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SUMMER SCHOOL AND CONFERENCE ON DYNAMICAL SYSTEMS

Generic Dynamics
(Lecture 1)

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1 Notions of recurrence

Consider a dynamical system, for instance a homeomorphism h of a compact set X. The *orbit* $\mathcal{O}(x, f)$ of a point $x \in X$ by f is the sequence $\{f^n(x), n \in \mathbb{Z}\}$; one speak on *forward* and *backward orbit* if one just consider the positive or negative iterates of f.

The aim of dynamical systems is to describe the behavior of the orbits when n goes to $\pm\infty$; not just the orbit of a given point, but if possible, the orbit $\mathcal{O}(x, f)$ of all point $x \in X$, and not only for a given f, but for every $f \in \text{Homeo}(M)$.

Certainly, it is too ambitious, so that we will have to replace *all* x and *all* f by *most of* $x \in X$ and *most of* f, what ever "most of" means!.

Other question is:

What means *describe the behavior of the orbits ?*

in other words:

When will we consider that we get a satisfactory description ?

For instance (as in the case of gradient flows), if the evolution of every initial data consists in reaching an equilibrium point (a fixed point) and if this fixed points are in finite number, the description of f is given by the collection of its fixed points, and by the positions of the *stable* and *unstable sets* of these fixed points. This may be the endpoint of the topological description, and maybe the begining of other studies, like the velocity of reaching the fixed points, and so on.

So, all points have not the same interest for describing the dynamics. Some of them allows to understand a large set of other orbits, and others are not so interesting. So in some sense, a first step consists in defining:

What is the heart of the dynamics ?

1.1 The limit sets

First of all, fixed or periodic points are clearly interesting for the dynamics. We denote by $\text{Fix}(f)$ and $\text{Per}(f)$ the sets of fixed and periodic point points of f, respectively. The set $\text{Fix}(f)$ is clearly compact and f invariant. The set $\text{Per}(f)$ is f invariant.

Exercise 1. *Build a homeomorphism f of a compact space X where $Per(f)$ is not compact.*

Exercise 2. *Prove that, if an orbit is compact, then it is a periodic orbit.*

For the other points, where is going the orbit? To a point x are associated its ω-limite set

$$\omega(x) = \bigcap_N \overline{\bigcup_N^{+\infty} f^n(x)},$$

that is the set of accumulation points oint its forward orbit. In the same way is defined the α-limit set $\alpha(x)$ being the set of accumulation points of the negative orbit. The sets $\alpha(x)$ and $\omega(x)$ are invariant compact sets.

The point x is α or ω *recurrent* if $x \in \alpha(x)$ or $x \in \omega(x)$, respectively. This means that the orbit of x comes back and back close to x (in the past or in the future, respectively). We denote by $\mathrm{Rec}(f) = \mathrm{Rec}_+(f) \cup \mathrm{Rec}_-(f)$ the sets of recurrent points. These set are clearly invariant by f.

Exercise 3. *Build f such that Rec(f) is not compact.*

Exercise 4. *Build f such that $Rec_+(f) \neq Rec_-(f)$*

The *limit set* $\mathrm{Lim}(f) = \mathrm{Lim}_+(f) \cup \mathrm{Lim}_-(f)$ of f is the union of all ω and α limit sets of points $x \in X$.

Around 1935, Birkhoff [Bi$_1$] introduced the following notion; A point x is called *wandering* if it admits a neighborhood U_x disjoint from all iterates $f^n(U_x)$, $n > 0$. By definition all point in U_x is non-wandering too. The set of non-wandering points, called the *non-wandering set of* f and denoted by $\Omega(f)$, is compact and f-invariant.

Finally, in the 70^{th}, Conley and Bowen (see [Bo, Co])considered a weaker notion of recurrence. A point x is called *chain recurrent* if for any $\varepsilon > 0$ there is a sequence $x = x_0, x_1, \ldots, x_n = x$, $n > 0$ such that for any $i \in \{0, \ldots, n-1\}$ one has $d(f(x_i), (x_{i+1}) < \varepsilon$ (such a sequence is called an ε-pseudo-orbit). The set of chain recurrent points, called *the chain recurrence set of f* and denoted by $\mathcal{R}(f)$, is an invariant compact set.

All these sets are invariant by f. Furthermore, $\mathrm{Fix}(f)$, $\Omega(f)$ and $\mathcal{R}(f)$ are compact. They are related by the following inclusions:

$$\mathrm{Per}(f) \subset \mathrm{Rec}(f) \subset \mathrm{Lim}(f) \subset \Omega(f) \subset \mathcal{R}(f)$$

Exercise 5. *Build examples showing that the closures of all these sets may be pairwize distinct.*

1.2 invariant subspaces

One would like to structurate the limit sets defined above using elementary pieces, that is in some sense dynamically indecomposable.

