SUMMER SCHOOL AND CONFERENCE
ON DYNAMICAL SYSTEMS

Flat surfaces,
Interval exchange Transformations
& Moduli Spaces of Abelian Differentials
(Lecture 5)

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These are preliminary lecture notes, intended only for distribution to participants
When the complex structures are different there is no conformal map which sends one to another. A map $f$ sends an infinitesimal circle at a point $x \in M^2$ to an infinitesimal ellipse:

Coefficient of quasiconformality of $f$ at $x \in M^2$ is the ratio $K_f(x) = \frac{a}{b}$.

**Definition**

Coefficient of quasiconformality of $f$ is

$$K(f) \overset{\text{def}}{=} \sup_{x \in M^2} K_f(x)$$
**Teichmüller Theorem**

Choose any two complex structures on a surface $M^2_g$ of genus $g$.

**Theorem (Teichmüller; developing ideas of Grotzsch)**
- There exist an extremal map $f_0$ which minimizes coef. of quasiconform. $K(f)$
- For this extremal map the coefficient of quasiconformality is constant:
  $$K_{f_0}(x) = K(f_0) \quad \forall x \in M^2_g \setminus \{finite \ points\}$$

- One can choose flat* metrics (compatible with the complex structures) in which the foliation along large (corresp. small) demi-axes of ellipses is the horizontal (corresp. vertical) foliation in the flat metric.

- In flat coordinates the map $f_0$ is just expansion-contraction with coeff. $\sqrt{K}$

* These flat metrics are slightly more general than those which we considered: they allow holonomy $\overrightarrow{v} \rightarrow -\overrightarrow{v}$ and correspond to quadratic differentials.
**Teichmüller Metric and Teichmüller Geodesics**

**Definition** The distance between two complex structures is measured as

\[
\frac{1}{2} \log K(\gamma_0),
\]

where \( K(\gamma_0) \) is the coefficient of quasiconformality of the extremal map.

It means that a quadratic differential (or a holomorphic 1-form) define a direction of deformation of complex structure and a geodesic in Teichmüller metric.

\[
\begin{pmatrix}
ed & 0 \\
0 & e^{-d}
\end{pmatrix}
\]

**Remark** Teichmüller metric is not a Riemannian but Finsler metric (unfortunately it is not defined by a quadratic form in the tangent space but just by a norm).
SL(2;IR) - action on flat surfaces in geometric terms

Space of pairs (complex, quadratic, differential)
is the (co)tangent bundle to the space of complex structures.

Space of pairs (complex, holomorphic)
is a subbundle of special directions;

subspace \( \mathcal{H}_1 \) of holomorphic 1-forms corresponding to flat surfaces of area one is a "subbundle of unit vectors".

An SL(2;IR) - orbit in \( \mathcal{H}_1 \) defines a diagram:

\[
\begin{align*}
\text{\{Unit tangent bundle to hyperbolic plane\}} & : \ SL(2;IR) \rightarrow \mathcal{H}_1 \ ("\text{unit tangent subbundle}) \\
\text{\{Hyperbolic plane\}} & : \ SL(2;IR) \bigg/ SO(2;IR) \rightarrow \mathcal{H}_1 \rightarrow J_g \ (\text{Teichmüller space})
\end{align*}
\]

The map \( \mathcal{H}_2 \rightarrow J_g \) is an isometry, so its image in Teichmüller space is a complex geodesic (called Teichmüller disk).
HOPE FOR "RATNER THEOREM"

THEOREM (N. Shah, 1990 as a corollary of Ratner theorem)
In a compact manifold of constant negative curvature, the closure of a totally geodesic, complete (immersed) submanifold of dim ≥ 2 is a totally geodesic immersed submanifold. (Generalization of J. Payne: target manifold of finite volume)

Moral: Complex geodesics are simple; (real geodesics are complicated).

Hope: The closure of any $SL(2;\mathbb{R})$-orbit in the space of holomorphic one forms (in the space of quadratic differentials) is a nice complex orbifold.

THEOREM (McMullen, 2003)
The hope is true for holomorphic 1-forms in genus two.

(UNPUBLISHED)

THEOREM (Kontsevich, ~1998)
If the closure of an $SL(2;\mathbb{R})$-orbit is a nice complex submanifold (suborbifold) it is represented by an affine subspace in cohomological coordinates.
SEARCH FOR INVARIANT SUBMANIFOLDS

Teichmüller geodesic flow (and SL(2;R)) preserve number and types of conical singularities (= degrees of zeroes of a holomorphic 1-form). So they preserve the strata \( \mathcal{H}_1(k_1, k_2, \ldots, k_n) \).

