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SMR.1573 - 13

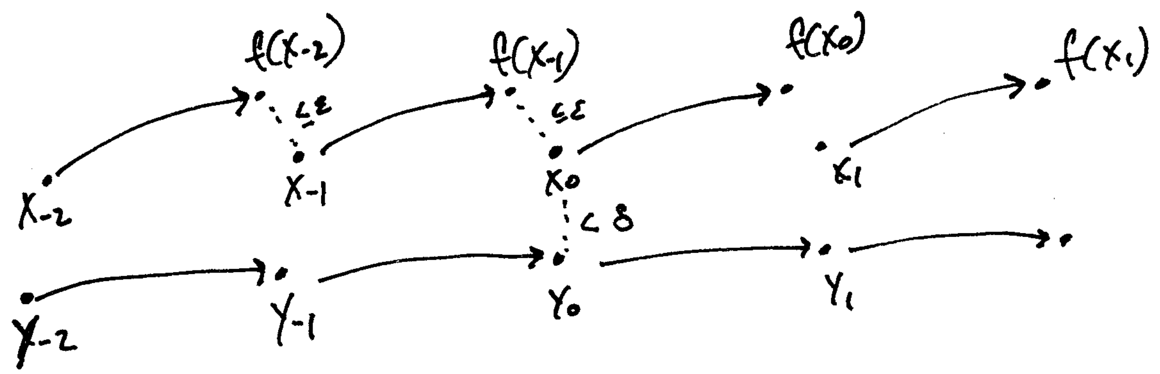
SUMMER SCHOOL AND CONFERENCE ON DYNAMICAL SYSTEMS

Polynomial Diffeomorphisms of C^2
(Lecture 2)

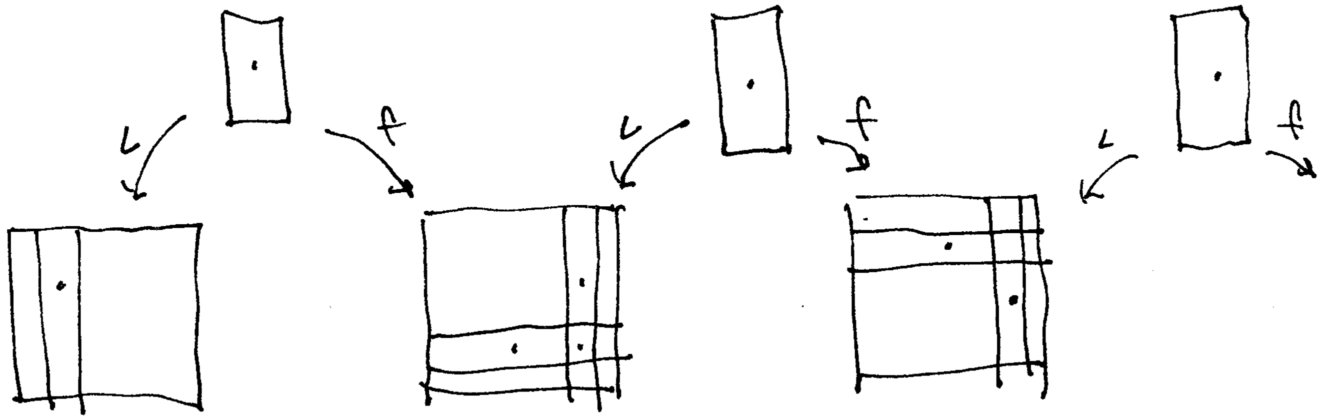
John D. Smillie
Department of Mathematics
Cornell University
Ithaca
USA

Definition. A diffeomorphism is Axiom A if its chain recurrent set is a hyperbolic set.

Shadowing Lemma. Given $\delta > 0$ there is an $\varepsilon > 0$ so that for any ε-pseudo orbit (x_j) there is an actual orbit (y_j) with $d(x_j, y_j) \leq \delta$.