The strongest notion of (topologica) indecomposability is the minimality:

Definition 1.1. *A compact invariant set $K \subset X$ is a* minimal set *if it is minimal for the inclusion in the set of compact invariant sets.*

Remark 1.2. *1. Every invariant compact set (in particular $\alpha(x)$ and $\omega(x)$) contains a minimal set (using Zorn lemma: just notice that $(\{K \subset X, K$ compact and$f(K) = K\}, \subset)$ is inductive: the intersection of a decreasing family of (non-empty) invariant compact set is a non empty invariant compact set)*

2. *Every orbit of a minimal set is dense in it;*

3. *There are examples of minimal set on which the dynamics is not uniquely ergodic (first example by Furstenberg).*

Exercise 6. *The shift on* $\{0,1\}^{\mathbb{Z}}$ *admits an uncountable family of minimal sets.*

A more flexible notion of undecomposability is the notion of transitivity:

Definition 1.3. *A compact invariant set* $K \subset X$ *is a* transitive set *(some authors use* topologically ergodic*) if one of the equivalent properties holds:*

1. *there is* $x \in K$ *such that* $K = \omega(x)$

2. *there is a countable intersection* $\mathcal{G} = \bigcap_{n\in\mathbb{N}} O_n$ *of dense open subset* O_n *ofX such that, for any* $x \in \mathcal{G}$,
$$\alpha(x) = \omega(x) = X.$$
In other words, the orbit of generic points *in* X *are positively and negatively dense in* X.

3. *for any (non-empty) open subsets* U, V *of* X *there is* $n > 0$ *such that* $f^n(U) \cap V \neq \emptyset$.

Let prove

Lemma 1.4. *The* 3 *properties in the definition of transitivity are equivalent.*

Démonstration : One has clearly b)$\Longrightarrow$ a).

a)$\Longrightarrow$ c): the point x has positive iterates accumulating on $f^{-1}(x)$ and so also on x so that x is positively recurrent, and so any $f^i(x)$ is positively recurrent. As a consequence, for any non empty open set V of X, there is a sequence $n_i \to +\infty$ such that $f^{n_i}(x) \in V$. So choose some n such that $y = f^n(x) \in U$. There is i such that $n_i > n$ so that $f^{n_i-n}(y) \in V$ that is$f^{n_i-n}(U) \cap V \neq \emptyset$.

c)$\Longrightarrow$ b) For any open set $U \subset X$ the set
$$O_+(U) = \{x \in X \mid \exists n > 0 f^n(x) \in U\} = \bigcup_{n>0} f^{-n}(U)$$
is open by definition and dense because c).

In the same way $O_-(U) = \{x \in X \mid \exists n > 0 f^n(x) \in U\}$ is open and dense. So $O(U) = O_+(U) \cap O_-(U)$ is a dense open subset of X

As X is a compact metric space, it admits a countable basis of open sent U_n. Then the announced set $\mathcal{G}$ is
$$\mathcal{G} = \bigcap_{n\in\mathbb{N}} O(U_n).$$

Any point in $\mathcal{G}$ admits positive and negative iterates in any open set, so that it positive and negative orbits are both dense in X. □

2 Filtrations

One very simple idea (typical of dissipative dynamics) for organizing the global topological dynamics, is to consider *strictly invariant regions* that is, open sets U such that $f(\bar{U}) \subset U$. The orbits passing nearby $\bar{U}$ enter in U and never come back. Then every orbit in $\bigcup_{n\geq 0} f^{-n}(U)$ has its ω-limit set contained in $\Lambda(U) = \bigcap_{n\geq 0} f^n(\bar{U})$, which is an invariant compact set.

This idea is the key of the famous Poincaré Bendixson theorem for vector fields on the sphere S^2.

Conley has pushed to its end this simple argument, explaining with what precision one see the dynamics if one just distinguish points x, y with $x \in U$ and $y \notin U$, for U strictly invariant open set.

2.1 Pairs attractors/repellors and Lyapunov functions

Definition 2.1. *An invariant compact set K is called a (topological) attractor if it admits an open neighborhood U with $f(\bar{U}) \subset U$, such that $K = \Lambda(U) = \bigcap_{n\geq 0} f^n(\bar{U})$. The open set U will be called an* isolating neighborhood *of K. A repellor is an attractor for f^{-1}.*

Remark 2.2. *If A is an attractor and U is an isolating neighborhood of A then $V = X - \bar{U}$ verifies $f^{-1}(\bar{V}) \subset V$ and $R = \bigcap_{n\geq 0} f^{-n}(\bar{V})$ is a repellor. So to any attractor A corresponds a pair (A, R) of attractor repellor..*

Exercise 7. *Build f such that $\Omega(f) \neq X$ but the unique attractors are X and $\emptyset$.*

Theorem 2.1. *Let (A, R) be a pair of attractor repellor of a homeomorphism f of a compac metric space X. Then there is a continuous fucton $\varphi \colon X \to [0, 1]$ such that:*

1. *$\varphi(R) = 1$ and $\varphi(A) = 0$.*
2. *for any $x \notin A \cup R$ one has $\varphi(f(x)) < \varphi(x)$*

A *Lyapunov function for f* is a function which is decreasing along the orbit ($\varphi(f(x)) \leq \varphi(x)$, $\forall x \in X$). A function φ verfying the conclusion of Theorem 2.1 is called a *Lyapunov function adapted to the pair (A, R)*.

