united nations educational, scientific and cultural organization

international atomic energy agency

the
abdus salam
international centre for theoretical physics

ICTP 40th Anniversary

SMR.1573 - 12

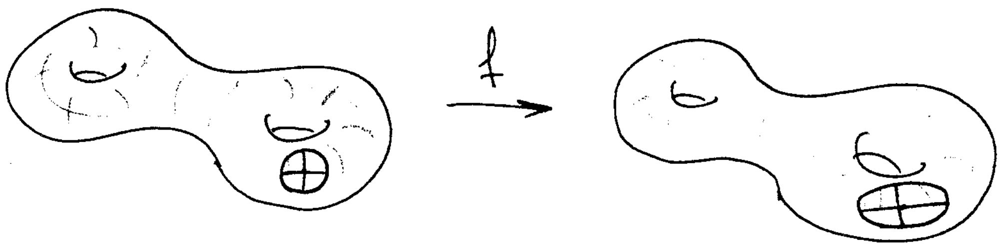
SUMMER SCHOOL AND CONFERENCE ON DYNAMICAL SYSTEMS

Flat surfaces, Interval exchange Transformations & Moduli Spaces of Abelian Differentials

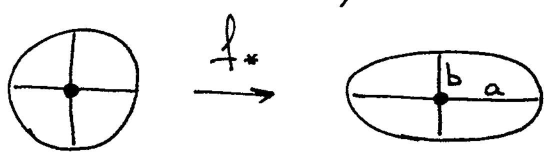
(Lecture 5)

Anton Zorich
Universite de Rennes 1
I.R.M.A.R.
Rennes, France
& MPI, Bonn, Germany

CRASH COURSE IN TEICHMÜLLER THEORY



When the complex structures are different there is no conformal map which sends one to another. A map f sends an infinitesimal circle at a poin $x \in M^2$ to an infinitesimal ellipse:



Coefficient of quasiconformality of f at $x \in M^2$ is the ratio $K_f(x) = \dfrac{a}{b}$.

DEFINITION Coefficient of quasiconformality of f is

$$K(f) \overset{\text{def}}{=} \sup_{x \in M^2} K_f(x)$$

TEICHMÜLLER THEOREM

Choose any two complex structures on a surface M_g^2 of genus g.

THEOREM (Teichmüller; developping ideas of Grotzsch)

- There exist an extremal map f_0 which minimizes coef. of quasiconform. $K(f)$
- For this extremal map the coefficient of quasiconformality is constant:

$$K_{f_0}(x) = K(f_0) \quad \forall\, x \in M_g^2 - \left\{\begin{matrix}\text{finite number} \\ \text{of points}\end{matrix}\right\}$$

- One can choose *flat** metrics (compatible with the complex structures) in which the foliation along large (corresp. small) demi-axes of ellipses is the horizontal (corresp. vertical) foliation in the flat metric.

- In flat coordinates the map f_0 is just expansion – contraction with coeff. $\sqrt{K}$

---

* These flat metrics are slightly more general than those which we considered: they allow holonomy $\vec{v} \longmapsto -\vec{v}$ and correspond to quadratic differentials.

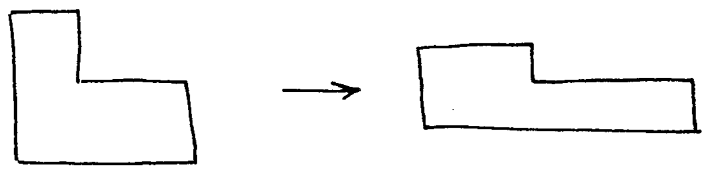
TEICHMÜLLER METRIC AND TEICHMÜLLER GEODESICS

<u>DEFINITION</u> The distance between two complex structures is measured as

$$\frac{1}{2} \log K(f_0) ,$$

where $K(f_0)$ is the coefficient of quasiconformality of the extremal map.

It means that a quadratic differential (or a holomorphic 1-form) define a direction of deformation of complex structure and a geodesic in Teichmüller metric.



$$\begin{pmatrix} e^{d} & 0 \\ 0 & e^{-d} \end{pmatrix}$$

REMARK TEICHMULLER metric is not a Riemannian but Finsler metric (unfortunately it is not defined by a quadratic form in the tangent space but just by a norm).

