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SUMMER SCHOOL AND CONFERENCE ON DYNAMICAL SYSTEMS

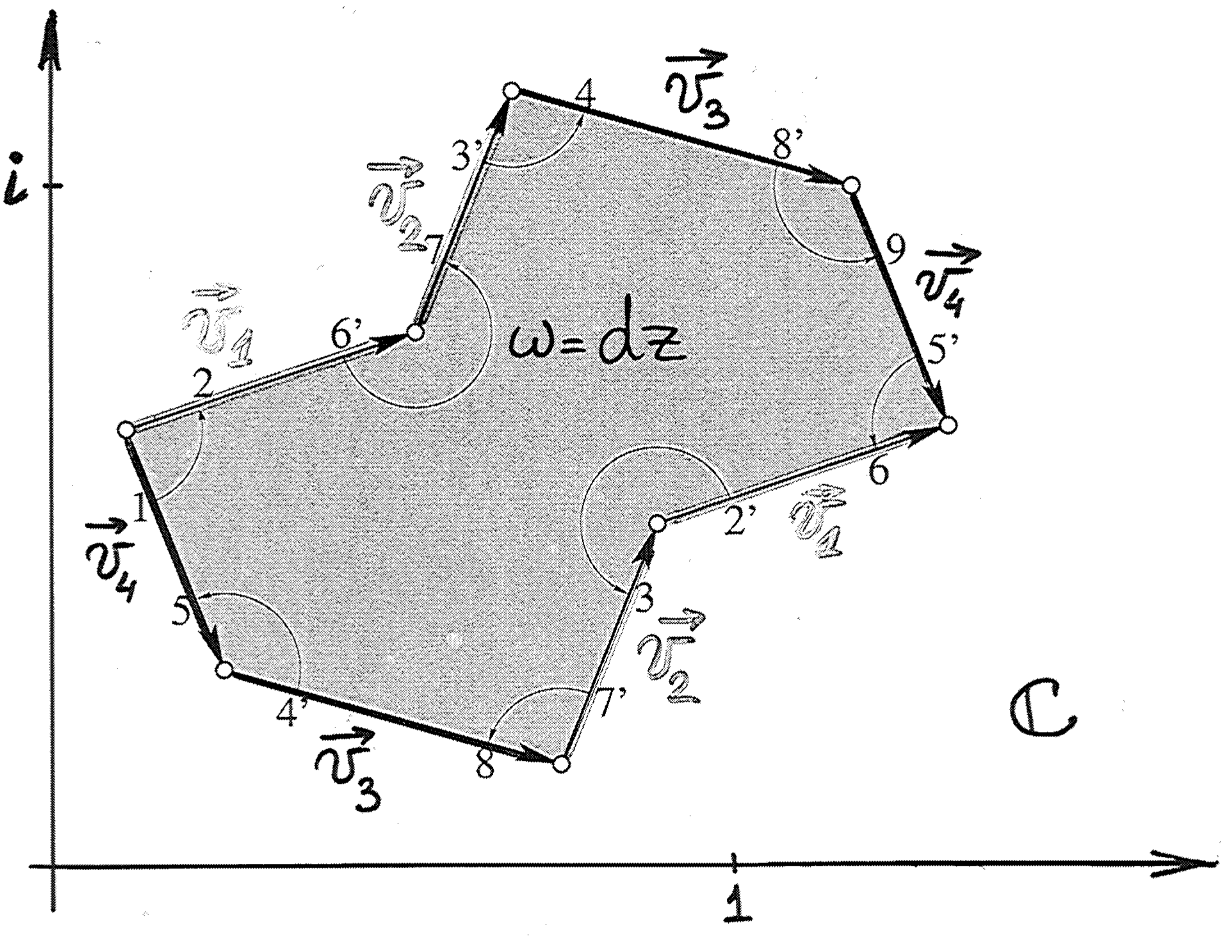
Flat surfaces, Interval exchange Transformations & Moduli Spaces of Abelian Differentials

(Lecture 4)

Anton Zorich
Universite de Rennes 1
I.R.M.A.R.
Rennes, France
& MPI, Bonn, Germany

These are preliminary lecture notes, intended only for distribution to participants

VERY FLAT (TRANSLATION) SURFACES

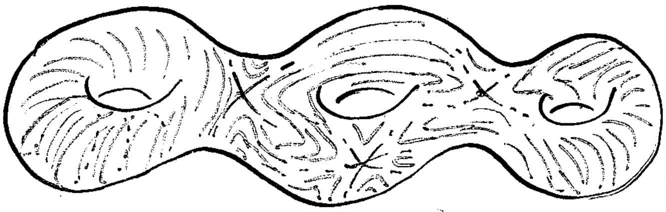


Closed geodesics on a surface of genus $g \geq 2$.

TYPICAL SITUATION: "irrational" direction

· Every trajectory is everywhere dense

(Actually, more: a.e. direction is uniquely ergodic (Kerckhoff, Masur, Smillie))

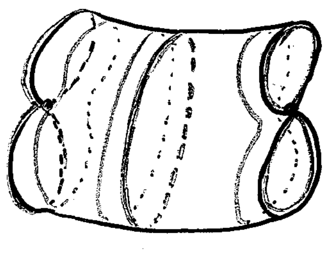


NONTYPICAL SITUATION: "rational" direction

There is a nonsingular closed geodesic γ_0 going in direction θ_0.

Trajectories going in the same direction θ_0 close to γ_0 are also closed.

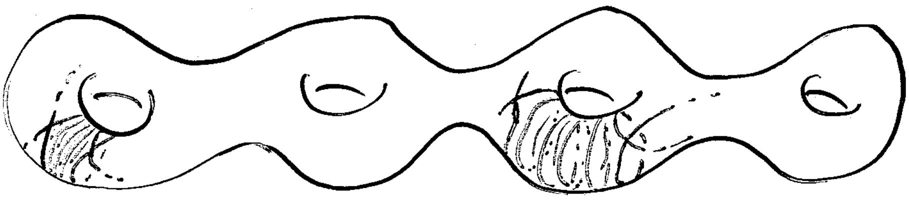
There is a whole cylinder of closed geodesics.



THEOREM. (A. Eskin, H. Masur (2001)) For a.e. translation surface in every connected component of any stratum

$$\begin{pmatrix}\text{the number of families of parallel closed} \\ \text{geodesics of length} \leq L \text{ on a surface of area } 1\end{pmatrix} \approx c \cdot \pi L^2$$

- How BIG is the chance to find another cylinder of closed geodesics
  - going in the same direction?
  - going in the same direction and having the same length?



- The chance is rather low!
- How low?
  - finately many cases?
  - $const \cdot L^{\alpha}$, then what is α?

<u>THEOREM</u> (A. Eskin, H. Masur, A. Zorich)

For the strata different from $\mathcal{H}(2g-2)$ and $\mathcal{H}(d_1, d_2)$

$$\left(\begin{array}{l}\text{number of two-cylinder families}\\ \text{of geodesics of length} \le L\end{array}\right) \approx c \cdot \pi L^2$$

where $c \neq 0$ is presented by an explicit formula.