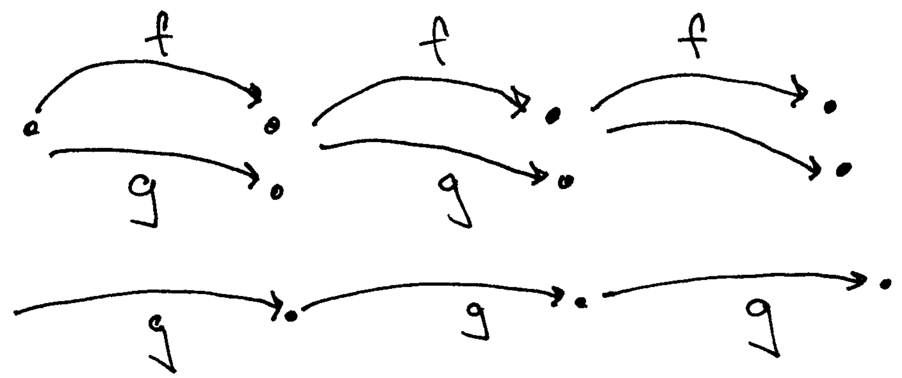
We can interpret our horseshoe discussion in terms of pseudo-orbits.



The Shadowing Lemma is the key to structural stability.

If f, g are nearby Axiom A maps with chain recurrent sets X and Y then $f|X$ is topologically conjugate to $g|Y$.

Why?



We ended our last lecture with a negative result:

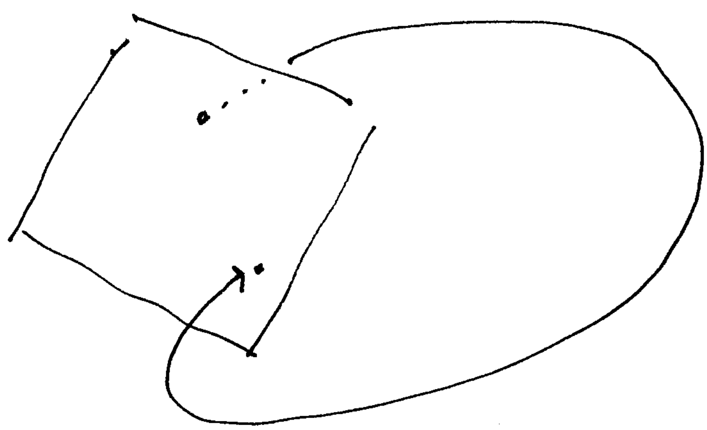
There are open sets of diffeomorphisms which the Axiom A theory cannot handle.

Are there other mechanisms that explain the dynamics of diffeomorphisms?

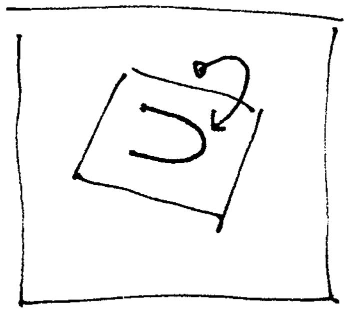
In the 70's the French astronomer Hénon was studying the equations for turbulent fluid flow.

These are partial differential equations which he simplified by dropping higher order Fourier coefficients. and approximating by an ordinary differential equation.

He computed a first return map to a section for the flow.



He observed a region which was taken into itself and folded.



There were two problems: time and large dissipation.

He replaced this first return map by a polynomial diffeomorphism in which he observed the same qualitative behavior.

$$H\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 + y - ax^2 \\ bx \end{bmatrix}$$

with $a = 1.4$ and $b = 0.3$.