$SL(2;\mathbb{R})$ - action on flat surfaces in geometric terms

Space of pairs $\left(\begin{array}{c}\text{complex}\\ \text{structure}\end{array}, \begin{array}{c}\text{quadratic}\\ \text{differential}\end{array}\right)$

is the (co)tangent bundle to the space of complex structures.

Space of pairs $\left(\begin{array}{c}\text{complex}\\ \text{structure}\end{array}, \begin{array}{c}\text{holomorphic}\\ \text{1-form}\end{array}\right)$

is a subbundle of special directions;

subspace $\mathcal{H}_1$ of holomorphic 1-forms corresponding to flat surfaces of area one is a "subbundle of unit vectors".

An $SL(2;\mathbb{R})$-orbit in $\mathcal{H}_1$ defines a diagram:

$$
\begin{tikzcd}
\left(\begin{array}{c}\text{Unit tangent}\\ \text{bundle to}\\ \text{hyperbolic plane}\end{array}\right) SL(2;\mathbb{R}) \arrow[r] \arrow[d] & \mathcal{H}_1 \left(\begin{array}{c}\text{"unit tangent}\\ \text{subbundle"}\end{array}\right) \arrow[d] \\
\left(\begin{array}{c}\text{Hyperbolic}\\ \text{plane}\end{array}\right) SL(2;\mathbb{R}) / SO(2;\mathbb{R}) = \mathbb{H}^2 \arrow[r] & \mathcal{T}_g \left(\begin{array}{c}\text{Teichmüller}\\ \text{space}\end{array}\right)
\end{tikzcd}
$$

The map $\mathbb{H}^2 \longrightarrow \mathcal{T}_g$ is an isometry, so its image in Teichmüller space is a complex geodesic (called Teichmüller disk).

HOPE FOR "RATNER THEOREM"

<u>THEOREM</u> (N. Shah, 1990 as a corollary of Ratner theorem)
In a compact manifold of constant negative curvature, the closure of a totally geodesic, complete (immersed) submanifold of dim ≥ 2 is a totally geodesic immersed submanifold.
(Generalization of T. Payne: target manifold of finite volume)

Moral: Complex geodesics are simple; (real geodesics are complicated).

HOPE The closure of <u>any</u> $SL(2;\mathbb{R})$-orbit in the space of holomorphic one forms (in the space of quadratic differentials) is a nice complex orbifold.

THEOREM (McMullen, 2003)
The hope is true for holomorphic 1-forms in genus two.

(unpublished)
THEOREM (Kontsevich, ~1999)
If the closure of an $SL(2;\mathbb{R})$-orbit is a nice complex submanifold (suborbifold) it is represented by an affine subspace in cohomological coordinates.

SEARCH FOR INVARIANT SUBMANIFOLDS

Teichmüller geodesic flow (and $SL(2;\mathbb{R})$) preserve number and types of conical singularities (= degrees of zeroes of a holomorphic 1-form). So they preserve the strata $\mathcal{H}_1(k_1, k_2, \dots, k_n)$.

EXAMPLE In genus $g=2$ there are two strata $\mathcal{H}_1(2)$ and $\mathcal{H}_1(1,1)$

In genus $g=3$ there are
$\mathcal{H}_1(4)$; $\mathcal{H}_1(1,3)$; $\mathcal{H}_1(2,2)$; $\mathcal{H}_1(1,1,2)$; $\mathcal{H}_1(1,1,1,1)$

REMINDER $\quad \sum_{i=1}^{n} k_i = 2g-2$

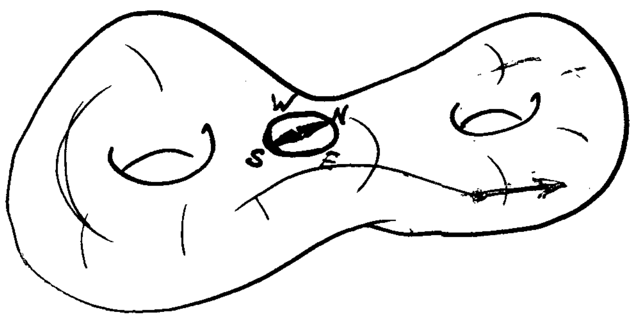
THEOREM (Masur, 1982 ; Veech, 1982)
$SL(2;\mathbb{R})$ (and Teichmüller geodesic flow) act ergodically on every connected component of every stratum $\mathcal{H}_1(k_1, \dots, k_n)$