THEOREM (A. Eskin, H. Masur, A. Zorich)

<... Explicit description of all typical (= = having quadratic asymptotics) configurations of closed geodesics and of saddle connections;

+ Explicit formula for the corresponding Siegel-Veech constants c in $c \cdot \pi L^2$ ...>

Example: Principal stratum $\mathcal{H}(1,\dots,1)$. Values of the constants c in $\dots \sim c\cdot\pi L^2$

	$g=1$	$g=2$	$g=3$	$g=4$
single cylinder	$\frac{1}{2}\cdot\frac{1}{\zeta(2)}\approx$ ≈ 0.304	$\frac{5}{2}\cdot\frac{1}{\zeta(2)}\approx$ ≈ 1.52	$\frac{36}{7}\frac{1}{\zeta(2)}\approx 3.13$ 96%	$\frac{3150}{377}\cdot\frac{1}{\zeta(2)}\approx 5.08$ 97,1%
two cylinders	—	—	$\frac{3}{14}\cdot\frac{1}{\zeta(2)}\approx 0.13$ 4%	$\frac{90}{377}\cdot\frac{1}{\zeta(2)}\approx 0.145$ 2.8%
three cylinders	—	—	—	$\frac{5}{754}\cdot\frac{1}{\zeta(2)}\approx 0.00403$ 0.1%

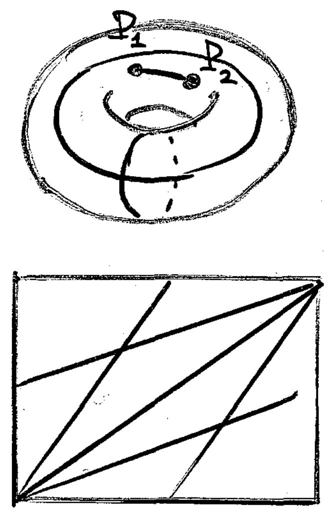
$$
\begin{aligned}
c = \frac{1}{|\Gamma_-|}\cdot\frac{1}{|\Gamma|}\cdot\prod_{m\in\alpha}\left(\frac{o(m)!}{\prod_{i=1}^{p} o_i(m)!}\right)\cdot \\
\cdot\prod_{\substack{1\le i\le p\\ z_i=w_i}} o_i(a_i)\cdot\prod_{\substack{1\le k\le p\\ z_k\neq w_k\\ b'_k\neq b''_k}} o_k(b'_k)o_k(b''_k)\cdot\prod_{\substack{1\le k\le p\\ z_k\neq w_k\\ b'_k= b''_k}} o_k(b'_k)(o_k(b'_k)-1)\cdot \\
\cdot\prod_{\substack{1\le i\le p\\ z_i=w_i}}(a_i+1)\cdot\prod_{\substack{1\le k\le p\\ z_k\neq w_k}}(b'_k+1)(b''_k+1)\cdot \\
\cdot\frac{1}{2^{p-1}}\cdot\frac{\prod_{i=1}^{p}(\frac{d_i}{2}-1)!}{(\frac{d}{2}-2)!}\cdot\frac{\prod_{i=1}^{p}\mathrm{Vol}(\mathcal{H}_1(\alpha'_i))}{\mathrm{Vol}(\mathcal{H}_1(\alpha))}
\end{aligned}
$$

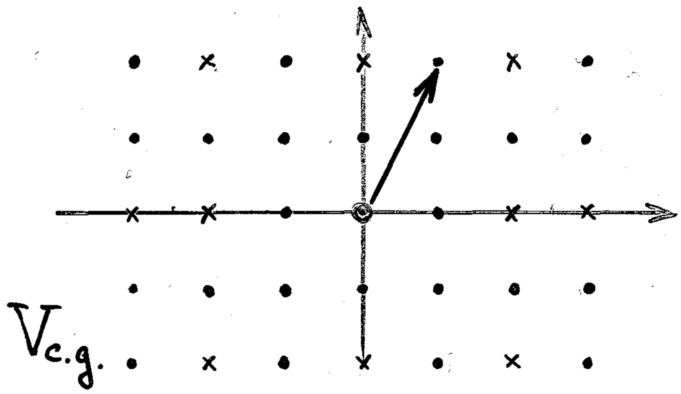
$$
c = \lim_{\varepsilon\to 0}\frac{1}{\pi\varepsilon^2}\,\frac{\mathrm{Vol}(\text{"}\varepsilon\text{-neighborhood of the cusp"})}{\mathrm{Vol}(\mathcal{H}_1(\alpha))}
$$

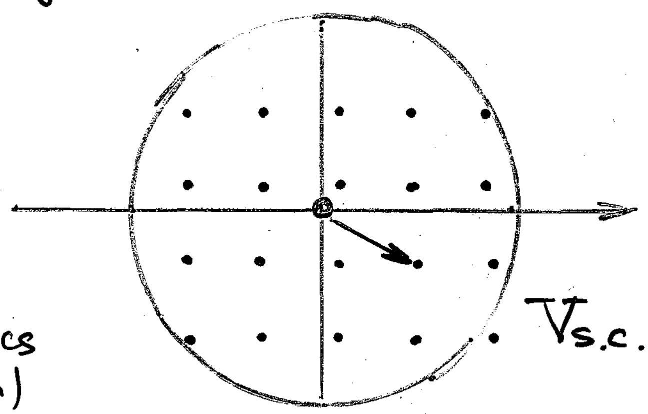
Counting closed geodesics and saddle connections in a model case of the torus.

$N_{c.g.}$:= number of (types of) regular closed geodesics of length $\leq L$

$N_{s.c.}$:= number of "saddle connections" (joining fixed points P_1 and P_2) of length $\leq L$







$V_{c.g.} \subset \mathbb{R}^2$ discrete set representing closed geodesics
(geodesci $\rightsquigarrow$ direction + length)

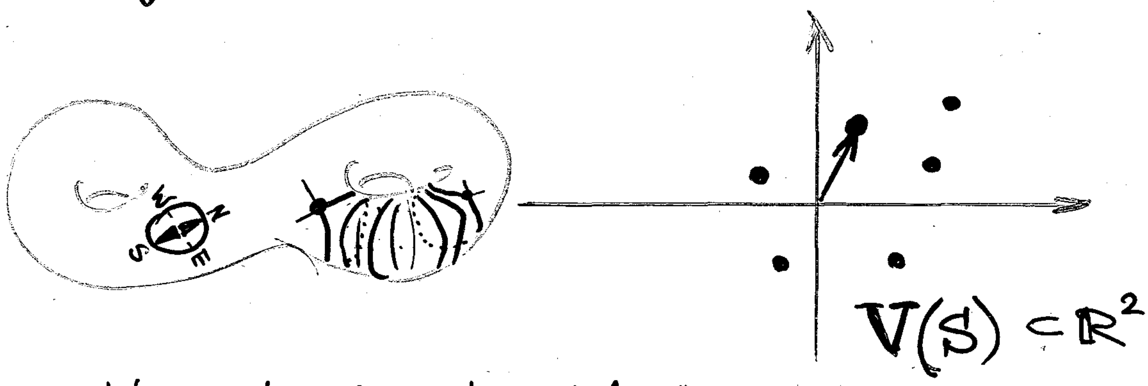
$V_{s.c.} \subset \mathbb{R}^2$ discrete set representing "saddle connections"

$$\lim_{L\to+\infty} \frac{\operatorname{card}(V \cap D_L^2)}{\pi L^2} = \text{const}$$

For closed geodesics $\quad$ const $= \dfrac{1}{\zeta(2)}$

For saddle connections $\quad$ const $= 1$

Siegel-Veech formula.



REMARK: We count only "straight line" simple geodesics or saddle connections and not the "broken line" ones.

Let f be a function on $\mathbb{R}^2$ with compact sup.

$$f \mapsto \hat{f}(S) = \sum_{\vec{v} \in V(S)} f(\vec{v})$$

$\hat{f}$ is now a function on the moduli space

LEMMA (VEECH) Functional

$$f \longmapsto \frac{1}{\operatorname{Vol} \mathcal{H}_1(k_1 \dots k_n)} \int_{\mathcal{H}_1(k_1 \dots k_n)} \hat{f}(S) \text{ "}dS\text{"}$$

is $SL(2;\mathbb{R})$ - invariant.