Example In genus \( g = 2 \) there are two strata \( \mathcal{H}_1(2) \) and \( \mathcal{H}_1(1,1) \).

In genus \( g = 3 \) there are \( \mathcal{H}_1(4) ; \mathcal{H}_1(1,3) ; \mathcal{H}_1(2,2) ; \mathcal{H}_1(1,1,2) ; \mathcal{H}_1(1,1,1,1) \)

Reminder \( \sum_{i=1}^{n} k_i = 2g - 2 \)

Theorem (Masur, 1982; Veech, 1982)

SL(2;R) (and Teichmüller geodesic flow) act ergodically on every connected component of every stratum \( \mathcal{H}_1(k_1, \ldots, k_n) \).

We shall discuss two extremal cases: 1) when the closure of SL(2;R)-orbit is entire connected component of a stratum; 2) when SL(2;R)-orbit is already closed.
CLASSIFICATION OF CONNECTED COMPONENTS

PARITY-OF-THE-SPIN-STRUCTURE:
ALTERNATIVE DEFINITION

Let $\omega \in \mathcal{H}(2l_1, \ldots, 2l_n)$. Let $\alpha$ be a smooth simple closed oriented curve on $C$. Consider the Gauss map $\alpha \rightarrow S^1 \subset \mathbb{C}$.

Total rotation angle = $2\pi \text{ind}_\alpha$

$\text{ind}_\alpha = \text{degree of the Gauss map}$

**Lemma.** The function

$$
\Omega([\alpha]) \overset{\text{def}}{=} (\text{ind}_\alpha + 1) \mod 2
$$

is well-defined on $H_1(C; \mathbb{Z}/2\mathbb{Z})$ and has the property

$$
\Omega(c_1 + c_2) = \Omega(c_1) + \Omega(c_2) + c_1 \cdot c_2
$$

**Lemma.** The value

$$
\varphi(\omega) = \sum_{i=1}^{g} \Omega(a_i)\Omega(b_i) \mod 2
$$

does not depend on the choice of a symplectic basis of cycles \{a_i, b_i\} and equals the parity-of-the-spin-structure determined by $\omega$. 
HYPERELLIPTIC CONNECTED COMPONENTS
(only in the strata $\mathcal{H}(2g-2)$ and $\mathcal{H}(g-1, g-1)$)

A flat surface $S$ belongs to a hyperelliptic connected component $\mathcal{H}^{hyp}(2g-2)$ or $\mathcal{H}^{hyp}(g-1, g-1)$ if it has an isometric involution $\tau: S \to S$, $\tau^2 = \text{id}$, such that $S/\tau = \text{sphere}$ and in the case of $S \in \mathcal{H}(g-1, g-1)$ the involution $\tau$ interchanges the conical points.
CLASSIFICATION THEOREM
( HOLOMORPHIC 1-FORMS )

THEOREM (M. Kontsevich + A.Z.) Let $g \geq 4$.
- There are two series of hyperelliptic connected components: $\mathcal{H}_{\text{hyp}}(2g-2)$ and $\mathcal{H}(g-1,g-1)$.

- When all degrees are even $\mathcal{H}(2k_1, \ldots, 2k_n)$ has exactly two nonhyperelliptic connected components $\mathcal{H}_{\text{odd}}(2k_1, \ldots, 2k_n)$ and $\mathcal{H}_{\text{even}}(2k_1, \ldots, 2k_n)$.

- When $g$ is even $\mathcal{H}(g-1,g-1)$ contains exactly one nonhyperelliptic connected component.

- All other strata are connected.

COROLLARY This classifies the extended Rauzy classes.
CLASSIFICATION THEOREM
(quadradic differentials)

THEOREM (E. Lanneau, 2003)
- there are three hyperelliptic series;
- otherwise, there are four exceptional strata containing exactly two
  mysterious connected components: Q(-1,9), Q(-1,3,6), Q(-1,3,3,3), Q(12)

REMARK Currently there are two ways to determine to which of the two connected
components of an exceptional stratum belongs a flat surface.
- Either find a “generalized permutation” and check to which of the two extended
  Rauzy classes it belongs;
- Or: to find a configuration of saddle connections which does not occur for one
  of the two components
  (Example: for one of the two components of Q(-1,9) as soon as we have a saddle
  connection between “-1” and “9” we get a closed geodesic going in the same direction.)
def Flat surface $S$ is a Veech surface if it has an enormous group of affine automorphisms, namely if Stabilizer$(S)$ is a lattice in $SL(2; \mathbb{R})$.