Proof : Let U_0 be an isolating neighborhood of A, and let U be the interior of $\bar{U}_0$. It is an isolating neighborhood of A, and verifies that $\mathrm{int}(\bar{U}) = U$. Notice that the orbit of any point $x \notin A \cup R$ intersect $U \setminus f(U)$ in exactly one point.

Consider a continuous fonction $\psi_0 \colon X \to [0, 1]$ such that $\psi_0(f(\bar{U})) = 0$, $\psi_0(X \setminus U) = 1$ and $\psi_0(x) \in]0, 1[$ if $x \in U \setminus f(\bar{U})$. Notice that ψ_0 is 1 on R and 0 one A, is decreasing along the orbits. More precisely, $\psi_0(f(x)) < \psi_0(x)$ if and only if $x \in X \setminus U$ and $f(x) \in U$ or $x \notin f(\bar{U})$ and $f(x) \inf(\bar{U})$. So for any $x \notin A \cup R$, the function ψ decreases just once or twice allong the orbit of x, according if the orbits of x meets the boundary ∂U or not.

Let denote $\psi_n(x) = \psi(f^n(x))$. Consider a sequence $a_n > 0$ such that $\sum_{-\infty}^{+\infty} a_n = 1$ and denote

$$\varphi = \sum_{-\infty}^{+\infty} a_n \psi_n.$$

Then φ is a continuous function, whose values are 1 on R and 0 on A, and which is strictly decreasing along the orbits of the point $x \notin A \cup R$: for such a point there is at least one n for which $\psi_n(f(x) < \psi_n(x)$.

□

Exercise 8. *Prove that, if X is a smooth compact manifold, f is an homeomorphism of X, and (A,R) is a pair of attractor repellor of f, then the function φ may be choosed C^∞.*

Consider a strictly invariant open set U (that is $f(\bar{U}) \subset U$) and let $V \subset U$ be any open set containing $f(U)$. Then V is also strictly invariant and define the same pair of attractor-repellor as U. As a consequence of this simple remark one get:

Lemma 2.3. *Let f be a homeomorphism of a compact metric space (X,d). Then the set of attractor-repellor pairs of f is at most countable.*

Proof : Consider a countable base of open sets $\mathcal{O} = \{\mathcal{O}_n\}$ of the topology of X. Then any attractor A admit a isolating open neighborhood V $(f(\bar{V}) \subset V)$ which is the union of finitely many open sets belonging to $\mathcal{O}$ (for this, consider any isolating open neighborhood U of A, and cover $f(\bar{U})$ by binitely many $\mathcal{O}_n$ contained in U).

Notice that the set of such open sets V is countable (because associated to a finite subset of $\mathcal{O}$ which is countable) and so the set of attractors is countable. □

Let $\alpha_i > 0$ be a sequence such that $\sum_i \alpha_i = 1$; we also assume that for any i $\sum_{i+1}^{+\infty} a_j < \frac{1}{2} a_i$: this condition ensures that, if two sequences $\mu_i, \nu_i \in \{0,1\}$ verify that $\sum_{i\in\mathbb{N}} \mu_i a_i = \sum_{i\in\mathbb{N}} \nu_i a_i$ then $\mu_i = \nu_i$; in particular the set of all $\sum_{i\in\mathbb{N}} \mu_i a_i$, $(\mu_i) \in \{0,1\}^{\mathbb{N}}$ is a Cantor set.

Let $\{(A_i, R_i), i \in \mathbb{N}\}$ be an indexation of the set of all pairs attractors repellors of f (we have seen that it is countable). For each $i \in \mathbb{N}$, let φ_i be a Lyapunov function adapted to the pair (A_i, R_i) and consider $\varphi = \sum_i \alpha_i$.

Theorem 2.2. *With the notation above one has:*

1. *for every $x \in X$ one has $\varphi(f(x)) \leq \varphi(x)$.*
2. *$\varphi(f(x)) = \varphi(x)$ if and only if $x \in \bigcap_{i\in\mathbb{N}}(A_i \cup R_i)$.*
3. *Let denote by $\mathcal{AR}(f)$ the set $\bigcap_{i\in\mathbb{N}}(A_i \cup R_i)$. We define an equivalence relation on $\mathcal{AR}(f)$ by :*
$$x \approx y \Longleftrightarrow (\forall i \in \mathbb{N}, x \in A_i \Leftrightarrow y \in A_i).$$
Then, for any $x, y \in \mathcal{AR}(f)$ one has
$$\varphi(x) = \varphi(y) \Longleftrightarrow x \approx y.$$
4. *$\varphi(\mathcal{AR}(f) \subset \mathbb{R}$ has empty interior.*

Démonstration : The function φ is decreasing allong the orbits as a sum of functions decreasing allong the orbits. Futhermore, as the a_i are all > 0, $\varphi(f(x)) = \varphi(x)$ if and only if $\psi_i(f(x)) = \psi_i(x)$ for all $i \in \mathbb{N}$, that is $x \in A_i \cup R_i$ for all $i \in NN$.