PROOF: $\widehat{(gf)}(S) = \hat{f}(g^{-1}S) \qquad g \in SL(2;\mathbb{R})$

since $V(gS) = g \circ V(S)$ ▨

COROLLARY. (VEECH)

$$\frac{1}{\operatorname{Vol} \mathcal{H}_1(k_1 \dots k_n)} \cdot \int_{\mathcal{H}_1(k_1 \dots k_n)} \hat{f}(S)\, dS = \text{Const} \cdot \int_{\mathbb{R}^2} f(x,y)\, dx\, dy,$$

where const <u>does not depend on</u> f!

EXAMPLE 1. f_L = char. function of a disk of rad. L

$$\hat{f}_L(S) = \operatorname{card}\left(V(S) \cap D_L^2\right) =$$

$= \#$ of closed geodesics (saddle connections) of length $\leq L$ on S

$$\frac{1}{\operatorname{Vol}\mathcal{H}_1} \int_{\mathcal{H}_1} \hat{f}_L(S)\, dS = \text{const} \cdot \int_{\mathbb{R}^2} f_L \, dx dy = \text{const} \cdot \pi L^2$$

THEOREM (A. ESKIN & H. MASUR)

For almost every surface S in $\mathcal{H}_1(k_1 \dots k_n)$

$$\lim_{L \to +\infty} \frac{\hat{f}_L(S)}{\pi L^2} = \text{const}$$

How can one compute const?

EXAMPLE 2. Let $L := \varepsilon \ll 1$. Then

$$\hat{f}_\varepsilon(S) = \begin{cases} 0 & \text{usually} \\ 1 & \left.\begin{array}{l}\text{if there is exactly one} \\ \text{closed geod. (sad. conn., ...) on } S\end{array}\right\} \mathcal{H}_1^{\varepsilon,\text{thick}} \\ >1 & \left.\begin{array}{l}\text{if there are several short} \\ \text{geodesics (sad. conn., ...) on } S\end{array}\right\} \mathcal{H}_1^{\varepsilon,\text{thin}} \end{cases}$$

$$\text{const} \cdot (\pi \varepsilon^2) = \text{const} \cdot \int_{\mathbb{R}^2} f_\varepsilon \, dx dy = \frac{1}{\operatorname{Vol}\mathcal{H}_1} \int_{\mathcal{H}_1} \hat{f}_\varepsilon(S)\, dS =$$

$$= \underbrace{\int_{\mathcal{H}_1^{\varepsilon,\text{thick}}} 1 \cdot dS}_{\operatorname{Vol}\mathcal{H}_1^{\varepsilon,\text{thick}}} + \underbrace{\int_{\mathcal{H}_1^{\varepsilon,\text{thin}}} \hat{f}_\varepsilon(S)\, dS}_{o(\varepsilon^2) \text{ (Eskin - Masur)}}$$

COROLLARY

$$\boxed{\text{const} = \lim_{\varepsilon \to 0} \frac{1}{\pi \varepsilon^2} \frac{\operatorname{Vol}\mathcal{H}_1^{\varepsilon}(k_1 \dots k_n)}{\operatorname{Vol}\mathcal{H}_1(k_1 \dots k_n)}}$$

Volume of a Neighborhood of a Cusp

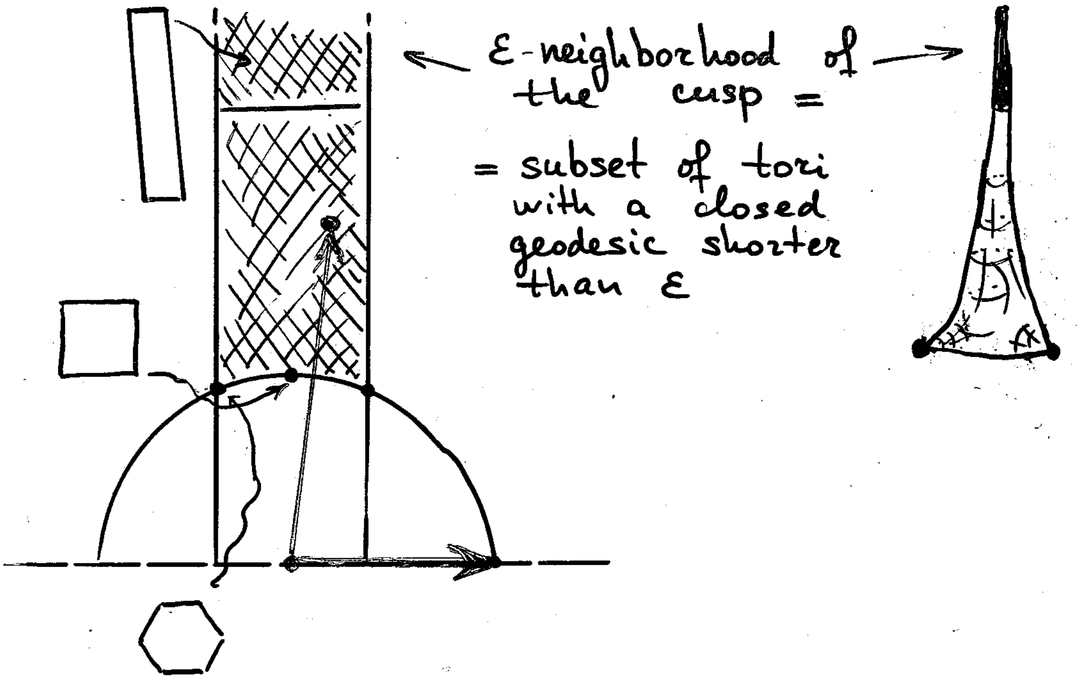
Theorem (H. Masur, J. Smillie). *There is a constant M such that for all $\epsilon, \kappa > 0$ the subset of $\mathcal{H}_1(\alpha, \mathcal{C})$ consisting of those flat surfaces, which have a saddle connection of length at most ϵ, has volume at most $M\epsilon^2$. The volume of the set of flat surfaces with a saddle connection of length at most ϵ and a nonhomologous saddle connection with length at most κ is at most $M\epsilon^2\kappa^2$.*

$$\mathcal{H}_1^{\epsilon}(\alpha, \mathcal{C}) = \mathcal{H}_1^{\epsilon,thick}(\alpha, \mathcal{C}) \sqcup \mathcal{H}_1^{\epsilon,thin}(\alpha, \mathcal{C}).$$

The *thick part* $\mathcal{H}_1^{\epsilon,thick}(\alpha, \mathcal{C})$ consists of surfaces S having *exactly one* saddle connection shorter than ϵ.

The *thin part* $\mathcal{H}_1^{\epsilon,thin}(\alpha, \mathcal{C})$ consists of surfaces S having at least one short saddle connection γ as above and at least one other short saddle connection β nonhomologous to γ.