**Theorem (Smillie)** $SL(2; \mathbb{R})$-orbit of $S$ is closed in the stratum $\iff S$ is a Veech surface.

Forgetting "polarizations" of flat surfaces we get a Teichmüller disc

$Stabilizer(S) \backslash SL(2; \mathbb{R}) / SO(2; \mathbb{R}) = Stabilizer(S) \backslash \mathbb{H}^2$

which is a Riemann surface (of finite area) with cusps.

Example $S$ is a torus glued from a unit square \((\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix})\) and \((\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix})\) are in stabilizer, so $SL(2; \mathbb{Z})$ is in the stabilizer. One can check that there is nothing else, so the corresponding Teichmüller disc is the modular curve $SL(2; \mathbb{Z}) \backslash \mathbb{H}^2$. 
Another "simple" example: Square-tiled surfaces and their Teichmüller disks

Three possible 3-square tiled surfaces live in \( \mathcal{H}(2) \) and belong to the same \( SL(2, \mathbb{R}) \)-orbit:

\[
\begin{pmatrix}
1 & 1 \\
0 & 1
\end{pmatrix}
\]  \hspace{1cm} two horizontal cylinders

\[
\begin{pmatrix}
0 & 1 \\
-1 & 0
\end{pmatrix}
\]  \hspace{1cm} single horizontal cylinder

The corresponding Teichmüller disk is a 3-fold covering over the modular curve:

Two cusps correspond to two possible configurations of closed geodesics.
**Primitive Veech Surfaces**

Any square-tiled surface (= arithmetic Veech surface) is a ramified covering over the torus with a marked point; all singularities project to the marked pt.

More generally: haven found a Veech surface one can construct other Veech surfaces in higher genera taking ramified coverings.

(and paying attention to positions of ramification points: Gutkin-Hubert-Schmidt; Vorobets)

**Problem:** What are the primitive Veech surfaces?

In genera $g > 2$ all known Veech surfaces come from triangular billiards and are given by the following list:

- $\left( \frac{\pi}{n} ; \frac{\pi}{n} ; \frac{n-2}{n} \pi \right)$ (Veech)
- $\left( \frac{\pi}{n} ; \frac{\pi}{n} ; \frac{2n-3}{2n} \pi \right)$ (Vorobets; stabilizers computed by Ward)
- $\left( \frac{\pi}{3} ; \frac{\pi}{4} ; \frac{5\pi}{12} \right)$ (Veech)
- $\left( \frac{\pi}{3} ; \frac{\pi}{5} ; \frac{7\pi}{15} \right)$ (Vorobets)
- $\left( \frac{2\pi}{3} ; \frac{3\pi}{9} ; \frac{4\pi}{9} \right)$ (Kenyon-Smillie)

Kenyon-Smillie; no other acute triangles Puncta which give Veech surfaces

Ward: no obtuse triangles satisfying

\[
\frac{1}{M} \pi < \frac{P}{M} \pi < \frac{2}{M} \pi \quad 4p < m \quad 1 + p + q = m
\]
Billiards in rational polygons
(Katok-Zemlyakov construction)

Billiard in a rectangle $\Rightarrow$ unfolding $\Rightarrow$ Directional flow on a torus (glued from 4 copies of the rectangle)

Billiard in a triangle $\left(\frac{\pi}{4} ; \frac{3\pi}{8} ; \frac{3\pi}{8}\right)$ $\Rightarrow$ unfolding $\Rightarrow$ Directional flow on a "flat" surface of genus two

Billiards in rational polygons $\Rightarrow$ Geodesics on VERY FLAT surfaces (translation)
Recent Progress (2003-2004)

Theorem (K. Calta, indie, C. McMullen)

- Complete classification of Veech surfaces in $\mathcal{H}(2)$ (genus $g=2$; single conical singularity)
  
  There are plenty of Veech surfaces.*

- Examples of $SL(2; \mathbb{R})$-invariant submanifolds of "intermediate" type (= bigger than Teichmüller discs, but smaller than the whole stratum)

M. Möller

- Algebra-geometric approach to Veech surfaces in any genus

Theorem (C. McMullen)

- Description of all possible types of $SL(2; \mathbb{R})$-invar. submanifolds in genus $g=2$.

However, even in genus $g=2$ there are plenty open problems!

