united nations educational, scientific and cultural organization

international atomic energy agency

the
abdus salam
international centre for theoretical physics

ICTP 40th Anniversary

SMR.1573 - 18

SUMMER SCHOOL AND CONFERENCE ON DYNAMICAL SYSTEMS

A global view of non-conservative dynamics

J. Palis
I. M. P. A.
Rio de Janeiro
Brazil

---

strada costiera, 11 - 34014 trieste italy - tel. +39 040 2240111 fax +39 040 224163 - sci_info@ictp.trieste.it - www.ictp.trieste.it

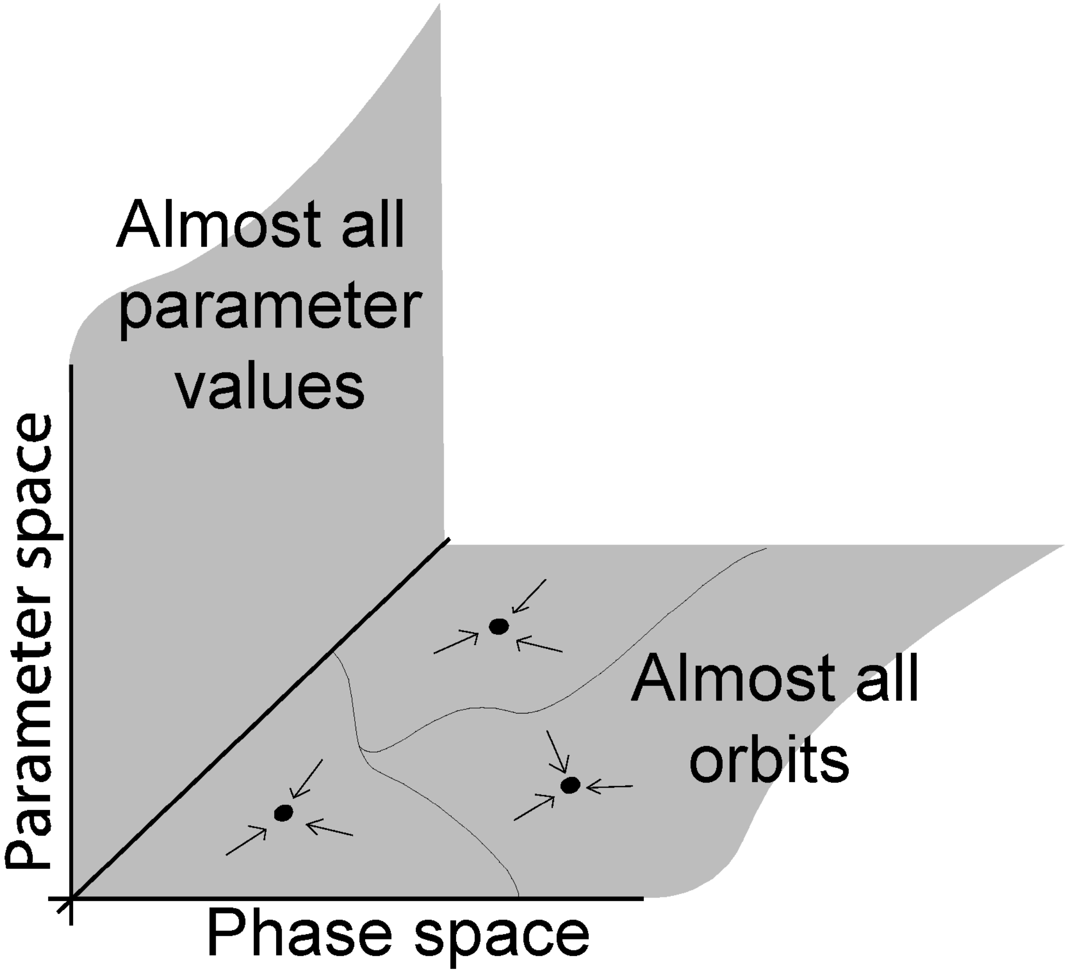
A Global View of Non-Conservative Dynamics
Recent Results

J. Palis ICTP 2004

Abstract **- We shall discuss a global conjecture on the finitude of large basin attractors and their stochastic stability: partial success, strategy and some related results on homoclinic bifurcations, dynamical robustness and partially hyperbolic systems. The aim of the conjecture is a description in a rather simple conceptual way of the long range behavior of a typical (positive) trajectory of a typical dynamical systems: a trajectory has only finitely many choices where to accumulate upon in the future.**

A global scenario is presented for dissipative dynamics, i.e. flows, diffeomorphisms and transformations.

The main focus is a conjecture on the denseness of systems having only finitely many attractors, the attractors being sensitive to initial conditions (chaotic) or just periodic sinks and the union of their basins of attraction having total probability. The attractors should be stochastically stable in their basins of attraction with respect to random perturbations in a finite dimensionally space of parameters.


Almost all
parameter
values
Parameter space
Almost all
orbits
Phase space

It is a probabilistic version of the once considered possible existence of an open and dense subset of systems with hyperbolic or dynamically stable structure, a main conjecture set up by Smale in the sixties, which evaporated by the end of that decade. Notice that

$$\text{hyperbolicity} \overset{C^1}{\sim} \text{structural (orbit) stability}$$

Most remarkably, around that time Kolmogorov when visiting IMPA – Rio de Janeiro, while I was visiting another institution abroad, stated that the global study of dynamical systems could not go very far without enriching it with new additional structures, like probabilistic ones.