Toy example: genus $g=1$



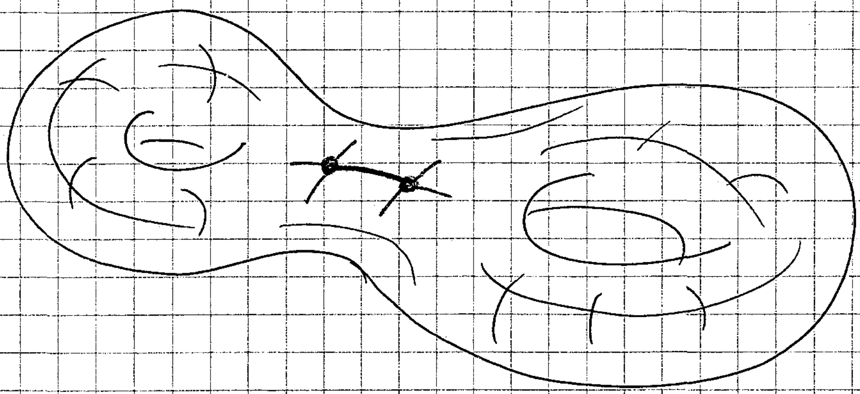
$$\cong \mathcal{H}(\emptyset)/\mathbb{C}^*$$

$$\text{const} = \lim_{\varepsilon\to 0} \frac{1}{\pi\varepsilon^2} \frac{\operatorname{Vol}\left(\begin{matrix}\text{tubular } \varepsilon\text{-neighborhood}\\ \text{of the cusp}\end{matrix}\right)}{\operatorname{Vol}\mathcal{H}(\emptyset)} =$$

$$= \lim_{\varepsilon\to 0} \frac{1}{\pi\varepsilon^2}\cdot \frac{\left(\begin{matrix}\text{Area of the tubular}\\ \varepsilon\text{-neighborhood of the cusp}\end{matrix}\right)}{\text{Area of the modular surface}} \cdot \frac{\left(\begin{matrix}\text{Length of the}\\ \text{circle } SO(2)/\pm 1\end{matrix}\right)}{\left(\begin{matrix}\text{Length of the}\\ \text{circle } SO(2)/\pm 1\end{matrix}\right)} =$$

$$= \lim_{\varepsilon\to 0} \frac{1}{\pi\varepsilon^2}\cdot \frac{\varepsilon^2}{\frac{\pi}{3}} \cdot \frac{\pi}{\pi} = \frac{3}{\pi^2} = \frac{1}{2}\cdot\frac{1}{\zeta(2)}$$

STRUCTURE OF A CUSP CORRESPONDING TO A SHORT SADDLE CONNECTION



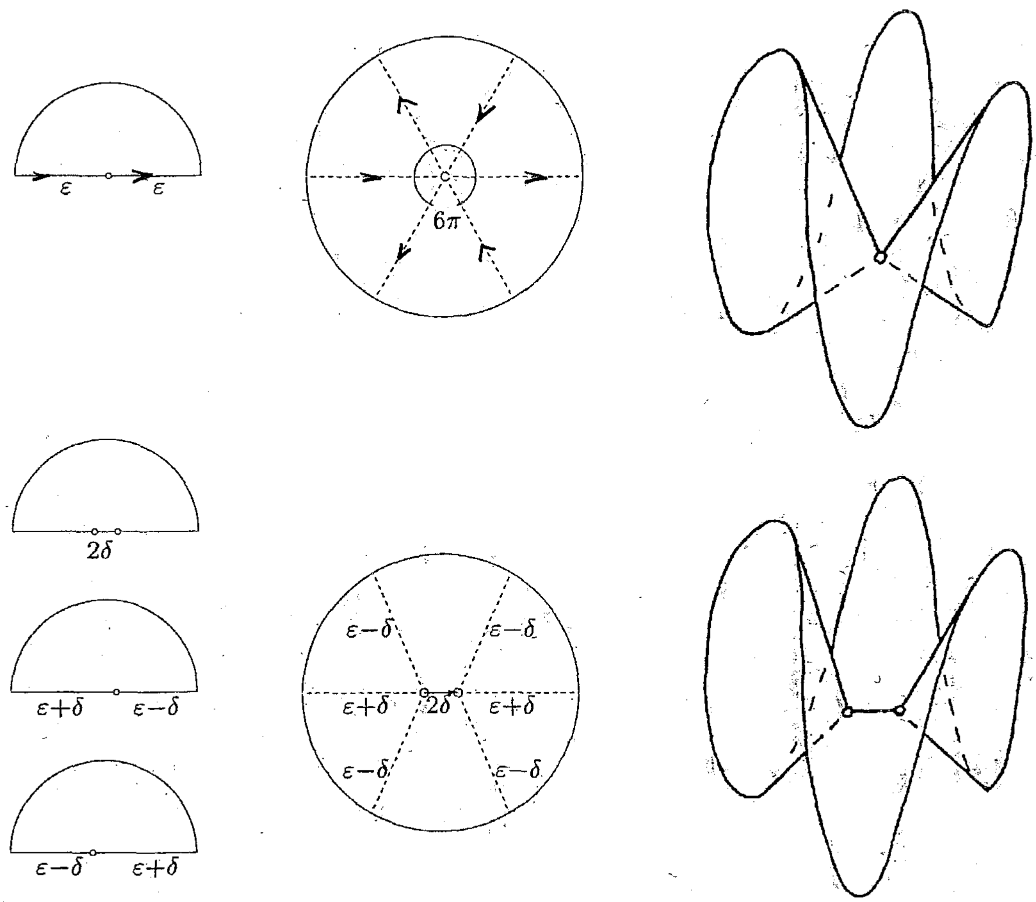
We are going to show that there is a canonical way to shrink the saddle connection on a flat surface $S \in \mathcal{H}_1^{\varepsilon,\,thick}(k_1, k_2, \ldots)$ coalescing two conical points into one.

We get "almost fiber bundle"

$$\begin{array}{c} \mathcal{H}_1^{\varepsilon,\,thick}(k_1, k_2, k_3, \ldots, k_n) \\ \Big\downarrow \tilde{\mathcal{D}}_\varepsilon^2 \\ \mathcal{H}_1(k_1 + k_2, k_3, \ldots, k_n) \end{array}$$

where $\tilde{\mathcal{D}}_\varepsilon^2$ is a (k_1+k_2+1) (ramified) cover over a disk of radius ε, and the measure desintegrates to a product measure.

Breaking up a Zero

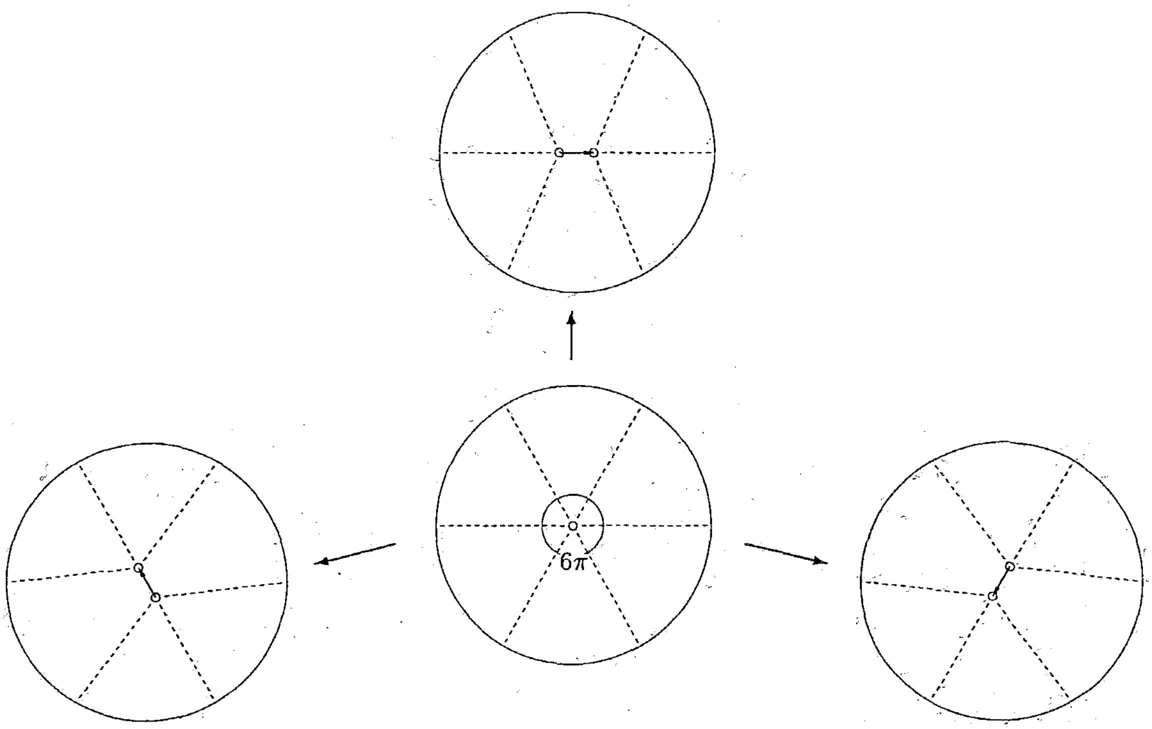


Parameters of surgery: length of the vector and angle $\varphi \in [0, 2\pi(k+1)[$. *The space of parameters is a $(k_1 + k_2 + 1)$-cover over a punctured disk.*

