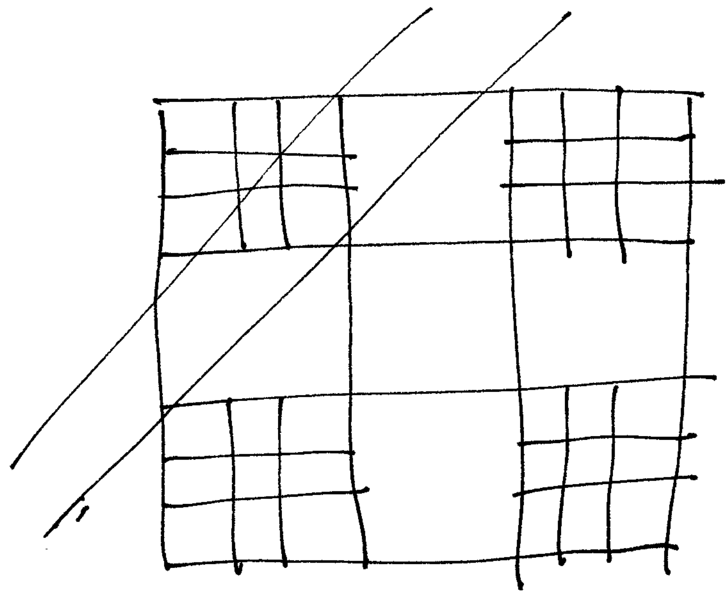
Equivalently

$$a = b+t \quad \text{for} \quad a, b \in C$$

or $$t = a-b$$

or $$t \in C - C.$$

The set $C - C$ is an interval.



$\pi: C \times C \to \mathbb{R}$
$\pi(a,b) = a - b.$

This suggests the following

Theorem. (Newhouse) Axiom A diffeomorphisms of $\mathbb{R}^2$ are not dense in the space of diffeomorphisms.

It would be useful to have a setting in which these questions could be investigated more carefully.

Next time: The Hénon family.