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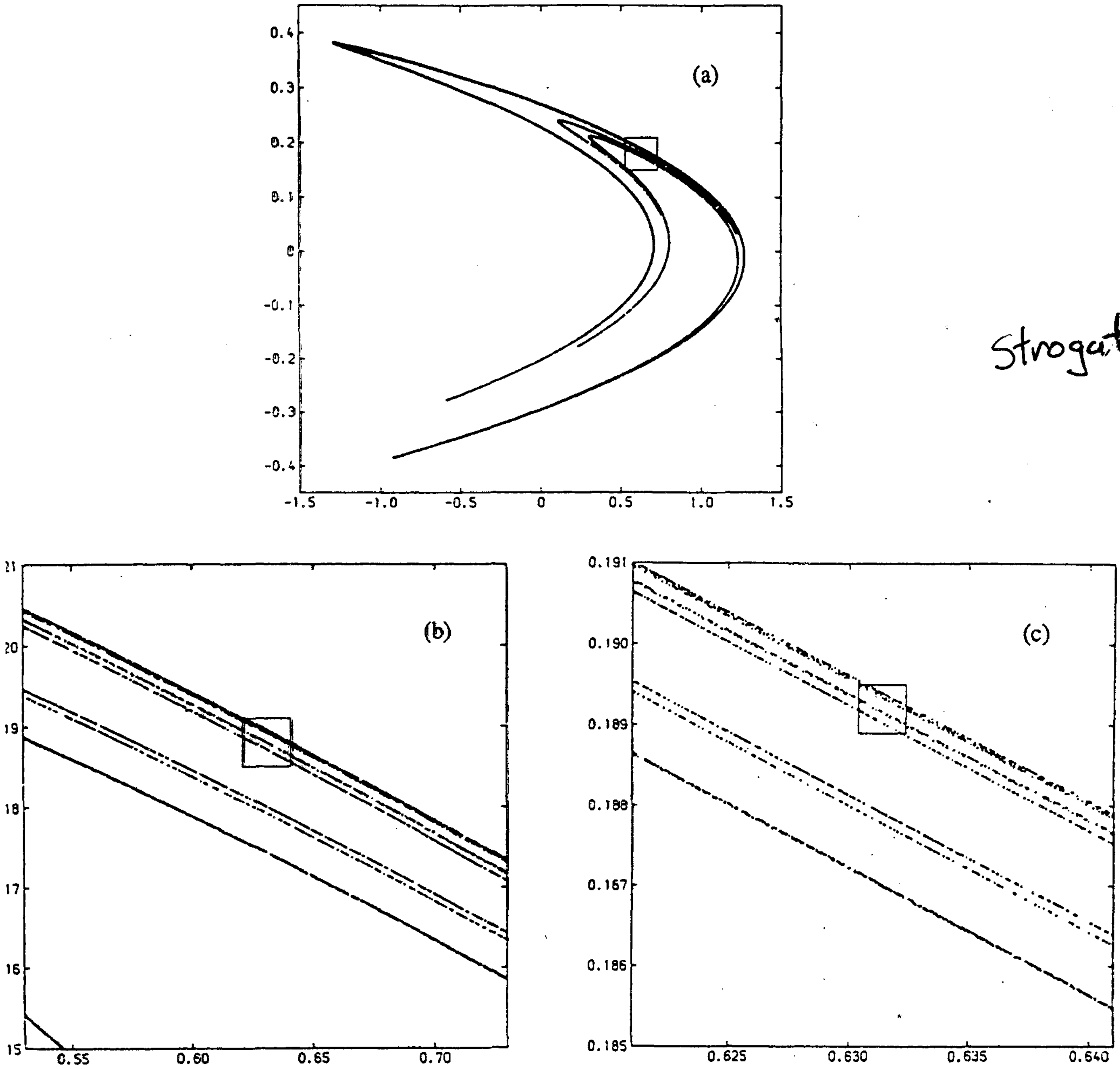
actor by computing ten thousand successive iterates of (1), starting from the ori-
n. You really must try this for yourself on a computer. The effect is eerie—the
oints (x_n, y_n) hop around erratically, but soon the attractor begins to take form,
ike a ghost out of the mist" (Gleick 1987, p.150).

The attractor is bent like a boomerang and is made of many parallel curves (Fig-
e 12.2.3a).



Strogatz

igure 12.2.3 Hénon (1976), pp 74–76

igure 12.2.3b is an enlargement of the small square of Figure 12.2.3a. The char-
cteristic fine structure of the attractor begins to emerge. There seem to be six par-
llel curves: a lone curve near the middle of the frame, then two closely spaced

Benedicks and Carleson established the existence of strange attractors with non uniform expansion and contraction for a set of parameters of positive measure and opened a new chapter in the field of dynamical systems.