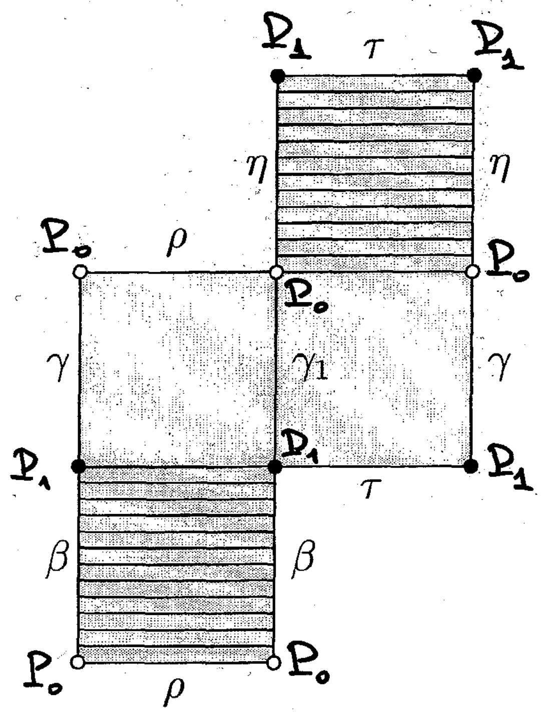
$$\mathcal{H}_1^{\varepsilon,\text{thick}}(k_1, k_2, \dots, k_n) = \mathcal{H}_1(k_1+k_2, k_3, \dots, k_n) \times \widetilde{\mathcal{D}}^2_\varepsilon$$

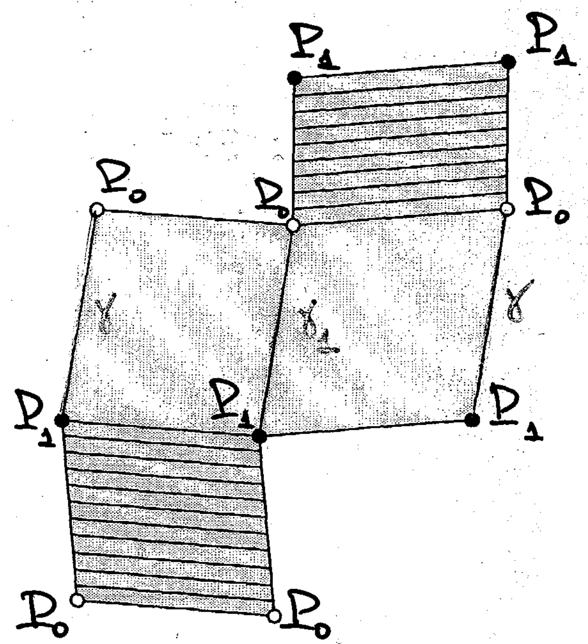
$$\operatorname{Vol} \mathcal{H}_1^{\varepsilon,\text{thick}}(k_1, k_2, \dots, k_n) =$$

$$= (k_1 + k_2 + 1) \cdot \pi \varepsilon^2 \cdot \operatorname{Vol} \mathcal{H}_1(k_1+k_2, k_3, \dots, k_n)$$



The cone angle corresponding to a zero of order $m = 2$ is equal to $(m+1) \cdot 2\pi = 6\pi$. Thus we have $(m+1) = 3$ different ways of breaking up a zero of order 2 in a direction $\vec{\gamma}$. In this way (generically) we get $m = 3$ different flat surfaces.

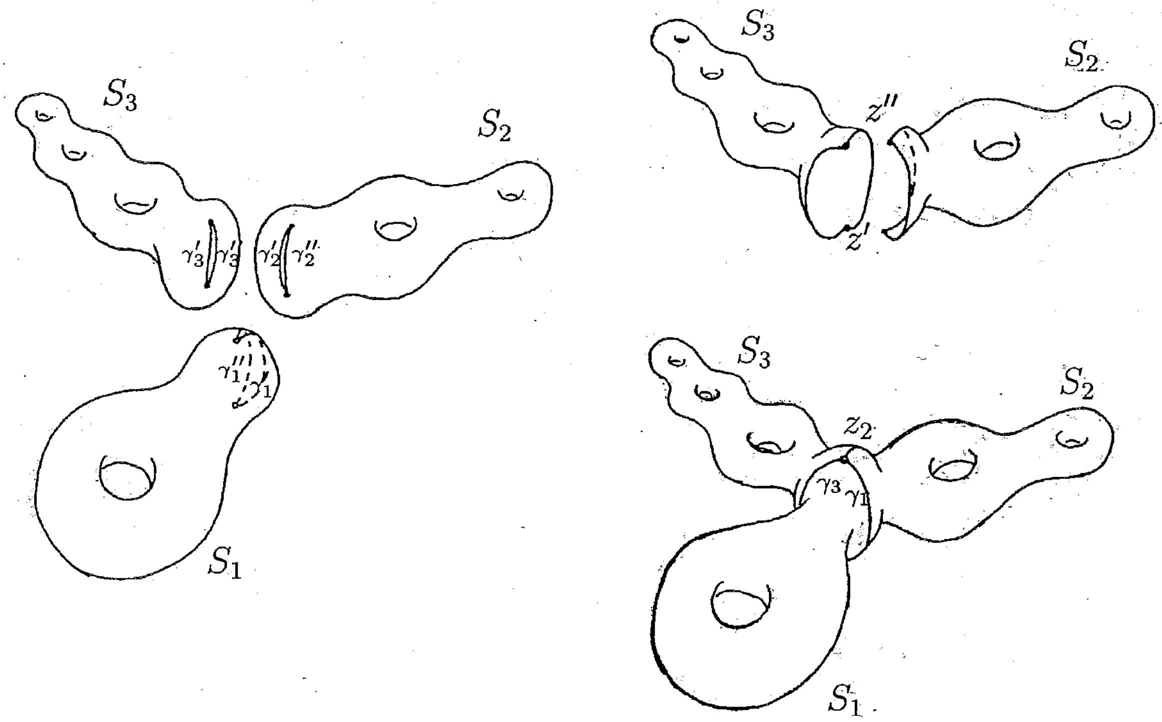




Nonhomologous saddle connections which have the same holonomy lose this property after a generic deformation of the surface, while homologous ones keep the same holonomy.

$$[\gamma]=[\gamma_1]\in H_1\left(M^2,\{P_0,P_1\};\mathbb{Z}\right)$$

(If you cut the surface M^2 along γ and γ_1 you will break it into two connected components.)



Multiple homologous saddle connections.