Of course, the work of Sinai, Arnold and Anosov, among others, had already at that point such a flavour, but I interpreted Kolmogorov's view, as expressed to me by Elon Lima sometime afterwards, as referring to Smale's proposal of a global scenario for dynamics, and I kept it in my mind for years to come.

I take the occasion to pay Kolmogorov and Smale my tribute.

The collapse of the previously cited conjecture excluded the case of one dimensional dynamics. In fact, it is true for real quadratic maps that the hyperbolic ones are dense, as proved by Swiatek - Graczyk, and independently Lyubich; subsequently Kozlovski extended the result to C^3 unimodal mappings. Very recently, Kozlovski-Shen-Van Strien announced the density of hyperbolicity among C^r (multimodal) maps of the interval, for $r \geq 2$.
Actually, for one-dimensional real or complex dynamics, our main conjecture goes even further :

For most values of the parameters, the corresponding dynamical system displays finitely many attractors which are periodic sinks or carry an absolutely continuous invariant probability measure.

Remarkable recent positive results concerning our main conjecture:
A first breakthrough is due to Lyubich, using results of Martens and Nowicki and previously Sullivan and McMullen, among others. To state his result, consider the quadratic family

$$f_a(x) = ax(1-x), \quad f_a : [0,1] \circlearrowleft, \quad 0 < a \le 4$$

that plays a key role in the theory of dynamics of interval maps.

Lyubich:

For almost every a in (0,4], f_a either has a hyperbolic attractor (sink) attracting almost all orbits or else it is chaotic, i.e. it has an absolute continuous invariant probability measure, which is unique and ergodic (and so an SRB measure).

Avila - de Melo - Lyubich:

Same conclusion in the analytic case assuming negative Schwarz derivative and making use of a key structure (laminations corresponding to the topological conjugacy classes) in the space of infinitely renormalizable maps.

Avila-Moreira:

The main conjecture is true for non-degenerate analytic families, in particular the above quadratic one, and even just C^k families, $k \geq 2$ of unimodal maps. Moreover, surprisingly, there exists an explicit formula that gives the eigenvalues of periodic orbits as a function of corresponding kneading sequences and of the kneading sequence of the critical orbit, valid for almost all parameters in any generic analytic family of unimodal maps. They also show that the main conjecture holds in the complement of a set of positive codimension.

Next: the multimodal case? I believe that this is a likely development in the near future.

Definition of Sinai-Ruelle-Bowen Invariant Measures

SRB – invariant measures

Let ***A*** be an attractor for ***f***, and μ an *f*-invariant on ***A*** – attracts sets of positive Lebesgue probability

The triple (*f*, A, μ) is an SRB measure if

$$\lim_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} g(f^i(x)) = \int g\,d\mu$$

$x \in E \subset B(A)$, *B* basin of attraction for ***A***

with $m(E) > 0$

where *m* denotes Lebesgue measure

Definition of Stochastic Stability

Let (f, A, μ), $\mu \in$ SRB,

finitely many-parameter families of maps

Random Lebesgue choice of parameters give rise to maps f_j

Let $z_j = f_j \circ \ldots \circ f_1 (z_0)$, $z_0 \in B(A)$

f_j ε - C^r near f, $\varepsilon > 0$

Definition:

(f,A,μ) is stochastically stable if given a neighborhood V of μ in the weak topology, the weak limit of

$$\frac{1}{n}\sum_{j=0}^{n-1} \delta_{Z_j}$$

is in V for a.a. $(z_0, f_1, f_2, \ldots)$ if ε is small, where δ indicates Dirac measure

Dissipative Systems

space of events

We look for limit sets for

$\left\{ \begin{array}{c} \text{many} \\ \text{almost all} \end{array} \right\}$ trajectories → attractors

Attractor:

attract in the future all, almost all, most nearby, trajectories ~ sets of positive probability

For all systems?

"most", "almost all"

"Small accidents are less relevant"

space of systems, parameters

Attractor :

- point
- circle
- all space of events

torus

hyperbolic attractors, 60'

$$\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$$

Other attractors:

- “butterfly” Lorenz 63, 70’
- Hénon, 70’: expansion and folding
  segment x fractal

strange attractor / chaotic

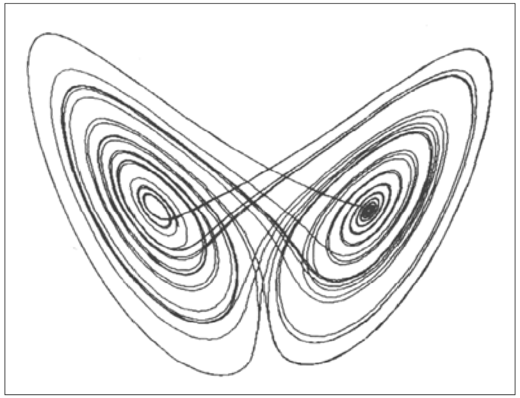
Pioneer work of Kolmogorov in fluid dynamics
Later, in the early 70’, May in the context of population growth

algebraically simple
cubic eqs. in 3 variables
Robinson-Rychlik

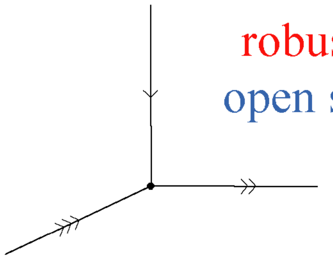
geometric examples, 1997
Guckenheimer, Williams
Afraimovich, Bykov, Shill'nikov

quadratic - Tucker (Viana : Math Intelligencer - V. 22, nº 3, 2000)