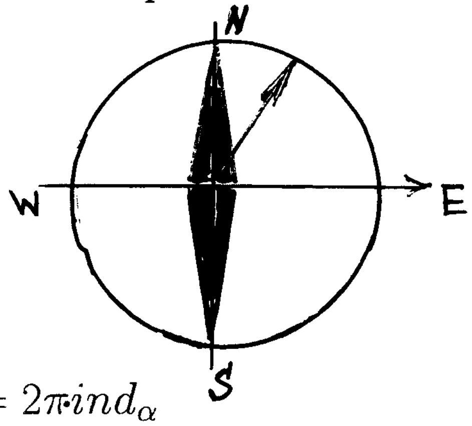
We shall discuss two extremal cases: **1**) when the closure of $SL(2;\mathbb{R})$-orbit is entire connected component of a stratum ; 2) when $SL(2;\mathbb{R})$-orbit is already closed.

CLASSIFICATION OF CONNECTED COMPONENTS

PARITY-OF-THE-SPIN-STRUCTURE: ALTERNATIVE DEFINITION

Let $\omega \in \mathcal{H}(2l_1, \ldots, 2l_n)$. Let α be a smooth simple closed oriented curve on C. Consider the Gauss map $\alpha \to S^1 \subset \mathbb{C}$.