We can chose as a basis of cycles in relative homology group $H_1(M, \{\text{conical points}\}; \mathbb{Z})$ of the compound surface a basis

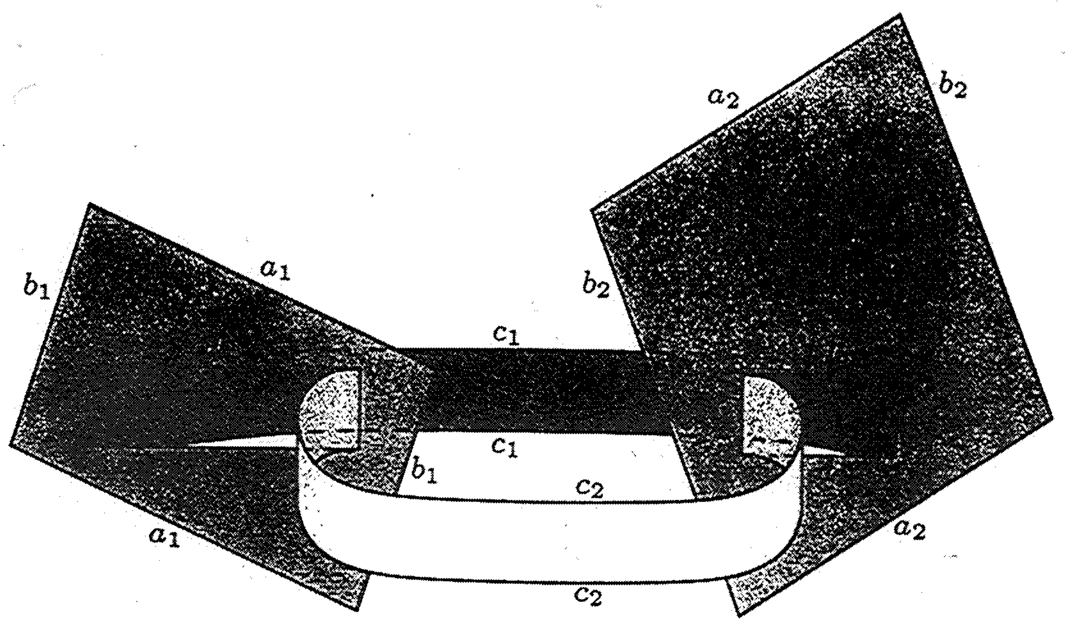
$$\begin{pmatrix}\text{basis of cycles}\\ \text{on closed surface}\\ S_1\end{pmatrix} \sqcup \cdots \sqcup \begin{pmatrix}\text{basis of cycles}\\ \text{on the closed}\\ \text{surface } S_K\end{pmatrix} \sqcup \gamma$$

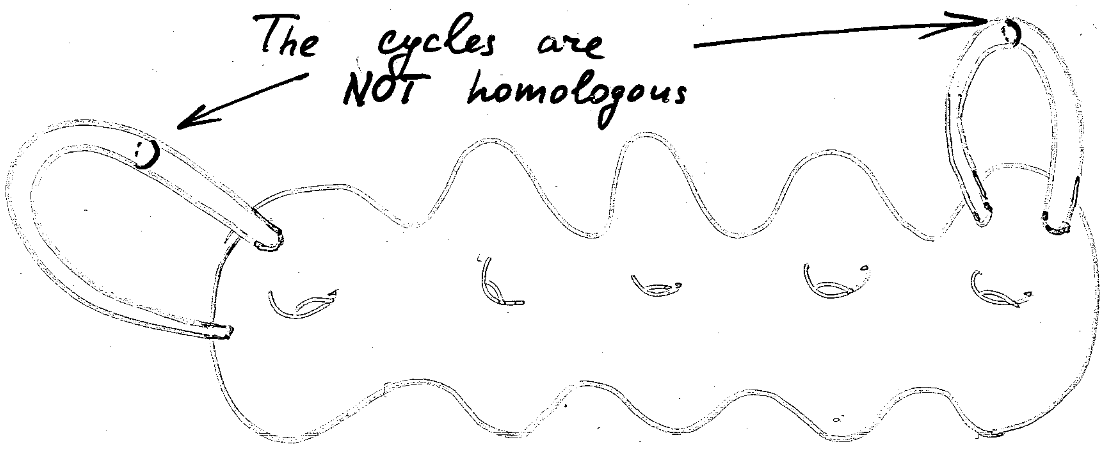
In cohomological coordinates the compound surface has only <u>one</u> short period, and we again obtain "almost fiber bundle"

$$\mathcal{H}_1^{\varepsilon,\,\text{thick}}\ \left(\begin{matrix}\text{particular}\\ \text{configuration}\end{matrix}\right)$$

$$\big\downarrow\ \widetilde{\mathcal{D}}^2_\varepsilon$$

$$\mathcal{H}_1\begin{pmatrix}\text{stratum}\\ \text{of } S_1\end{pmatrix} \times \cdots \times \mathcal{H}_1\begin{pmatrix}\text{stratum}\\ \text{of } S_K\end{pmatrix}$$


a_2
b_2
b_1
a_1
b_2
c_1
c_1
b_1
c_2
a_1
a_2
c_2



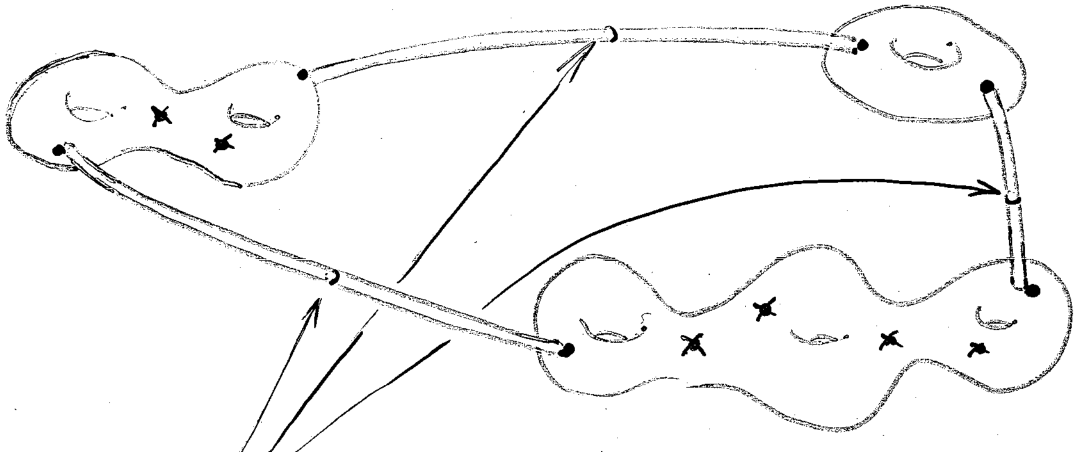
We get two "independent" short geodesics of length $\leq \varepsilon$. $\Rightarrow$ two small $\mathbb{C}$-parameters in the parameter space (two small periods) $\Rightarrow$ this gives a subset of measure $\sim (\varepsilon^2)^2$

Example of an "edge" of the compactification.

Moral: To compute the volume of the tubular ε-neighborhood ($\approx$ "surface area") of the compactification one has to know only the "areas" of the (complex) "faces" (= typical degenerations).

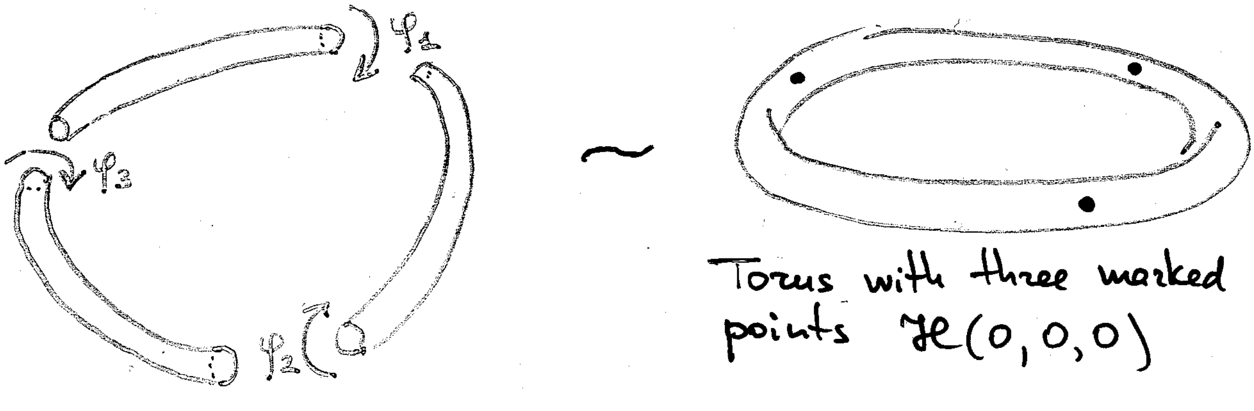
One does not need information about "edges", "vertices", etc.