Lorenz, 1963

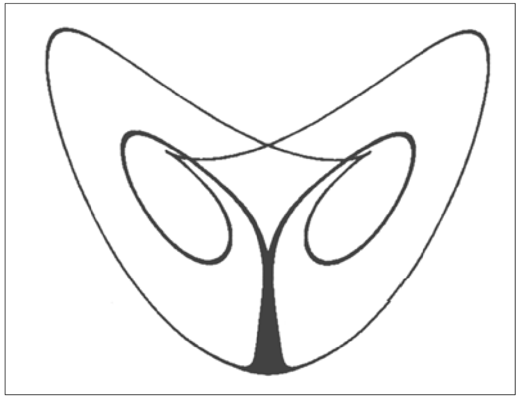


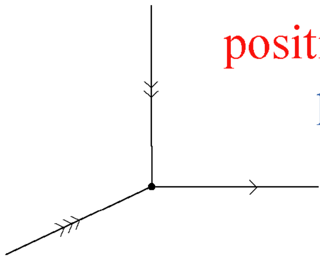
$$\begin{cases} \dot{x} = 10x + 10y \\ \dot{y} = 28x - y - xz \\ \dot{z} = -\frac{8}{3}z + xy \end{cases}$$



robust
open set

Rovella, 1992
(Arneodo - Coullet - Tresser)





positive probability
persistent

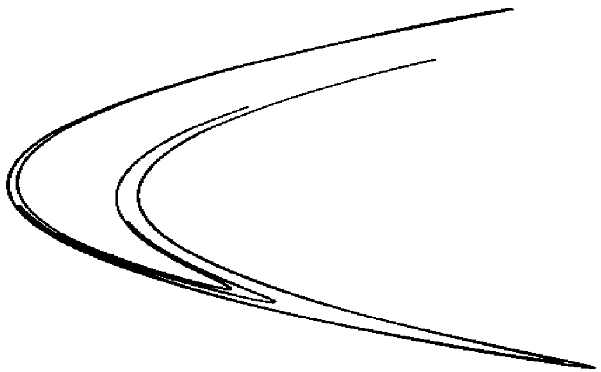
higher dimension : Bonatti-Pumariño-Viana

Hénon Transformation

$$f_{a,b}(x,y)=(1-ax^2+y,bx) \qquad a\approx 1.4, b\approx 0.3$$

1D : Feigenbaum, Coullet-Tresser period doubling

Jakobson 1D, Benedicks-Carleson, Mora-Viana, Viana, Benedicks-Young, Benedicks-Viana*



Hénon Attractor

*No holes in the basin of attraction, stochastically stable.

Ruelle-Sinai' question

Hyperbolic Diffeomorphism f

hyperbolic limit set L

$$T_L M = E^s \oplus E^u$$

$$df \left| E^s, \; df^{-1} \right| E^u$$ contractions

$$X_t, \; t \in R$$

Flow $\quad$ **$C > 0, \quad 0 < \lambda < 1$**

$$\left\| dX_t \mid E^s \right\|, \left\| dX_{-t} \mid E^u \right\| \leq C \, e^{\lambda t}, \; t \in R$$

$$\text{hyperbolicity} \quad \overset{C^1}{\sim} \quad \text{structural (orbit) stability}$$

$\Rightarrow$ Anosov, Palis-Smale, Robbin, de Melo, Robinson

$\Leftarrow$ Mañé, Hayashi

Liao, Sannami, Pliss, Doering, Hu, Wen

Smale´s Conjecture in the 60' :

Every system can be approximated by a hyperbolic one

Counter-Examples

Smale, Abraham-Smale, Simon

More striking, Newhouse: arithmetic difference of "thick" Cantor sets generating infinitely many simultaneous sinks.

Exceptions :

Flows on disks

Andronov-Pontryagin (1937)

Flows on orientable surfaces

Peixoto (1960)

1D real, recent result of (Kozlovski-Shen-Van Strien)

1D complex ??

Persistent attractor :

Exists with positive probability (Lebesgue) in parameter space

1D *quadratic transformation* Jakobson

2D *quadratic diffeomorphism* Benedicks-Carleson

3D flow Lorenz-Rovella

- They all carry physical or SRB (Sinai-Ruelle-Bowen) invariant measures
  Jacobson, Benedicks-Young
  Sinai, Kifer, Metzger
- They are also stochastically stable
  Hénon-like attractor: Benedicks-Viana
  Lorenz, Lorenz-Rovella: Kifer, Metzger

Global Conjecture on Finitude of Attractors and Metric Stability

(I) There is a dense set ***D*** of dynamics such that any element of ***D*** has finitely many attractors whose union of basins of attraction has total probability;

(II) The attractors of the elements in ***D*** support a physical (SRB) measure;

(III) For any element in ***D*** and any of its attractors, for almost all small perturbations in generic ***k-parameter*** families of dynamics, $k \in N$, there are finitely many attractors whose union of basins is nearly (Lebesgue) equal to the basin of the initial attractor; each such perturbed attractor supports a physical measure;

(IV) Stochastic stability of attractors - the attractors of elements in ***D*** are stochastically stable in their basins of attraction;

(V) For generic families of one-dimensional dynamics, with total probability in parameter space, the attractors are either periodic sinks or carry an absolutely continuous invariant measure. The same can be asked in higher dimensions.