Total rotation angle $= 2\pi \cdot ind_\alpha$

$ind_\alpha =$ degree of the Gauss map

Lemma. *The function*

$$\Omega([\alpha]) \stackrel{def}{:=} (ind_\alpha + 1) \ mod\ 2$$

is well-defined on $H_1(C; \mathbb{Z}/2\mathbb{Z})$ and has the property

$$\Omega(c_1 + c_2) = \Omega(c_1) + \Omega(c_2) + c_1 \cdot c_2$$

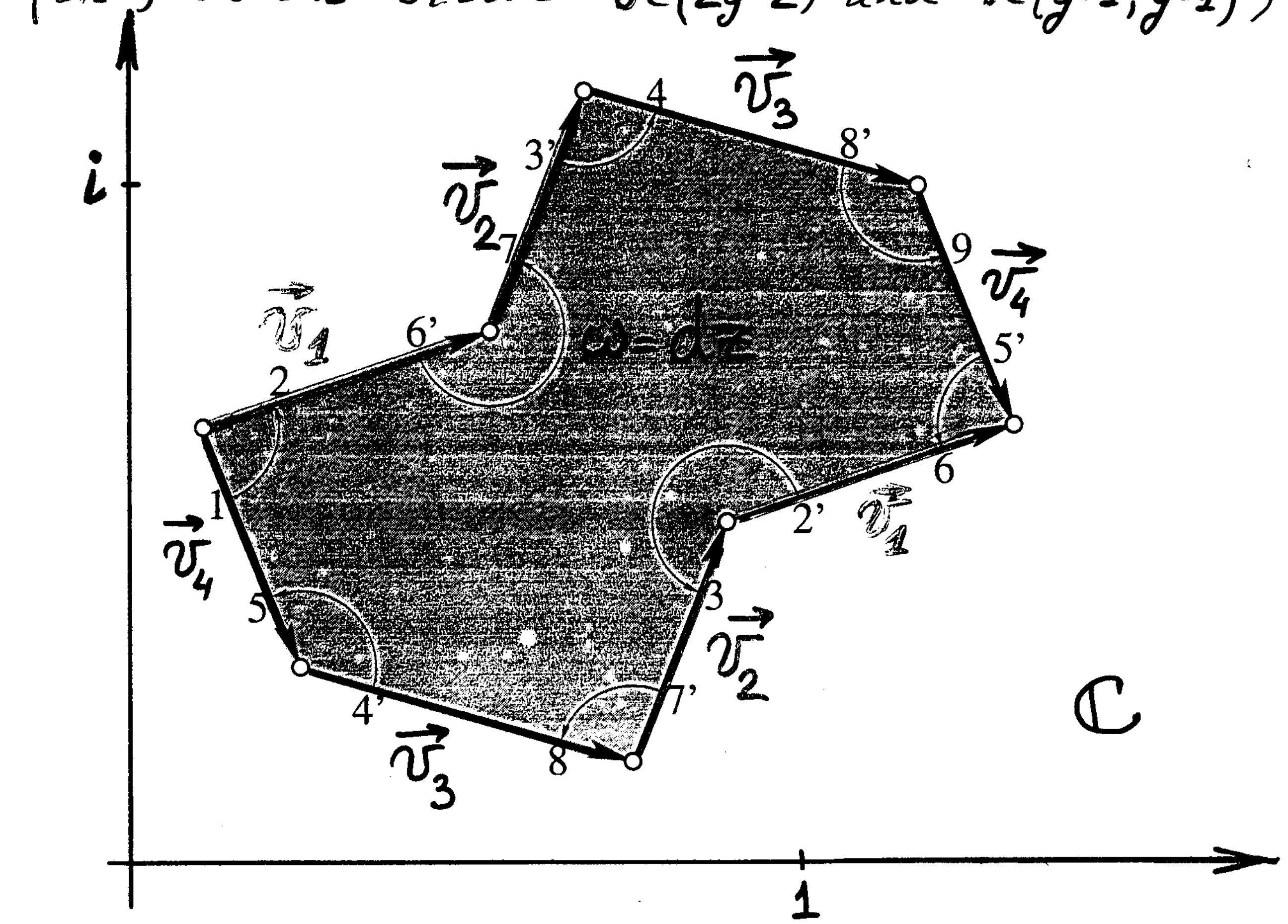
Lemma. *The value*

$$\varphi(\omega) = \sum_{i=1}^{g} \Omega(a_i)\Omega(b_i) \ mod\ 2$$

does not depend on the choice of a symplectic basis of cycles $\{a_i, b_i\}$ and equals the parity-of-the-spin-structure determined by ω.

HYPERELLIPTIC CONNECTED COMPONENTS

(only in the strata $\mathcal{H}(2g-2)$ and $\mathcal{H}(g-1,g-1)$)



A flat surface S belongs to a ~~hyperelliptic connected component~~ $\mathcal{H}^{hyp}(2g-2)$ or $\mathcal{H}^{hyp}(g-1,g-1)$ if it has an isometric involution $\tau : S \to S$, $\tau^2 = id$, such that

$$S/\tau = \text{sphere}$$

and in the case of $S \in \mathcal{H}(g-1,g-1)$ the involution τ interchanges the conical points.

CLASSIFICATION THEOREM
(HOLOMORPHIC 1-FORMS)

<u>THEOREM</u> (M. Kontsevich + A.Z.) Let $g \geq 4$.

- There are two series of hyperelliptic connected components: $\mathcal{H}^{hyp}(2g-2)$ and $\mathcal{H}^{hyp}(g-1, g-1)$

- When all degrees are even $\mathcal{H}(2k_1, \ldots, 2k_n)$ has exactly two nonhyperelliptic connected components $\mathcal{H}^{odd}(2k_1, \ldots, 2k_n)$ and $\mathcal{H}^{even}(2k_1, \ldots, 2k_n)$

- When g is even $\mathcal{H}(g-1, g-1)$ contains exactly one nonhyperelliptic connected component

- All other strata are connected.

<u>COROLLARY</u> This classifies the <u>extended Rauzy classes.</u>

CLASSIFICATION THEOREM
(quadratic differentials)

THEOREM (E. Lanneau, 2003)
- there are three hyperelliptic series;
- otherwise, there are four exceptional strata containing exactly two misterious connected components:

$Q(-1,9)$, $Q(-1,3,6)$, $Q(-1,3,3,3)$, $Q(12)$

REMARK Currently there are two ways to determine to which of the two connected components of an exceptional stratum belongs a flat surface.

- Either find a "generalized permutation" and check to which of the two extended Rauzy classes it belongs;
- OR: to find a configuration of saddle connections which does not occur for one of the two components

(Example: for one of the two components of $Q(-1,9)$ as soon as we have a saddle connection between "-1" and "9" we get a closed geodesic going in the same direction.)