The nature of higher multiplicities.

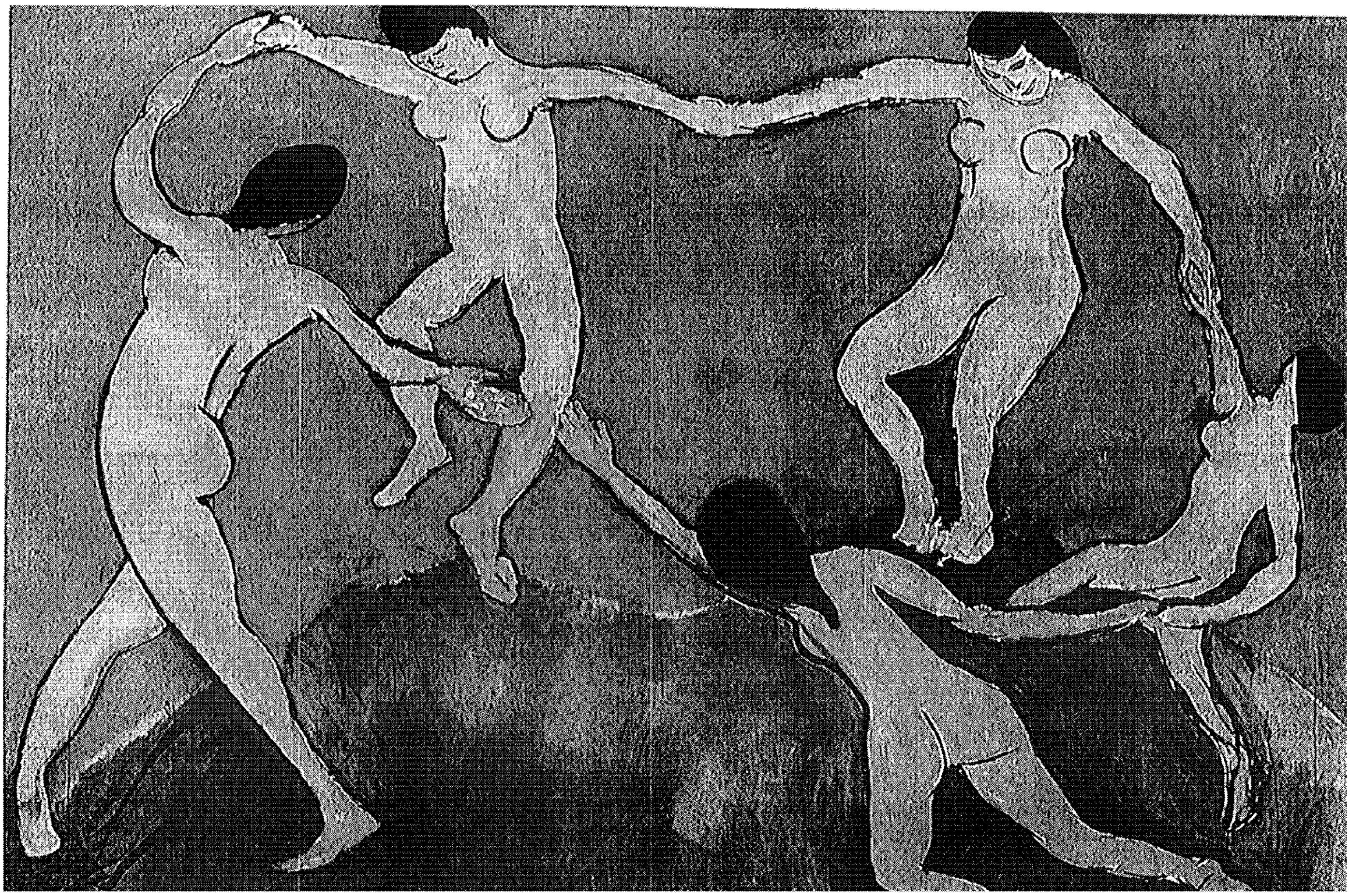


The cycles are <u>homologous</u> We get three short geodesics having the same length $\leq \varepsilon$ and going in the same direction. However, this configuration is described by only <u>one</u> small parameter (period) in the parameter space. Such subset has measure $\sim \varepsilon^2$ in $\mathcal{H}(1, \ldots, 1)$.

This complex "face" of compactification of $\mathcal{H}(\underbrace{1, \ldots, 1}_{12})$ is expressed in terms of $\mathcal{H}(1,1,0,0) \times \mathcal{H}(0,0) \times \mathcal{H}(1,1,1,1,0,0)$



Torus with three marked points $\mathcal{H}(0,0,0)$



Stratum $\mathcal{H}(5,1)$. Configurations of closed geodesics

Degeneration pattern	$\lvert\Gamma_-\rvert$	$\lvert\Gamma\rvert$	M	$c\cdot\zeta(2)$	c approx.
$\Rightarrow(\mathbf{0},\mathbf{4};)\Rightarrow$	1	1	5	$\frac{38125}{15552}$	1.4903
$\rightarrow(\mathbf{0},\mathbf{3};\mathbf{1})\rightarrow$	1	1	4	$\frac{2240}{243}$	5.60394
$\Rightarrow(\mathbf{0}+\mathbf{3}=\mathbf{3};\mathbf{1})\Rightarrow$	1	1	4	$\frac{320}{243}$	0.800562
$\Rightarrow(\mathbf{1}+\mathbf{2}=\mathbf{3};\mathbf{1})\Rightarrow$	1	1	4	$\frac{320}{243}$	0.800562
$\rightarrow(\mathbf{1},\mathbf{2};\mathbf{1})\rightarrow$	1	1	6	$\frac{175}{18}$	5.9104
$\rightarrow(\mathbf{0}+\mathbf{0}=\mathbf{0};)\Rightarrow(\mathbf{0},\mathbf{2};)\rightarrow$	1	1	3	$\frac{35}{288}$	0.07388
$\rightarrow(\mathbf{0},\mathbf{0};)\Rightarrow(\mathbf{0}+\mathbf{2}=\mathbf{2};)\rightarrow$	1	1	3	$\frac{35}{576}$	0.03694
$\rightarrow(\mathbf{0},\mathbf{0};)\Rightarrow(\mathbf{1}+\mathbf{1}=\mathbf{2};)\rightarrow$	1	1	3	$\frac{35}{576}$	0.03694
$\rightarrow(\mathbf{0},\mathbf{0};)\Rightarrow(\mathbf{2}+\mathbf{0}=\mathbf{2};)\rightarrow$	1	1	3	$\frac{35}{576}$	0.03694
$\rightarrow(\mathbf{0}+\mathbf{0}=\mathbf{0};)\rightarrow(\mathbf{0},\mathbf{1};\mathbf{1})\rightarrow$	1	1	2	$\frac{175}{486}$	0.218904
$\rightarrow(\mathbf{0},\mathbf{0};)\rightarrow(\mathbf{0}+\mathbf{1}=\mathbf{1};\mathbf{1})\rightarrow$	1	1	2	$\frac{35}{243}$	0.0875615
$\rightarrow(\mathbf{0}+\mathbf{0}=\mathbf{0};)\Rightarrow(\mathbf{0}+\mathbf{1}=\mathbf{1};\mathbf{1})\rightarrow$	1	1	2	$\frac{35}{486}$	0.0437808
$\rightarrow(\mathbf{0}+\mathbf{0}=\mathbf{0};)\Rightarrow(\mathbf{1}+\mathbf{0}=\mathbf{1};\mathbf{1})\rightarrow$	1	1	2	$\frac{35}{486}$	0.0437808
$\rightarrow(\mathbf{0}+\mathbf{0}=\mathbf{0};)\rightarrow(\mathbf{0}+\mathbf{0}=\mathbf{0};)\Rightarrow(\mathbf{0},\mathbf{0};)\rightarrow$	1	1	1	$\frac{175}{7776}$	0.0136815