Possible Strategy: to focus on homoclinic bifurcations.

Paraphrasing Poincaré (Les Méthodes Nouvelles de la Mécanique Céleste): Homoclinic behavior is at the heart of the difficult problems in dynamics.

Indeed, there is a better chance of proving the Global Conjecture when there are no "homoclinic tangencies" in a robust way.

The reason is that robust absence of tangencies implies a certain amount of hyperbolicity:

Let $\Lambda \subset \mathbf{L}(f)$ limit set compact, maximal, invariant subset in a neighborhood.

Suppose that for any C^1 perturbation g of f the continuation Λ_g of Λ contains no tangencies between stable and unstable manifolds of periodic orbits, THEN Λ has a dominated decomposition

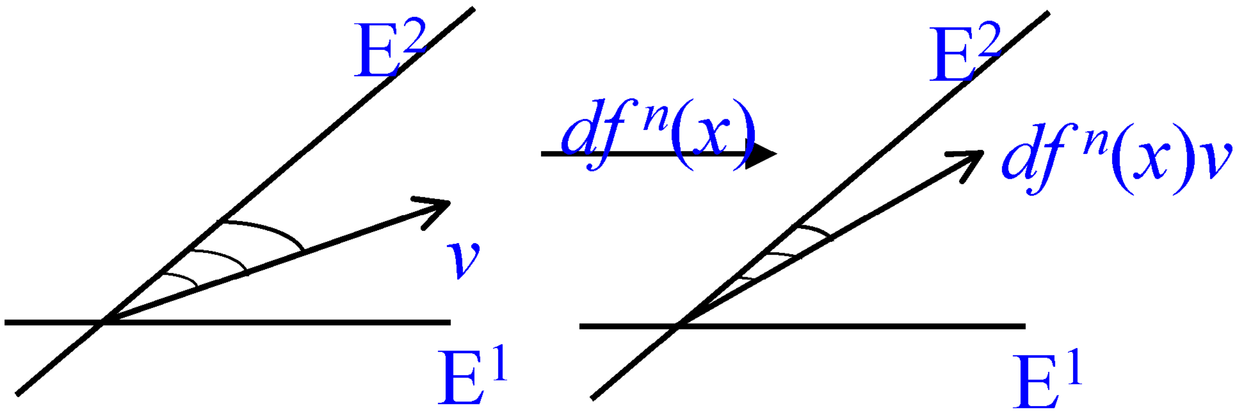
[Pujals-Sambarino: dim 2, Wen: general]

Let us define dominated decomposition:

Definition: Dominated Decomposition

$$T_\Lambda M = E^1 \oplus E^2 \qquad \begin{array}{l} \exists\, C > 0 \\ \exists\, \lambda < 1 \end{array}$$

$$\frac{\left\| df^n(x) v^1 \right\|}{\left\| df^n(x) v^2 \right\|} \le c\lambda^n \frac{\left\| v^1 \right\|}{\left\| v^2 \right\|} \qquad \forall\, n \ge 1$$



Definition: Partial Hyperbolicity

$$T_\Lambda M = E^s \oplus E^c \oplus E^u$$

$$E^{cs} = E^s \oplus E^c, \quad E^{cu} = E^c \oplus E^u$$

$$E^s \text{ unif contracting}, \quad E^u \text{ unif expanding}$$

$$E^u \text{ dominates } E^{cs}, \quad E^{cu} \text{ dominates } E^s$$

Many New Results on SRB Probability Measures.

We mention the following one:

$$K \subset M \text{ compact posit invar for } C^2 \text{ diffeo } f$$

Theorem:

E^{cs} uniformly contract E^{cu} non-uniformly expanding on $H \subset K$, H positive Lebesgue. Then, H covered *mod 0*, by finitely many SRB measures (Alves-Bonatti-Viana; some extension by Vasquez)

Remarkable Result by Tsujii:

$C^r, r \geq 20$, surfaces endomorphisms which are partially hyperbolic with 1D central subbundle, generically carry finitely many ergodic SRBs whose union of basins of attraction has total probability.

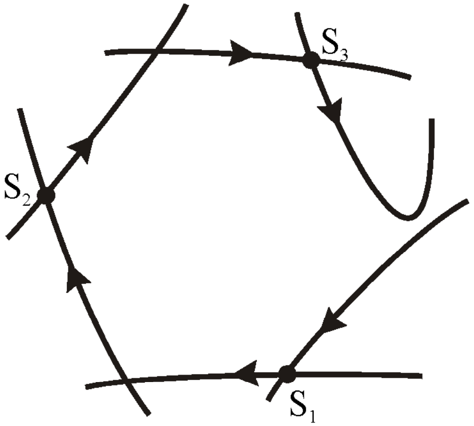
Probably, Tsujii's results can be substantially generalized in higher dimensions.