Finally if $x \in \mathcal{AR}(f)$ then $\varphi(x) = \sum_{i\in\mathbb{N}} \mu_i a_i$ where $\mu_i = 0$ if $x \in A_i$ and $\mu_i = 1$ if $x \in R_i$. The third and fourth property now follow from the choice of the numbers a_i. □

Lemma 2.4. $\mathcal{R}(f) \subset \mathcal{AR}(f)$.

Démonstration : Let ψ be a Lyapunov function adapted to a pair (A, R). Let $\varepsilon > 0$, and defined the sets $A_\varepsilon = \psi^{-1}([0, 2\varepsilon[)$ and $R_\varepsilon = \psi^{-1}(]1-\varepsilon, 1])$.

Consider $\delta_0 = \inf\{\psi(x) - \psi(f(x)) \mid \psi(x) \in [\varepsilon, 1-\varepsilon]\}$. Notice that $\delta_0 > 0$. Let $\delta = \frac{1}{2}\inf\{\delta, \varepsilon\}$

Notice that, if $x_0 \in \psi^{-1}([2\varepsilon, 1-\varepsilon])$ and if $x_0, \ldots, x_n$ is a δ-pseudo-orbite then $\psi(x_n) \leq \psi(x_0) - \frac{1}{2}$.

Let $x \in \mathcal{R}(f)$. Then $x \dashv_\delta x$; the argument above shows that $x \in A_\varepsilon \cup R_\varepsilon$. So $x \in A \cup R = \bigcap_\varepsilon (A_\varepsilon \cup R_\varepsilon)$.

□

2.2 The chain recurrent classes

For any $\varepsilon > 0$ one defines a relation $\dashv_\varepsilon$ on X as follows: $x \dashv_\varepsilon y$ if there is $n > 0$ and a ε-pseudo-orbit $x_0 = x, \ldots, x_n = y$ (one tell that y is *ε-accessible* from x).

One tell that y is *accessible from* x and we write $x \dashv y$ if it is ε-accessible for any $\varepsilon > 0$.

Remark 2.5. *1. The relation $\dashv_\varepsilon$ is a transitive relation:*

$$x \dashv_\varepsilon y \text{ and } y \dashv_\varepsilon z \Longrightarrow x \dashv_\varepsilon z$$

As a consequence, $\dashv$ is a transitive relation.

2.

$$x \dashv x \Longleftrightarrow x \in \mathcal{R}(f).$$

We define now a relation on $\mathcal{R}(f)$ by $x \vdash\!\dashv y \Longleftrightarrow x \dashv y$ and $y \dashv x$.

Corollary 2.6. *The relation $\vdash\!\dashv$ on $\mathcal{R}(f)$ is an equivalence relation.*

The equilavence classes of $\vdash\!\dashv$ are called *the chain recurrence classes.*

Exercise 9. *The chain recurrence classes are compact invariant sets.*

Exercise 10. *Let f be a homeomorphism of a compact connected metric set X, and assume that $\Omega(f) = X$. Show that X is the unique chain recurrence class.*

2.3 The fundamental theorem of dynamical systems

Theorem 2.3. *Let f be an homeomorphism of a compact metric space X, d. Then there is a continuous function $\varphi \colon X \to \mathbb{R}$ verifying the following properties:*

- *For every $x \in X$ one has $\varphi(f(x)) \leq \varphi(x)$ (i.e. φ is a Lyapunov function);*
- $$\varphi(f(x) = \varphi(x) \Longleftrightarrow x \in \mathcal{R}(f)$$
- *$forall x, y \in \mathcal{R}(f)$ one has*
$$\varphi(x) = \varphi(y) \Longleftrightarrow x \vdash\!\dashv y$$
- *The compact set $\varphi(\mathcal{R}(f)) \subset \mathbb{R}$ has empty interior.*

A function verifying the conclusion of Theorem 2.3 will be called a *Lyapunov function adapted to* $\mathcal{R}(f)$.

Lemma 2.7. *If* $x \notin \mathcal{R}(f)$ *then there is a pair* (A, R) *of attractor repellor such that* $x \notin A \cup R$. *In other words* $\mathcal{AR}(f) \subset \mathcal{R}(f)$ *that is* $\mathcal{R}(f) = \mathcal{AR}(f)$.