$$\sum\lambda_i = 2$$

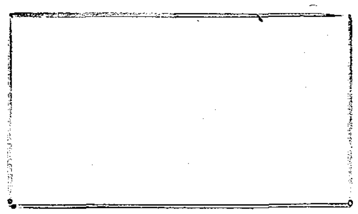
What do the Siegel–Veech constants serve for?

Having computed the Siegel–Veech constants let us count the closed billiard trajectories or trajectories going from one corner to another (for rational billiards)!

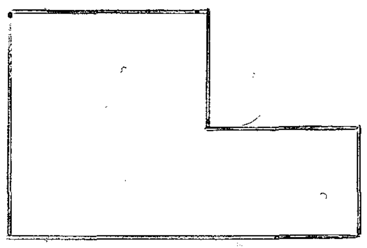
... No way... Families of rational polygons with fixed angles give rise to families of flat surfaces (obtained after unfolding) of large codimension. Our theory does not see them. We need to have classification of all $SL(2;\mathbb{R})$-orbit closures.

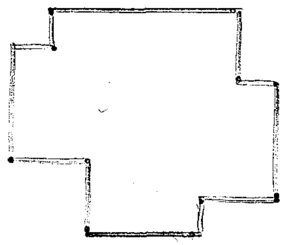
However, configurations of saddle connections and Siegel–Veech constants give some invariants of $SL(2;\mathbb{R})$-orbits. For example, some configurations cannot occur in some connected components.

Rectangular polygons.

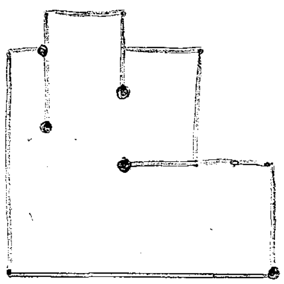


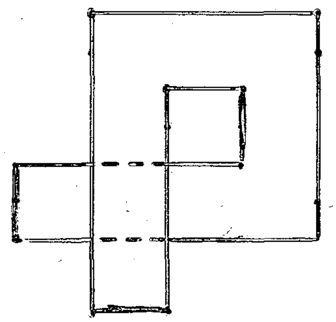
An example of a rectangular polygon.



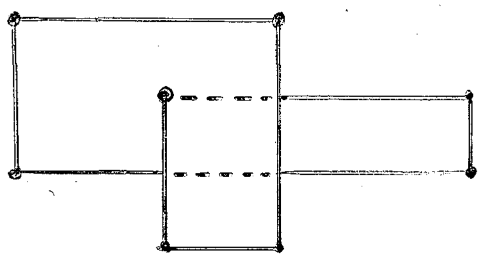


and more examples...





and even more...



and even this!..

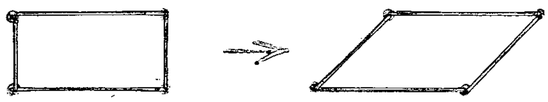
Definition A rectangular polygon = topological disk + flat metric such that

i) no singularities in the interior

ii) the boundary is a finite broken line of geodesics

iii) an angle between two conseq. sides is $k \cdot \frac{\pi}{2}$, $k \in N$ $(k \neq 2)$

$SL(2;\mathbb{R})$-action. Ergodicity.



The flow induced by the action of the diagonal subgroup $\begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}$ is called Teichmüller geodesic flow.

THEOREM (H. Masur ; W. Veech)
$SL(2;\mathbb{R})$ acts ergodically (with resp. to some finite absol. contin. natural measure) on every connected component of every stratum of the moduli space of quadratic differentials.

Yes, but a family coming from billiards has hight codimension inside the moduli space (stratum)!

TRICK (G. Margulis; A. Eskin + C. McMullen)
If we have a nice hyperbolic system, and a subvariety transversal to the stable foliation, we can (almost) pretend that the subvariety is an open subset.

LEMMA Any subvariety corresponding to rectanglular polygons with fixed angles IS transversal to the stable foliation.

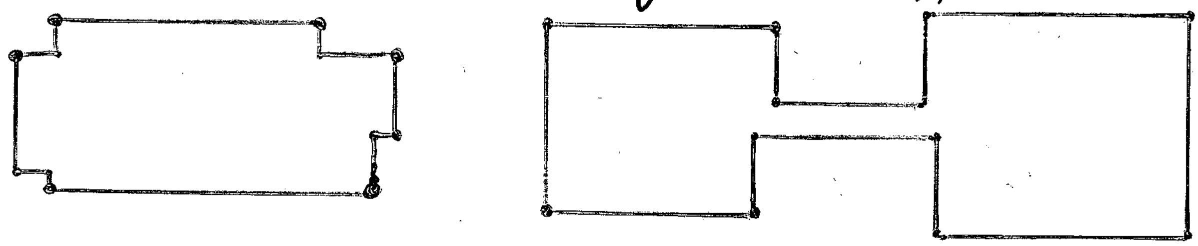
THEOREM (A. Eskin, A. Zorich) Consider a family of rectangular polygons having exactly $K \geq 0$ angles of the type $\frac{3\pi}{2}$ and all the other $(= K+4)$ angles of the type $\frac{\pi}{2}$. Choose a generic (mes. th.) billiard table. Fix any two corners P_i, P_j with the angles $\frac{\pi}{2}$. Let $N_1(L)$ be the number of trajectories of length at most L joining P_i with P_j.

$$N_1(L) \approx \frac{1}{2\pi} \cdot \frac{L^2}{\text{area of the table}}$$

PROOF: Idea of Margulis on a similar subject $\rightsquigarrow$ Developement of Eskin-McMullin $\rightsquigarrow$ ...

REMARK

The shape might be different.

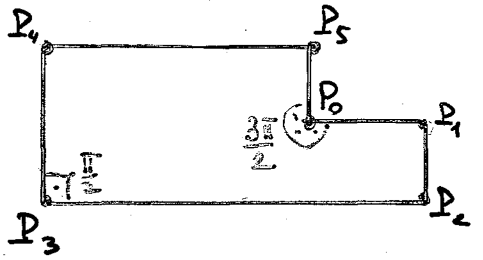


REMARK

I do not expect any elementary proof of the Theorem since for more general rectangular polygons the constant is much more complicated.

REMARK Intuition does not help...

For example, in



the number of trajectories joining P_0 with any of P_i, $i=1,\dots,5$ is "approximately"

$$\frac{2}{\pi}\cdot\frac{L^2}{\text{area of the table}}$$

which is 4 times (and not 3 times!) more than the corresponding number for P_iP_j, $i,j\geq 1$.

REMARK

We can obtain numerous other counting formulae (families of closed trajectories, generalised diagon,...) They are expressed in terms of the volumes of the corresponding strata in the moduli spaces of quadratic differentials.