VEECH SURFACES

def Flat surface S is a <u>Veech surface</u> if it has an enormous group of affine automorphisms, namely if
Stabilizer (S) is a lattice in $SL(2;\mathbb{R})$

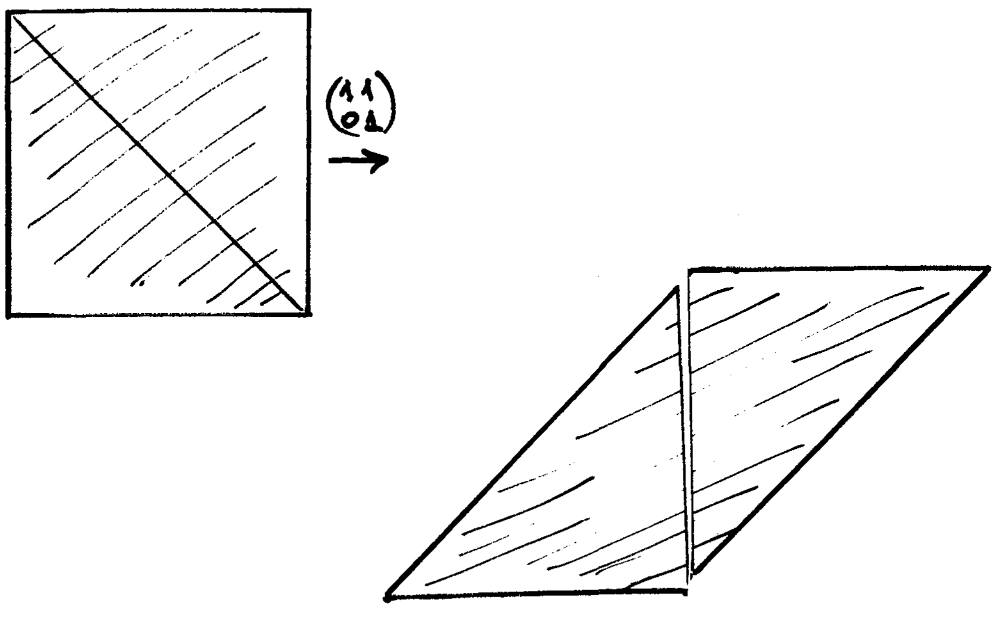
THEOREM $\left(\begin{smallmatrix}\text{Smillie}\\ \text{Veech}\end{smallmatrix}\right)$ $SL(2;\mathbb{R})$-orbit of S is closed in the stratum $\Longleftrightarrow$ S is a Veech surface

Forgetting "polarizations" of flat surfaces we get a <u>Teichmüller disc</u>

$$\text{Stabilizer}(S) \big\backslash SL(2;\mathbb{R}) \big/ SO(2;\mathbb{R}) = \text{Stabilizer}(S) \big\backslash \mathbb{H}^2$$

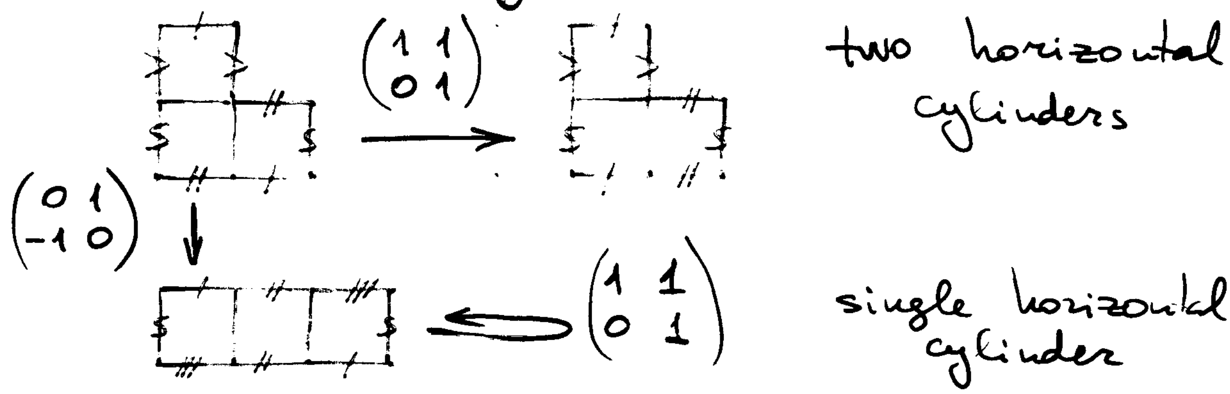
which is a Riemann surface (of finite area) with cusps.