Conjecture (Viana, 1997):

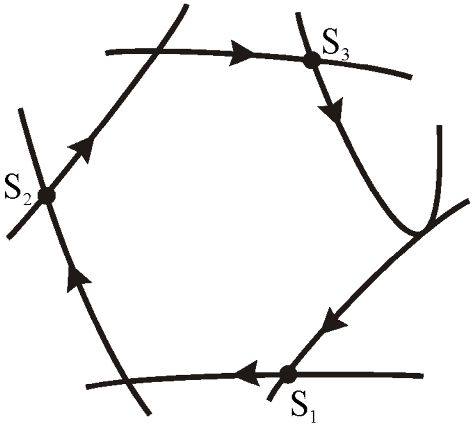
Generically (densely), partially hyperbolic diffeos with a uniformly expanding subbundle carry finitely many SRB measures whose union of basins of attraction has total Lebesgue probability.

Homoclinic Behavior

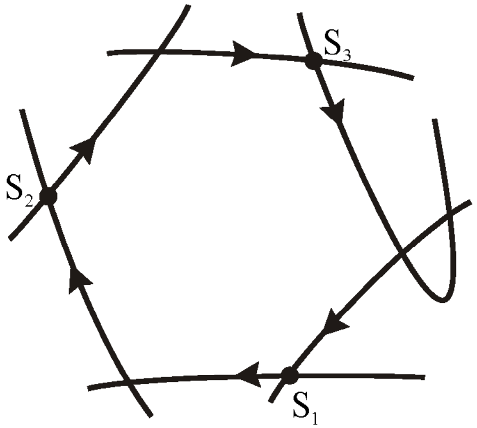
Poincaré	Initiator, provided some of the basic facts
Birkhoff Smale	Birkhoff transversal homoclinic orbits on surfaces. Van der Pol, Cartwright-Littlewood, Littlewood, Levinson, Levi, Smale's horseshoe in general.
Andronov	pioneer work on structure stability (with Pontryagin) and bifurcation theory
Arnold and school	basic work on bifurcation theory
Shil'nikov and school	transversal homoclinic orbits and homoclinic bifurcations
Newhouse-Palis- Takens-Viana- Yoccoz-Moreira	bifurcation theory – homoclinic bifurcations



No cycle



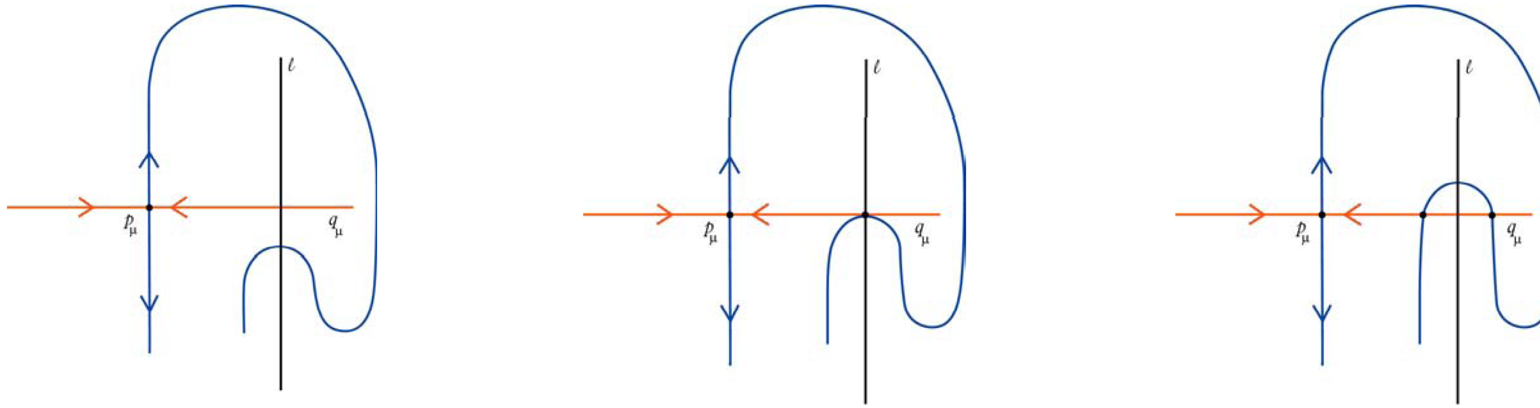
Creating an unstable cycle



A stable cycle

We were generally baffled by cycles in dynamics. Creation of cycles unavoidable when varying parameters. Should occupy a small part of the space of events.