Conjecture (C. McMullen)

Regular decagon is the only primitive Veech surface in $\mathcal{H}(1,1)$ (genus $g=2$; principal stratum)

* All these Veech surfaces come from L-shape billiards.
Teichmüller discs of Veech surfaces

Problem Which square-tiled surfaces belong to the same Teichmüller disc (SL(2;R) - orbit)

Example. Genus g = 2, stratum \( \mathcal{H}(2) \)

\( n = 3 \) There are 3 3-square-tiled surfaces same SL(2;R) orbit

\( n = 4 \) There are 9 4-square-tiled surfaces same SL(2;R) orbit

\( n = 5 \) There are 27 5-square-tiled surfaces two distinct SL(2;R) - orbits!

\[ \cdots \]

Theorem (P. Hubert + S. Lelievre: prime N + conjecture for pm. case)

(C. McMullen: generalization to any N in \( \mathcal{H}(2) \) + nonarithmetic Veech surfaces in \( \mathcal{H}(2) \))

A \( n \)-square-tiled surfaces in \( \mathcal{H}(2) \)

(which cannot be tiled with \( p \times q \)-rectangles for \( p, q > 1 \)) get to the same SL(2;R)-orbit when \( n \) is even and get to exactly two SL(2;R)-orbits when \( n \geq 5 \) is odd.

Problem What happens in other strata?
Probability $P(N)$ of $N=1,2,3,4$ bands of trajectories.
Permutation $(7,6,5,4,1,3,2)$ (and 769 other ones)

$0.19 \approx P(1) = \frac{3 \zeta(7)}{16 \zeta(6)}$

$0.47 \approx P(2) = \frac{55 \zeta(1,6) + 29 \zeta(2,5) + 15 \zeta(3,4) + 8 \zeta(4,3) + 4 \zeta(5,2)}{16 \zeta(6)}$

$0.30 \approx P(3) = \frac{1}{32 \zeta(6)} (12 \zeta(6) - 12 \zeta(7) + 48 \zeta(4) \zeta(1,2) + 48 \zeta(3) \zeta(1,3) + 24 \zeta(2) \zeta(1,4) + 6 \zeta(1,5) + 250 \zeta(1,6) - 6 \zeta(3) \zeta(2,2) - 5 \zeta(2) \zeta(2,3) + 6 \zeta(2,4) - 52 \zeta(2,5) + 6 \zeta(3,3) - 82 \zeta(3,4) + 6 \zeta(4,2) - 54 \zeta(4,3) + 6 \zeta(5,2) + 120 \zeta(1,1,5) - 30 \zeta(1,2,4) - 120 \zeta(1,3,3) - 120 \zeta(1,4,2) - 54 \zeta(2,1,4) - 34 \zeta(2,2,3) - 29 \zeta(2,3,2) - 88 \zeta(3,1,3) - 34 \zeta(3,2,2) - 48 \zeta(4,1,2))$

$0.04 \approx P(4) = \frac{\zeta(2)}{8 \zeta(6)} (\zeta(4) - \zeta(5) + \zeta(1,3) + \zeta(2,2) - \zeta(2,3) - \zeta(3,2))$
VOLUMES OF SOME STRATA

\[
\begin{align*}
\text{Vol}(\mathcal{H}(2)) &= \frac{\pi^4}{240} \\
\text{Vol}(\mathcal{H}(1,1)) &= \frac{\pi^4}{540} \\
\text{Vol}(\mathcal{H}^{\text{hyp}}(4)) &= \frac{\pi^6}{13440} \\
\text{Vol}(\mathcal{H}^{\text{odd}}(4)) &= \frac{\pi^6}{9720} \\
\text{Vol}(\mathcal{H}(1^8)) &= \frac{23\,357\cdot\pi^{10}}{25\,421\,189\,713\,920\,000} \quad (\text{A. Eskin, A. Okounkov})
\end{align*}
\]

SUMS OF LYAPUNOV EXPONENTS

\[
\begin{align*}
\mathcal{H}(2) \quad \nu_1 + \nu_2 &= \frac{4}{3} \\
\mathcal{H}(1,1) \quad \nu_1 + \nu_2 &= \frac{3}{2} \\
\mathcal{H}^{\text{hyp}}(4) \quad \nu_1 + \nu_2 + \nu_3 &= \frac{9}{5} \\
\mathcal{H}^{\text{odd}}(4) \quad \nu_1 + \nu_2 + \nu_3 &= \frac{8}{5} \\
\mathcal{H}(1^8) \quad \nu_1 + \cdots + \nu_5 &= \frac{235\,761}{93\,428}
\end{align*}
\]