Example S is a torus glued from a unit square
$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ are in stabilizer, so $SL(2;\mathbb{Z})$ is in the stabilizer. One can check that there is nothing else, so the corresponding Teichmüller disc is the modular curve $SL(2;\mathbb{Z}) \backslash \mathbb{H}^2$

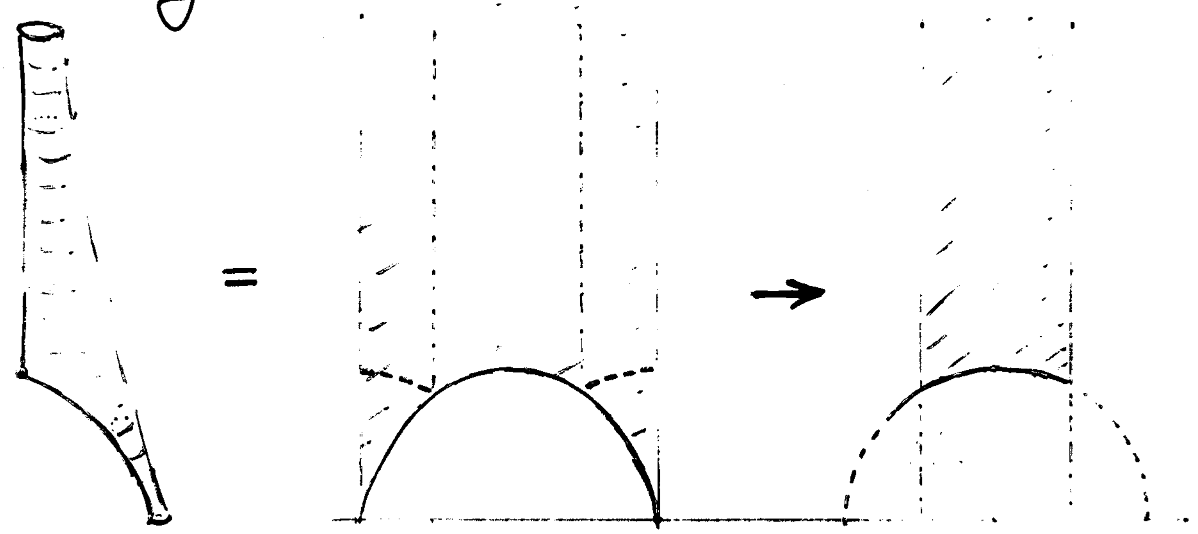


ANOTHER "SIMPLE" EXAMPLE: SQUARE-TILED SURFACES AND THEIR TEICHMÜLLER DISKS

Three possible 3-square tiled surfaces live in $\mathcal{H}(2)$ and belong to the same $SL(2;\mathbb{R})$-orbit:



The corresponding Teichmüller disk is a 3-fold covering over the modular curve:



Two cusps correspond to two possible <u>configurations</u> of closed geodesics.

Primitive Veech surfaces

Any square-tiled surface (= arithmetic Veech surface) is a ramified covering over the torus with a marked point; all singularities project to the marked pt.

More generally: haven found a Veech surface one can construct other Veech surfaces in higher genera taking ramified coverings.

(and paying attention to positions of ramification points: Gutkin-Hubert-Schmidt; Vorobets)

Problem:
What are the primitive Veech surfaces?

In genera $g > 2$ all known Veech surfaces come from triangular billiards and are given by the following list:

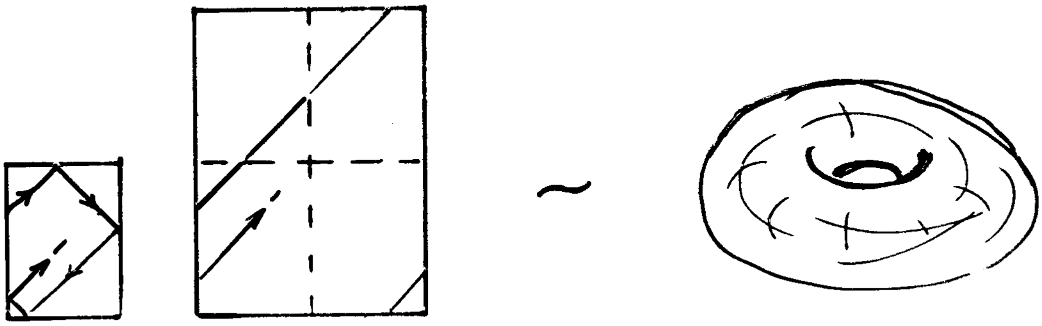
- $\left(\frac{\pi}{n}; \frac{\pi}{n}; \frac{n-2}{n}\pi\right)$ (Veech)
- $\left(\frac{\pi}{n}; \frac{\pi}{2n}; \frac{2n-3}{2n}\pi\right)$ (Vorobets; Stabilizer computed by Ward)
- $\left(\frac{\pi}{3}; \frac{\pi}{4}; \frac{5\pi}{12}\right)$ (Veech)
- $\left(\frac{\pi}{3}; \frac{\pi}{5}; \frac{7\pi}{15}\right)$ (Vorobets)
- $\left(\frac{2\pi}{9}; \frac{3\pi}{9}; \frac{4\pi}{9}\right)$ (Kenyon-Smillie)