Démonstration : Consider $W^u_\varepsilon(x) = \{y \mid x \dashv_\varepsilon y\}$. By debinition, it is an open subset of X. Notice that, by definition of $\dashv_\varepsilon$ the $\varepsilon/2$ neighborhood of $f(W^u_\varepsilon(x)$ in contained in $W^u_\varepsilon(x)$. As a consequence $f(\overline{W^u_\varepsilon(x)}) \subset W^u_\varepsilon(x)$. In other word, $W^u_\varepsilon(x)$ is a strictly invariant open set.

Assume now $x \notin \mathcal{R}(f)$. Then there is ε such that $x \notin W^u\varepsilon(x)$, but $f(x) \in W^u_\varepsilon(x)$. Then x does not belongs to the attractor or the repellor associated to $W^u_\varepsilon(x)$. □

The following lemma finishes the proof of the theorem:

Lemma 2.8. *For* $x, y \in \mathcal{R}(f)$ *one has:*

$$x \vdash\!\dashv y \Longleftrightarrow x \approx y.$$

Démonstration : If x, y are not equivalent for $\vdash\!\dashv$, that is for example $y \notin W^u_\varepsilon x$. Then for the corresponding pair of attractor repellor, x belong to the attractor and y to the repellor, so that x is not equivalent to y for $\approx$. Conversally, if x is an attractor A and y in the repellor R of the pair (A, R), there is ε such that y is not ε accessible fron x. □

Exercise 11. *Build a homeomorphism and Lyapunov function (that is, a function which is decreasing along the orbits) which s not constant on one chain recurrence class.*

Let me make an ingenuous[1] question:

Naïve question 1. *Consider the following equivalence relation on* $\mathcal{R}(f)$: $x \simeq y$ *if for any Lyapunov function* ψ *one has* $\psi(x) = \psi(y)$. *What is the dynamical interpretation of the equivalence classes?*

2.4 Global dynamics

Theorem 2.3 give a nice description of the global dynamics similar to the descrition of gradient-like dynamics , the chain recurrence classes taking the place of the critical points: for any x the orbit of x decrease for a Lyapunov function from a chain recurrent chain to another chain recurrent chain.

Now, the description of the toplogical global dynamics of a homeomorphism or a dffeomorphism consists in:

- give the description of the dynamic in resctriction to each chain recurrence class
- describe the relative positions of the invariant manifolds of the chain recurrence classes.

Exercise 12. *Prove that if a homeomorphism* f *of a compact metric space* X *has a unique chain-recurrent class then* $\mathcal{R}(f) = X$.

[1] In this text, *naïve question* means that I did not think enough on it or that I do not feel myself competent enough for asking seriously this question: maybe the answer is already well known by specialists or at the contrary the question is known to be a real difficulty! Anyway, I will be gratefull to anybody giving me the answer or any information (references) on it!

Exercise 13. *Let f be a homeomorphism of a compact metric space X, d and let $\varphi\colon X \to \mathbb{R}$ be a Lyapunov function adapted to $\mathcal{R}(f)$. Let C be a chain recurrence class oand U be a neighborhood of C.*

Prove that there is $\varepsilon > 0$ such that any chain recurrence class $\tilde{C}$ with $|\varphi(C) - \varphi(\tilde{C})| < \varepsilon$ is contained in U.

In other word, the map $t \mapsto \varphi^{-1}(t) \cap \mathcal{R}(f)$ is upper semicontinuous.

2.5 Hyperbolic dynamics

I cannot do here a survey of hyperbolic dynamics : it could be the aim of a whole course. Let me just make some comments, in the spirit of this course.

Let f be a diffeomorphism of a compact manifold M. An invariant compact set K is called *hyperbolic* if the tangent space of M over K splits in the direct sum $TM|_K = E^s \oplus E^u$ such:

1. the spliting is invariant under the natural action of the differential Df: $E^s(f(x) = D_x f(E^s(x))$ and $E^u(f(x)) = D_x f(E^u(x))$

2. the vectors in E^s are uniformly contracted and the vectors in E^u are uniformly expanded: there are $C > 0$ and $0 < \lambda < 1$ such that for any $x \in K$ and any $u \in E^s(x)$ and $v \in E^u(x)$) and any $n > 0$ one has:

$$\|Df^n(u)\| \leq C\lambda^n \|u\|$$
and
$$\|Df^{-n}(v)\| \leq C\lambda^n \|v\|$$

Exercise 14. *Prove that the plitting $T_x M = E^s(x) \oplus E^u(x)$ is unique and continuous on K.*

A diffeomorphism f verifies the Axiom A and the no-cycles condition if $\mathcal{R}(f)$ is hyperbolic. I will not explain here the hyperbolic theory which need a whole course to be presented.

just recall that, if $\mathcal{R}(f)$ is hyperbolic then:

- there are finitely many chain recurrence classes
- each chain recurrence class is transitive, semi-conjugated to a subshift of finite type, and admits a dense subset of periodic orbits which are all homoclinically related.
- the local stable manifold of a class is a union a continuous family of disks tangent to E^s
- for each point x in M there is a point x^ω (in the chain recurrence class containing $\omega(x)$ such that $x \in W^s(x^\omega)$; then $d(f^n(x), f^n(x^\omega)) \to 0$.
- finally the same description holds for all diffeomorphisms g in a C^1-neighborhood of f. More precisely,