It is useful to write the Hénon family of diffeomorphisms in different coordinates than those used by Hénon.

$$f_{a,b}(x,y) = (-x^2+a-by, x)$$

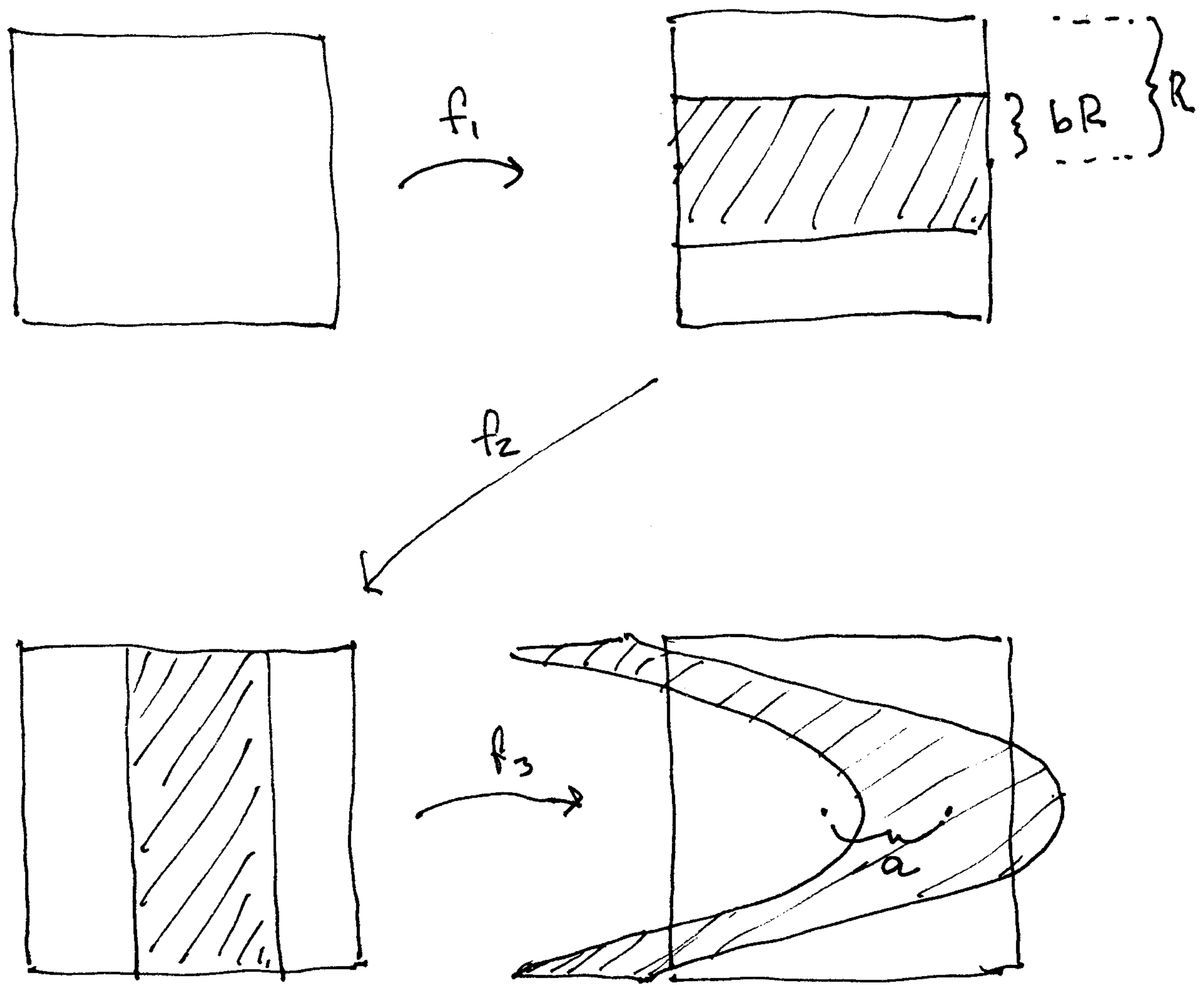
We can write f as the composition of 3 simpler diffeomorphisms:

$$f_1(x,y) = (x, by)$$

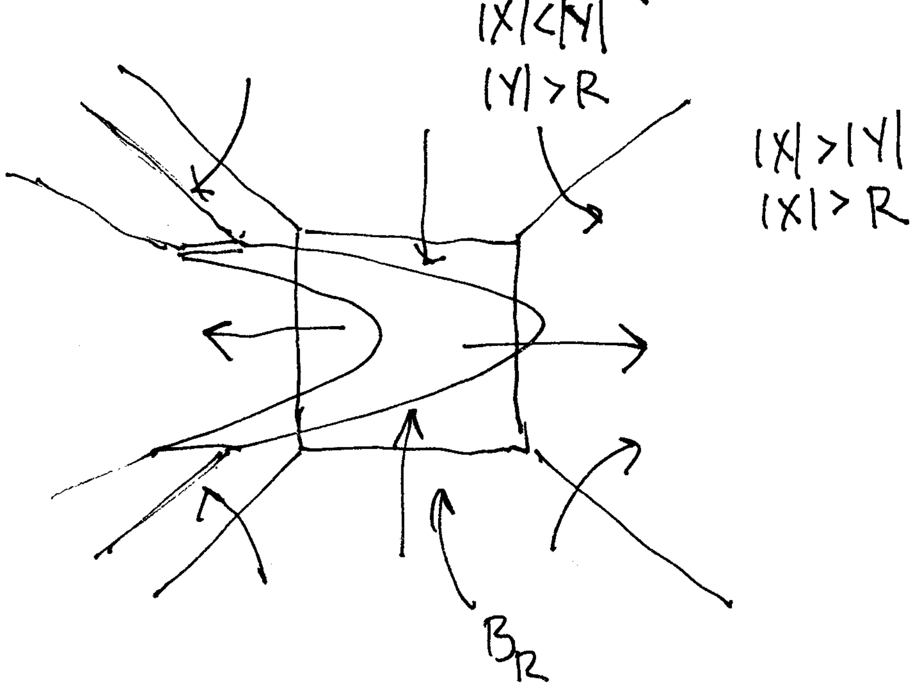
$$f_2(x,y) = (-y, x)$$

$$f_3(x,y) = (x+(-y^2+a), y)$$

$B_R = \{(x, y) : |x| \le R, |y| \le R\}$

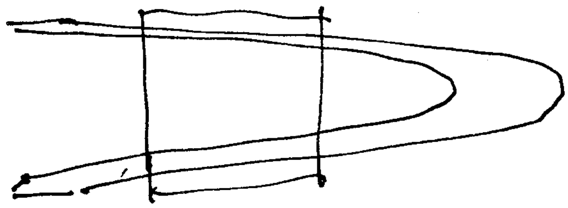


Exercise. There exists an $R = R(a,b)$ so that the following holds



This family of diffeomorphisms provides an excellent laboratory in which to study the dynamics of diffeomorphisms of $\mathbb{R}^2$.

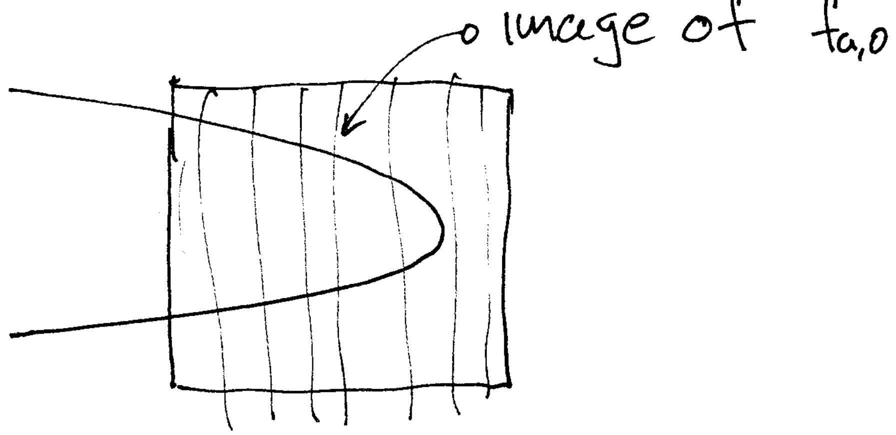
Horseshoes appear when $a \gg 0$ (Devaney-Nitecki).



The Newhouse persistent tangency appears. There are $f_{a,b}$ with ∞ many sink orbits.

Strange attractors appear.

The case of extreme dissipativity $b=0$ corresponds to one dimensional dynamics.