Kenyon-Smillie; Puchta: no other acute triangles which give Veech surfaces

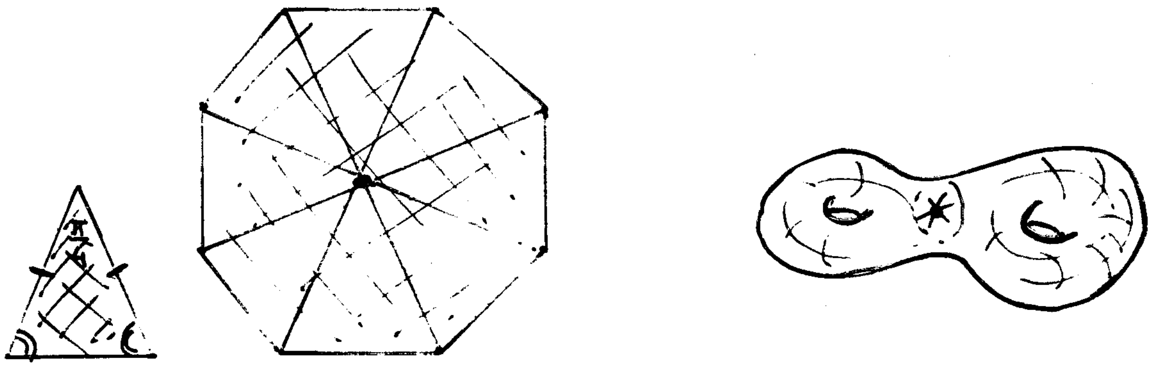
Ward: no obtuse triangles satisfying

$$\frac{1}{m}\pi < \frac{p}{m}\pi < \frac{q}{m}\pi \qquad 4p < m \qquad 1+p+q = m$$

BILLIARDS IN RATIONAL POLYGONS
(Katok - Zemlyakov construction)



Billiard in a rectangle → unfolding → Directional flow on a torus (glued from 4 copies of the rectangle)



Billiard in a triangle $\left(\frac{\pi}{4}; \frac{3\pi}{8}; \frac{3\pi}{8}\right)$ → unfolding → Directional flow on a "flat" surface of genus two

Billiards in rational polygons → Geodesics on VERY FLAT (translation) surfaces

RECENT PROGRESS (2003 - 2004)

THEOREM (K. Calta indep. C. McMullen)

- Complete classification of Veech surfaces in $\mathcal{H}(2)$ (= genus $g=2$; single conical singularity)
  There are plenty of Veech surfaces! *
- Examples of $SL(2;\mathbb{R})$ - invariant submanifolds of "intermediate" type (= bigger than Teichmüller discs, but smaller than the whole stratum)

M. Möller

- Algebro-geometric approach to Veech surfaces in any genus

THEOREM (C. McMullen)

- Description of all possible types of $SL(2;\mathbb{R})$-invar. submanifolds in genus $g=2$.

However, even in genus $g=2$ there are plenty open problems:

CONJECTURE (C. McMullen)

Regular decagon is the only primitive Veech surface in $\mathcal{H}(1,1)$ (genus $g=2$; principal stratum)

---

* All these Veech surfaces come from L-shape billiards.

TEICHMÜLLER DISCS OF VEECH SURFACES

PROBLEM Which square-tiled surfaces belong to the same Teichmüller disc ($SL(2;\mathbb{R})$ - orbit)

Example. Genus $g=2$, stratum $\mathcal{H}(2)$

$n=3$ There are 3 3-~~square~~-tiled surfaces — same $SL(2;\mathbb{R})$ orbit

$n=4$ There are 9 4-square-tiled surfaces — same $SL(2;\mathbb{R})$ orbit

$n=5$ There are 27 5-square-tiled surfaces — two distinct $SL(2;\mathbb{R})$ - orbits !