  Theorem 2.4. *(Smale-Palis) The two following properties are equivalent:*

  1. *$\mathcal{R}(f)$ is hyperbolic*
  2. *there is a C^1-neighborhood of f such that, for any g in this neighborhood, the restriction of g to $\mathcal{R}(g)$ is conjugated to the restriction of f to $\mathcal{R}(f)$.*

One tells that f verifies the Axiom A and the strong transversality condition if $\mathcal{R}(f)$ is hyperbolic and if for any points $x, y \in \mathcal{R}(f)$, the invariant manifolds $W^s(x)$ and $W^u(y)$ are transverse.

Theorem 2.5. *(Robbin, Robinson, Mañé) The two following properties are equivalent:*

1. *f verifies the Axiom A and the strong transversality condition*
2. *there is a C^1-neighborhood of f such that, any g in this neighborhood is conjugated to f (one tells that f is C^1-structurally stable.*

The hyperbolic dynamics represent the part of $Diff(M)$ for which on has a satisfactory descrition. However, the hyperbolic systems represent an open but not dense subset of $Diff^r(M)$, for any $r \geq 1$ if $dimM \geq 3$ and $r \geq 2$ if $dimM = 2$.

One of the important problem remaining open is

Conjecture 2.9. *(Smale) Let S be a compact surface. The set of diffeomorphism f of S for which $\mathcal{R}(f)$ is hyperbolic is a dense subset of $Diff^1(S)$.*

Exercise 15. *Build a diffeomorphism f on a compact manifold such that $\Omega(f)$ is hyperbolic but $\mathcal{R}(f)$ is not hyperbolic.*

3 C^1-Generic diffeomorphisms: motivations

3.1 Definition and discussion

We want to study non-hyperbolic diffeomorphisms and more precisely the complement of the closure of hyperbolic diffeomorphisms. In this set, no diffeomorphisms is $\mathcal{R}$-stable, that is, there are everywhere bifurcations. Then there is no hope of describing all systems. The hope is that, even if some systems presents very complicated behavior, "most of them" could present some regularity. In other word, we will avoid fragil pathological behavior.

The topological way to say "most of all " consist in focus on generic diffeomorphisms. Let me first recall this notion :

For any complete metric space (X, d), Baire proved that the countable intersection of dense open subset are dense. A *residual* set is a set containing such countable intersection of dense open subsets.

The sentense:

any generic point $x \in X$ verifies property $\mathcal{P}$

means:

There is a residual subset $\mathcal{R}$ of X on which property $\mathcal{P}$ holds.

Example 1. *For any manifold M, the set $Diff^r(M)$ (endowed with the C^r topology, is a Baire space.*

Digression 1: There are other attempts to define a notion of "most of". One of the most popular notion is the following:

Consider generic families with k parameter (that is, generic smooth pas from $\mathbb{R}^k \to Diff(M)$). A property $\mathcal{P}$ is called *prevalent* if for any generic family with k-parameters, the Lebesgues measures of parameter for which $\mathcal{P}$ does not hold is 0. The problem is that many generic properties,are not prevalent. For instance, generic diffeomorphismson S^1 have rationnal rotation number but in generic 1-parameter family of analytic diffeomorphism of S^1 the measure of those with irrationnal rotation number may be positive.

Let me make an ingenuous question:

Naïve question 2. *If a property is prevalent for family with k parameters, is it prevalent for family with $k+1$-parameters?*

My feeling is that the answer is no... The stupid example is the notion of being prevalent for family with 0 parameters is the usual notion of being generic!

Anoter way to formulate this question is

Naïve question 3. *Does it exist a subset $\mathcal{E}$ of $\mathbb{R}^2$ having 0 Lebesgue measure, such that any generic path γ: $[0, 1]$ in $\mathbb{R}^2$ meets $\mathcal{E}$ in a full (positive?) set of parameters $t \in [0, 1]$?*

This time, my intuition is that the answer would be "no", contracdicting my intuition for the previous question!

Other notion use families with infinitely many parameters, using a probability on the Hilbert cube.

This different notions lead to the philosophical question:

What is the good notion of considering "most of all diffeomorphisms"...?

Digression 2: Many people ask *Why the C^1-topology?*, and most of them means, why not C^2, C^r C^∞, analytic?

There is nothing philosophical here..It is just that, for the C^r-topology ($r > 1$) only few phenomena are understood. In contrast, for the C^1 topology, one has now some hope to present a global picture of the dynamic of generic diffeomorphisms.