Unfolding homoclinic tangencies



– Infinitely many simultaneous sinks residually in intervals in parameter line (Newhouse)
– Hénon-like attractor (Mora-Viana extending Benedicks-Carleson)
– Infinitely many simultaneous Hénon-like attractors (Colli)
– Arbitrary fast growth of number of periodic points generically in open Newhouse set (Kaloshin, based on Shil'nikov-Gonchenko-Turaev)

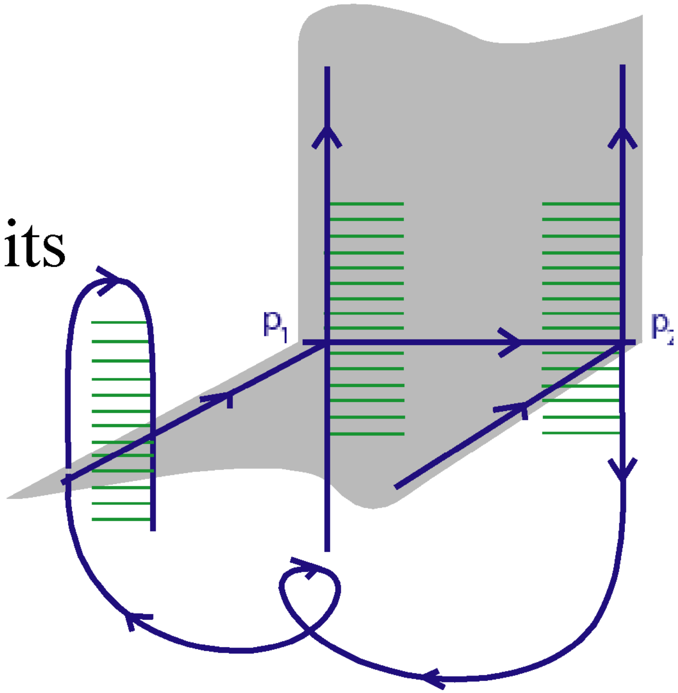
Koloshin communicates progress in the conjecture that infinitely many sinks should be of zero probability in parameter line.

Following the cited strategy, we pose

Conjecture: In any dimension, every diffeomorphism can be approximated by a hyperbolic one or one exhibiting a homoclinic tangency or by one with a (finite) cycle of hyperbolic periodic orbits with different stable dimensions, i.e. heterodimensional cycle.

Heterodimensional cycle:

Creation of (transverse) homoclinic orbits without going through tangencies of stable and unstable manifolds, see fig.

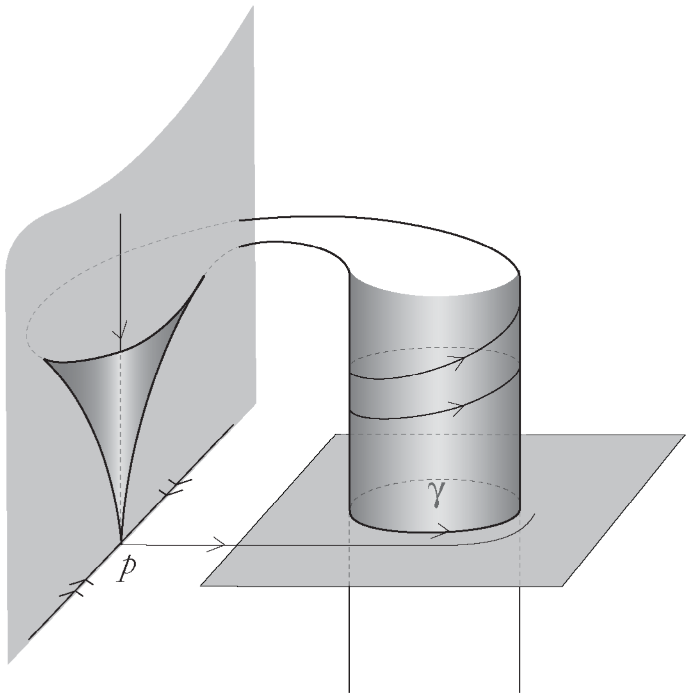


Remarkable Result

- True for C^1 surface diffeomorphisms,
Pujals and Sambarino, Annals of Math. 2000

The authors are announcing the same result in 3D.

For flows, we can formulate a similar conjecture adding as key elements Lorenz-like attractors and singular cycles.



A singular cycle

Definition: A singular hyperbolic invariant set is a partially hyperbolic one with volume expanding or contracting central bundle.

Conjecture for Flows:

Every flow in 3D can be C^k, $k \geq 1$, can approximated by

- a hyperbolic one, no cycles or
- one with homoclinic tangency or
- a singular hyperbolic one (Lorenz-like), no cycles

Recent Results, C^1

Theorem: Lorenz-like attractors in 3D are characterized as robust, transitive with a singularity. This is a beautiful result of Morales, Pacifico and Pujals, Annals of Math. 2004

Theorem (Arroyo, Rodriguez-Hertz):

Every flow in 3D can be approximated by

- hyperbolic one, or
- one with a homoclinic tangency, or
- one with a singular cycle (has singularity)

Annales Inst. H. Poincaré, Anal. Non Lineaire, 2004

Other results for diffeomorphisms

Robustness-definition

f diffeomorphism, Λ max invariant, transitive

Λ_g continuation of Λ, ***g*** C^1 near ***f***, implies Λ_g transitive

Results:

Mañé 2 dim, robustness

$\Rightarrow$ hyperbolicity

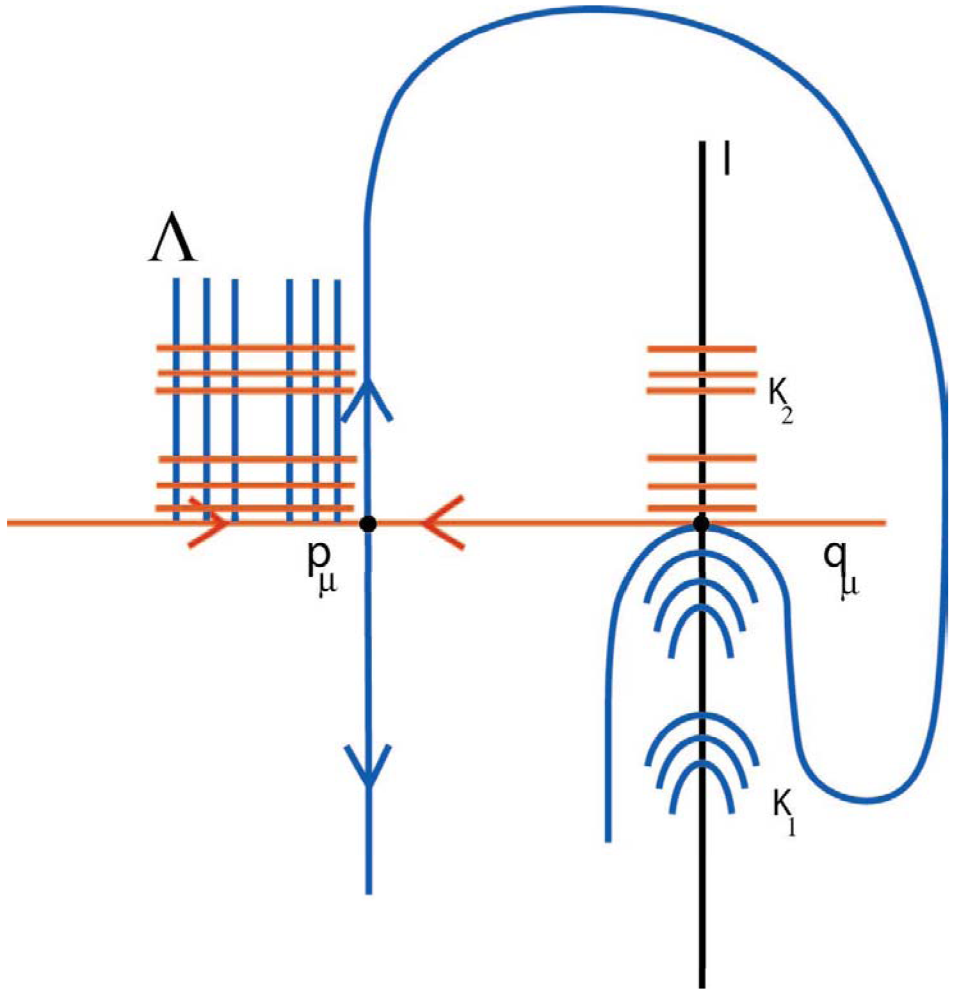
Diaz-Pujals-Ures 3 dim, robustness

$\Rightarrow$ partial hyperbolicity

Bonatti-Diaz-Pujals *n* dim, robustness

$\Rightarrow$ dominated decomposition

We have now a better understanding of the (complicated) dynamics when bifurcating homoclinic tangencies, say associated to a hyperbolic set Λ


Λ
l
K_2
p_μ
q_μ
K_1

(a) If the Hausdorff dimension of Λ, HD(Λ), is smaller than one, then hyperbolicity is fully prevalent at the bifurcating parameter value $\mu = 0$. (Palis-Takens)

(b) If HD(Λ) is bigger than one, then hyperbolicity is not fully prevalent at $\mu = 0$. (Palis-Yoccoz)

(c) If HD(Λ) is bigger than one, then K2 - K1 contains nonempty intervals. Moreira-Yoccoz and it implies b) in a strong sense.

(d) If HD (Λ) is bigger than but close to one, then the parameter values such that the corresponding diffeomorphism display attractors have density zero at $\mu = 0$.

(Palis-Yoccoz in a work about to appear - Resumé, CRAS, 2001)

Actually, for many parameters values (total density at $\mu = 0$), the continuation of the positive maximal invariant set in a neighborhood has HD < 2.

Similarly for the negative maximal invariant set. It's to be remarked that although we may have tangencies in intersection of these maximal positive and negative invariant sets, we don't have attractors/repellers.

The proof goes by showing that, for most parameters, the limit set is hyperbolic, in a delicate, non-uniform sense. Essentially, although the limit set may contain tangencies, these correspond to very special points: at "most" points there are transversal directions which are (asymptotically) contracted by forward and backward iterates, respectively.

The proof of this fact requires a very careful analysis of how trajectories return close to the tangencies and, even, the very definition of what a “tangency” is. To ensure hyperbolic behavior, such returns should not be too frequent nor too close. This is achieved by parameter exclusions, which turns out to be less and less significant near the original tangency parameter (Lebesgue density). The rate of formation of tangencies is a crucial ingredient, and it is closely related to the Hausdorff dimension of the original horseshoe. The assumption that this dimension is not far from 1 ensures that the number of tangencies that must be considered at each stage grows fairly slowly, so that a fairly small amount (in measure) of parameter exclusions is needed. Returns close to the tangencies yield quadratic type folds. The condition on the frequency and depth of returns is used to ensure that folds always are “ironed-out” before a new return occurs. In this way, one never has to deal with contacts of order bigger than 2. One can certainly expect to have a similar general result without imposing any restriction on the Hausdorff dimension of the original horsehsoe, at the price of having to deal with higher order contacts.