...

THEOREM (P. Hubert + S. Lelievre: prime n + conjecture for gen. case) (C. McMullen: generalization to any n in $\mathcal{H}(2)$ + nonarithmetic Veech surfaces in $\mathcal{H}(2)$)

A n-square-tiled surfaces in $\mathcal{H}(2)$ (which cannot be tiled with $p \times q$-rectangles for p or $q > 1$) get to the same $SL(2;\mathbb{R})$-orbit when n is even and get to exactly two $SL(2;\mathbb{R})$ - orbits when $n \geq 5$ is odd.

PROBLEM What happens in other strata?

PROBABILITY $P(N)$ OF N=1,2,3,4 BANDS OF TRAJECTORIES.
Permutation (7,6,5,4,1,3,2) (and 769 other ones)

$$0.19 \approx P(1) = \frac{3\,\zeta(7)}{16\,\zeta(6)}$$

$$0.47 \approx P(2) = \frac{55\,\zeta(1,6) + 29\,\zeta(2,5) + 15\,\zeta(3,4) + 8\,\zeta(4,3) + 4\,\zeta(5,2)}{16\,\zeta(6)}$$

$$\begin{aligned} 0.30 \approx P(3) = \frac{1}{32\,\zeta(6)}&(12\,\zeta(6) - 12\,\zeta(7) + 48\,\zeta(4)\,\zeta(1,2) + 48\,\zeta(3)\,\zeta(1,3) + \\ &+ 24\,\zeta(2)\,\zeta(1,4) + 6\,\zeta(1,5) - 250\,\zeta(1,6) - 6\,\zeta(3)\,\zeta(2,2) - \\ &- 5\,\zeta(2)\,\zeta(2,3) + 6\,\zeta(2,4) - 52\,\zeta(2,5) + 6\,\zeta(3,3) - 82\,\zeta(3,4) + \\ &6\,\zeta(4,2) - 54\,\zeta(4,3) + 6\,\zeta(5,2) + 120\,\zeta(1,1,5) - 30\,\zeta(1,2,4) - \\ &120\,\zeta(1,3,3) - 120\,\zeta(1,4,2) - 54\,\zeta(2,1,4) - 34\,\zeta(2,2,3) - \\ &29\,\zeta(2,3,2) - 88\,\zeta(3,1,3) - 34\,\zeta(3,2,2) - 48\,\zeta(4,1,2)) \end{aligned}$$

$$0.04 \approx P(4) = \frac{\zeta(2)}{8\,\zeta(6)}\,(\zeta(4) - \zeta(5) + \zeta(1,3) + \zeta(2,2) - \zeta(2,3) - \zeta(3,2))$$

VOLUMES OF SOME STRATA

$$
\begin{aligned}
\mathrm{Vol}(\mathcal{H}(2)) &= \frac{\pi^4}{240}\\
\mathrm{Vol}(\mathcal{H}(1,1)) &= \frac{\pi^4}{540}\\
\mathrm{Vol}(\mathcal{H}^{hyp}(4)) &= \frac{\pi^6}{13\,440}\\
\mathrm{Vol}(\mathcal{H}^{odd}(4)) &= \frac{\pi^6}{9\,720}\\
\dots \quad & \dots \quad \dots\\
\mathrm{Vol}(\mathcal{H}(1^8)) &= \frac{23\,357\cdot\pi^{10}}{25\,421\,189\,713\,920\,000} \quad \text{(A.Eskin, A.Okounkov)}\\
\dots \quad & \dots \quad \dots
\end{aligned}
$$

SUMS OF LYAPUNOV EXPONENTS

$$
\begin{array}{lll}
\mathcal{H}(2) & \nu_1+\nu_2 & = \frac{4}{3}\\
\mathcal{H}(1,1) & \nu_1+\nu_2 & = \frac{3}{2}\\
\mathcal{H}^{hyp}(4) & \nu_1+\nu_2+\nu_3 & = \frac{9}{5}\\
\mathcal{H}^{odd}(4) & \nu_1+\nu_2+\nu_3 & = \frac{8}{5}\\
\dots & \dots & \dots\\
\mathcal{H}(1^8) & \nu_1+\dots+\nu_5 & = \frac{235\,761}{93\,428}\\
\dots & \dots & \dots
\end{array}
$$