Recently, Fançois Béguin ask me: *Why not the C^0-topology ?* In fact I never though deeply on this, but my intuition was that everything was very unstable in the C^0 topology, so that nothing interesting could be told on C^0-generic homeomorphisms. Preparing this course, tried to organize my arguments, but indeed I was completely wrong. Let me begin by some words on C^0-generic homeomorphisms.

3.2 C^0-generic homeomorphism

3.2.1 In dimension 1

$Homeo(M)$ is a complete space for the metric $sup\{d(x, f(x) + d(x, f^{-1}(x)), x \in M\}$, that is, for the uniform convergence of f and f^{-1}. As a consequence it is a Baire space. Moreover it admits a countable basis of open set.

The dynamics of generic homeomorphisms is very complicated, but in some sense, uniformly complicated:

Theorem 3.1. *There is a residual subset $\mathcal{R}_0$ of orientation preserving homeomorphism of the circle S^1 with the following properties:*

1. *All $f \in \mathcal{R}$ has a rational rotation number;*
2. *Let $f, g \in \mathcal{R}$ and $p, q \in \mathbb{Z}$ such that the rotation numbers of f^p and g^q are equal. Then f^p and g^q are conjugated by an orientation preserving homeomorphism;*

Lemma 3.1. *There is a open and dense subset of $Homeo_+(S^1)$ of homeomorphisms having a rationnal rotation number, and this rotation number is locally constant on this open set.*

Démonstration : Just notice that any homeomorphism f may be perturbed in a homeomorphism g having a periodic point x; let π denotes the period of x. Up to considering another perturbation, one can assume that the graph of g^π cuts transversally the diagonal at x. Now a neighborhood of g has the same rotation number has g. □

Lemma 3.2. *Let f be a generic homeomorphism with rotation number $\frac{p}{q}$ with $p \wedge q = 1$. Then the set of periodic points of f is a Cantor set.*

Démonstration : One kows that all the periodic points have the same period (q) so that Per(f) is compact. It sufices to verify that Per(f) has no isolated points. For any homeomorphism f with $\rho(f) = \frac{p}{q}$, let denote $\delta(f) = \max\{d(x, \text{Per}(f) \setminus \{x\} \mid x \in \text{Per}(f)\}$. It suffices to show that $\delta(f) = 0$ for generic f.

For that one show that $\{f \mid \delta(f) < \varepsilon\}$ contains an open and dense subset in the set of homeomorphisms with rotation number $\frac{p}{q}$. This is easily obtained by a local argument: any isolated periodic points may be perturbed in a set of periodic points wery close one to each others. □

For a homeomorphism f of S^1 having rational $\frac{p}{q}$ rotation number, the sign of $f - id$ is well defined between two successive periodic points (that is, in any connected component of $S^1 \setminus \text{Per}(f)$).

Lemma 3.3. *Assume that I is an interval of S^1 and that $J_1, J_2 \subset J$ are two distinct components of $S^1 \setminus Per(f)$ on which the sign of $f - id$ is th same. Then there is $J_3 \subset I$, between J_1 and J_2, on which the sign of $f - id$ is different from the sign of $f - id$ on J_1 and J_2*

Démonstration : Just notice that, for generic homeomorphism, any periodic point is accumulate by intervals of both signs for $f - id$. □

One conclude the prof of the theorem by the following lemma:

Lemma 3.4. *Two homeomorphisms f and g with the same rotation number, whose sets of periodic points are Cantor sets, and verifying the property of Lemma 3.3 are conjugated.*

I do not know if a so strong statments may hold in higher dmension. Let present a generalisation that holds in dimension 2(and may be in any dimension):

3.2.2 In dimension 2

Let S a compact surface and $\varphi_i\colon U_i \to \mathbb{R}^2$ a covering of S by charts (relatively compact in larger charts for avoidin boundary-like problems). Denote by $\mathbb{D}^2$ the unit disk of $\mathbb{R}^2$. We will say that a homeomorphism f is *k-universal* if it admits two families $\mathcal{D}^+$ and $\mathcal{D}^-$ of disks, each embedded is one U_i, verifying the following properties:

1. each disk $D \in \mathcal{D}^+$ is dsjoint from its iterates $f(D), \dots, f^{k-1}(D)$ and $f^k(D) \subset D$.
2. each disk $D \in \mathcal{D}^+$ is dsjoint from its iterates $f(D), \dots, f^{k-1}(D)$ and $f^k(D) \subset D$.
3. for any two distinct disks $D_1, D_2 \in \mathcal{D}^+ \cup \mathcal{D}^-$ the unions of their k first iterates are disjoint.
4. for any $D \in \mathcal{D}^+$,(resp. $\mathcal{D}^-$) there is $i(D)$ such that $D \subset U_{i(D)}$ and D is a ball for the euclidian metrics in the coordinates of $U_{i(D)}$. One denote by $f_D\colon \mathbb{D}^2 \to \text{int}(\mathbb{D}^2)$ the homeomorphism obtained by renormalization (conjugation by a homothety-translation) of the restriction of f^k (resp. f^{-k}) to D.
5. for any open set $\mathcal{O}$ of $\text{Homeo}_+(\mathbb{D}^2, \text{int}(\mathbb{D}^2))$ there is $D^+ \in \mathcal{D}^+$ and $D^- \in \mathcal{D}^-$ such that $f_{D^+} \in \mathcal{O}$ and $f_{D^-} \in \mathcal{O}$.