If $g_a(x) = -x^2+a$ then

$$f_{a,0}(x,y) = (-x^2+a,\, x) = (g_a(x),\, x)$$

$$f^n_{a,0}(x,y) = (g^n_a(x),\, g^{n-1}_a(x)).$$

Hénon chose to work with polynomial diffeomorphisms because they were easy to iterate by computer. An unintended consequence of this choice is that these diffeomorphisms have holomorphic extension to $\mathbb{C}^2$.

In the mid 80's John Hubbard gave a number of lectures making this point.

There are a number of reasons why one might want to work with holomorphic mappings. We focus on one of these now.

Let us start with the family $f_c(z) = z^2 + c$, $f_c : \mathbb{C} \circlearrowleft$.

It can be a somewhat daunting task to prove that a diffeomorphism of $\mathbb{R}^2$ is Axiom A.

For the family $f_c : \mathbb{C} \circlearrowleft$ there is a complete characterization of Axiom A.

A holomorphic is a differentiable map where the derivative is $\mathbb{C}$-linear. In many situations holomorphic maps $f: U \overset{\subset \mathbb{C}}{\longrightarrow} \mathbb{C}$ behave more like <u>affine</u> maps $g: I \overset{\subset \mathbb{R}}{\longrightarrow} \mathbb{R}$ than like C^1 maps.

The Schwartz-Pick Lemma is an example of this.

If $I \subset \mathbb{R}$ is an interval and $f: I \to I$ is an affine map then either f decreases distance or f is a bijection.

Consider the Poincaré metric $\frac{ds}{1-|z|^2}$ on the unit disk D_1.

Schwartz-Pick Lemma. If $f: D_1 \to D_1$ is holomorphic then Df does not increase distance. If Df is an isometry at some point then f is a bijection.

More sophisticated version:

Let M, N be proper subsets of $\mathbb{C}$. We can define Poincaré metrics on M and N by identifying their universal covers with D_1.

Lemma. Let $f: M \to N$ be holomorphic then Df does not increase distance. If Df is an isometry at some point then f is a covering map.

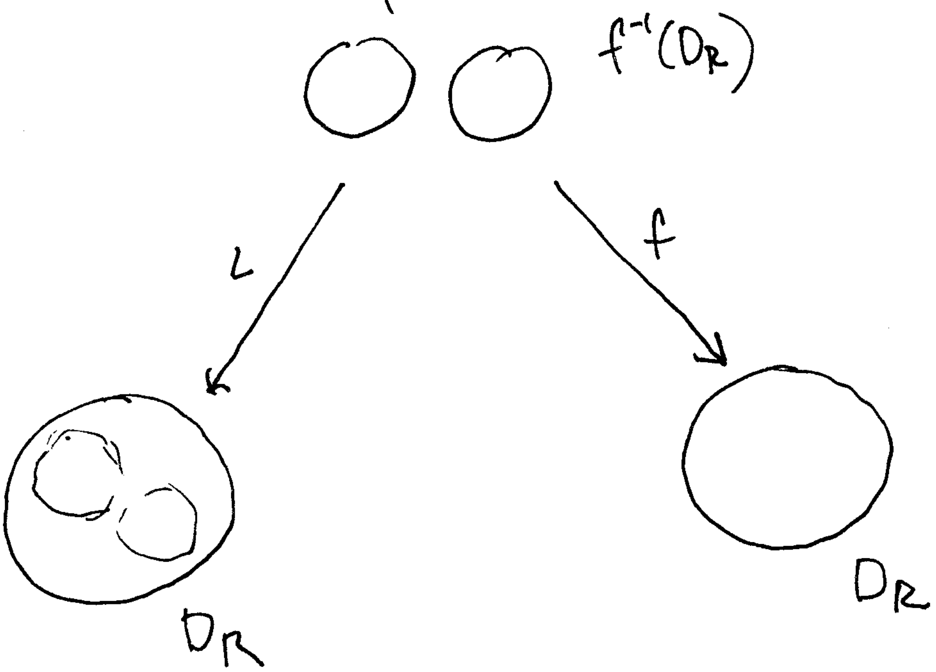
Let $f_c : \mathbb{C} \circlearrowleft$. Let $K_c = \{z : f_c^n(z) \not\to \infty \text{ as } n \to \infty\}$.
Let $J_c = \partial K_c$. J_c is the Julia set.