On the horizon lies the case of area preserving maps, like the standard map, for which the limit set has (the whole ambient space) Hausdorff dimension 2 and where one should have to deal with contacts of all order simultaneously.

Recent Result in Higher Dimensions:

the principle hyperbolicity prevails

$\Leftrightarrow$

Hausdorff dimension is smaller than 1

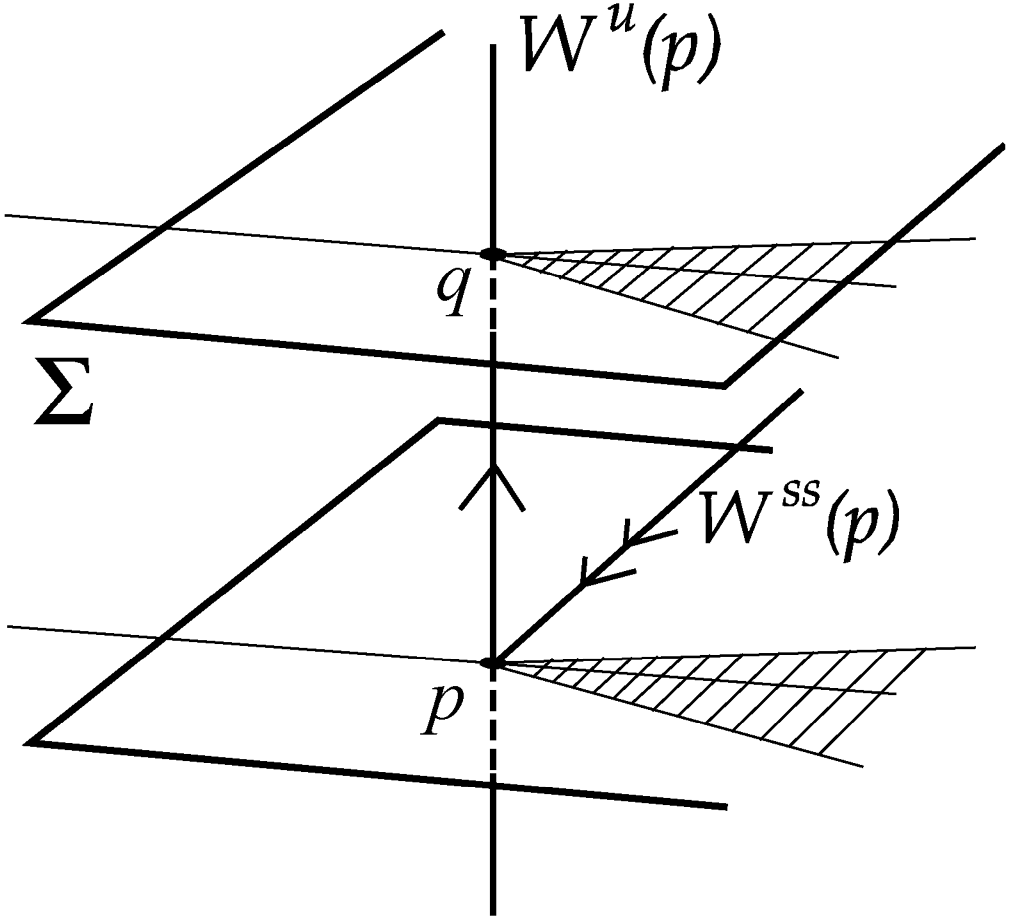
Remains true in higher dimensions. Namely:

There exist open subsets R_1, R_2 in $Dif^k (M)$, $k \geq 2$ such that

a) $\mathbf{R_1} \cup \mathbf{R_2}$ it is open and dense in the set of diffeomorphism exhibiting a “first” homoclinic tangency

b) for f in $\mathbf{R_1}$, f exhibits a horseshoe **HD < 1** and f is a density point of hyperbolic diffeomorphisms

c) for f in $\mathbf{R_2}$, f exhibits a horseshoe with **HD > 1** and f is NOT a density point of hyperbolic diffeomorphisms



$W^u(p)$
q
Σ
$W^{ss}(p)$
p

Part (b) is reminiscent of the Palis-Takens' result on surfaces. The proof here goes by ensuring that, up to small perturbation, the horseshoe avoids the strong (stable and unstable) directions. This is possible because its Hausdorff dimension is small. It has the geometric consequence that the horseshoes is roughly two-dimensional, which permits to mimic the two-dimensions arguments to obtain uniform transversality of stable and unstable foliations of the continuation of the original horseshoe for most parameters.

Part (c) is considerably more delicate. The main ingredient is the construction, after perturbation of f, of strong-stable and strong-unstable foliations of codimension 1 for hyperbolic subsets of Λ with almost the same upper Hausdorff dimensions.

These foliations are used to reduce the study of the geometries of the stable and unstable foliations near the initial homoclinic tangency to the 2-dimensional case. Another key step is to obtain stable tangencies in terms of parameters of stable and unstable manifolds of periodic orbits as in Moreira-Yoccoz, also cited above

Moreira-Palis-Viana, CRAS, Paris, 2000

Global References:

- J.Palis, Global View of Dynamics, Astérisque, 2000, vol. in honour of Adrien Donady.

- C. Bonatti, L. Diaz and M. Viana, Dynamics Beyond Uniform Hyperbolicity, book by Springer, August 2004.