Theorem 3.2. *Let S a compact surface and $\varphi_i\colon U_i \to \mathbb{R}^2$ a covering of S by charts. There is a residual subset $\mathcal{R} \subset Homeo_+(S)$ of diffeomorphism f verifying the following properties:*

1. *f has a periodic orbit x;*
2. *for each $k > 0$, period of a periodic orbit of f, f is k-universal.*

Démonstration : The C^0 closing lemma allows to build f_1 arbitrarily close to f ad having a periodic orbit x of period p. Then there is f_2 arbitrarily close to f_1 having a periodic disk D_0, disjoint from $f_2(D_0), \cdots, f_2^{p-1}(D_0)$ on which f_2^p is the identity. Finally there is f_3 arbitrarilly close to f_2 and having a strictly invariant disk $D \subset D_0$, $f_3(D) \subset \text{int}(D)$ and $f_3|D$ is conjugated to an element of $\mathcal{O}$, by the composition of a chart φ and an homothety-translation A. This property is open and we just proved that it is dense.

Using a countable basis of the topology of $\text{Homeo}_+(D^2, \text{int}(D^2))$ one gets that generic dhomeomorphisms verify this property simultanuously for all open set $\mathcal{O}$.

□

Corollary 3.5. *Any generic homeomorphism of a compact surface S has infinite topological entropy.*

Question 1. *Are all generic homeomorphisms of the sphere S^2 conjugated?*

On any surface, are the generic homeomorphisms structurellement stable behind the world of generic homeomorphisms*?*

In this direction, Sylvain Crovisier notices that Theorem 3.2 implies that the conjugacy class of any generic (orientation preserving) homeomorphism of S^2 is dense in $\text{Homeo}_+(S^2)$.

3.3 Genericity in C^r-topology $r \geq 2$

Very few is known! The following open problem show how large is our ignorance[2]:

Problem 1. *Among C^2 diffeomorphisms of the torus T^2 which are isotopic to identity, are those having a periodic orbit dense for the C^2 topology?*

Let just mention an important result on C^r-generic diffeomorphisms, $r \geq 1$: the Kupka-Smale theorem:

Theorem 3.3. *(Kupka-Smale) For any $r \geq 1$, there is a residual subset $\mathcal{R}$ of $Diff^r(M)$ such that, for any $f \in \mathcal{R}$ any periodic point x is hyperbolic and for any periodic points x, y the invariant manifolds $W^s(x)$ and $W^u(y)$ are transverse.*

This theorem is the generic consequence for diffeomorphisms of Thom's transversality theorem.

Notice that the set of Kupka-Smale diffeomorphisms is disjoint from the set of generic homeomorphisms described in Theorem 3.2.

3.4 semi-continuity and genericity

Let M be a compact metric space and $\mathcal{K}(M) \subset \mathcal{P}(M)$ be the set of compact subsets of M, endowed with the Hausdorff metric. Let X be a topological space and $\varphi\colon X \to \mathcal{K}(M)$ be a map.

Definition 3.6. 1. *The map φ is* lower semi-continuous *if for any $x \in X$ and any $\varepsilon > 0$ there is a neighborhood U of x such that, for any $y \in U$ the compact set $\varphi(x)$ is contained in the ε-neighborhood of $\varphi(y)$. In other word:*

$$\forall p \in \varphi(x) \exists q \in \varphi(y), d(p, q) < \varepsilon$$

2. *The map φ is* upper semi-continuous *if for any $x \in X$ and any neighborhood $\mathcal{V}$ of $\varphi(x)$ there is a neighborhood U of x such that, for any $y \in U$ the compact set $\varphi(y)$ is contained in $\mathcal{V}$. In other word, if y_n converges to x and $p_n \in \varphi(y_n)$ and $p_n \to p$ then $p \in \varphi(x)$.*

Exercise 16. 1. *The closure of set of hyperbolic periodic points of f varies lower semi-continuously*

2. *The chain recurrent set varies upper-semi continuously.*

Theorem 3.4. *Let X be a complete metric space with a countable basis of open sets, and M be a compact metric space. Let $\varphi\colon X \to \mathcal{K}(M)$ be a semi-continuous function. Then there is a residual subset $\mathcal{R}$ of X such that φ is continuous at each point of $\mathcal{R}$.*

[2] The answer is positive among conservative diffeomorphisms of the torus T^2: the proof consisit in noticing that, componing the diffeomorphisms by a translation, one changes the rotation vector of Lebesgue measure, getting a rational rotation vector. Then one uses the non trivial fact that such a a diffeomorphisms admits a periodic orbit.