Exercise. The interior of K_c consists of points that are <u>Lyapunov stable</u>. p is Lyapunov stable if for every $\varepsilon > 0$ there is a $\delta > 0$ so that $d(p,q) \le \delta \Rightarrow d(f^n(p), f^n(q)) \le \varepsilon$.
(Hint: Schwarz-Pick.)

In the Axiom A case f_c is expanding on J_c and int K_c consists of basins of finitely many sink orbits.

Examples of hyperbolic maps f_c.

If $|c|>2$ then there is an R so that $K \overset{\text{int}}{\subset} D_R$ but the image of the critical point is not in D_R.



L decreases distance

f is an isometry

$f \circ L^{-1}$ expands distance

f_c is Axiom A.

(One sided horseshoe.)

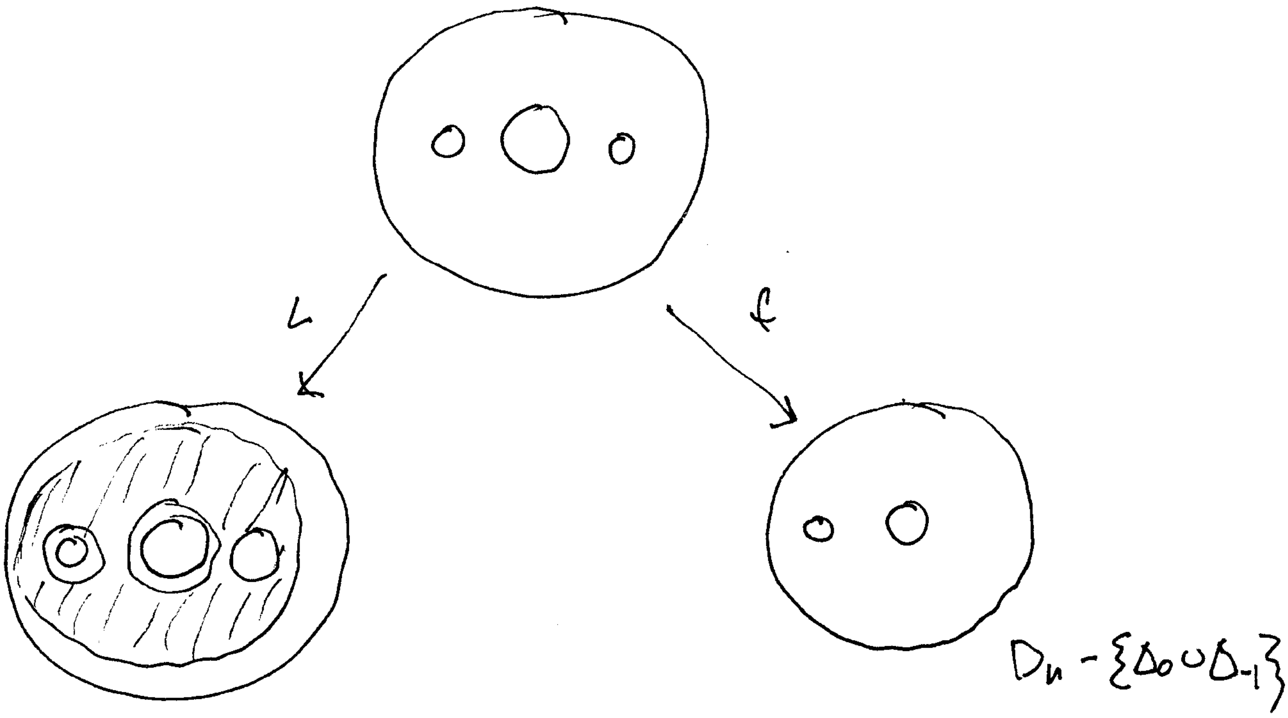
Example 2.

Consider $f_{-1}(z) = z^2 - 1$.

$f(0) = -1 \qquad f(-1) = 0$.

Thus we have a sink of period 2 which contains the critical point.

Choose disks Δ_0, Δ_{-1} around 0 and -1 so that $f(\Delta_0) \subset \Delta_1$ and $f(\Delta_{-1}) \subset \Delta_0$.



f is a proper map of degree 2 with no critical point so f is a covering. f preserves distance, ι decreases distance $f \circ \iota^{r}$ is expanding.

c = -1