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SUMMER SCHOOL AND CONFERENCE ON DYNAMICAL SYSTEMS

The global dynamics of generic diffeomorphisms
(Lecture 3)

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Chapter V Hyperbolic properties

K invariant ~~compact~~ set.

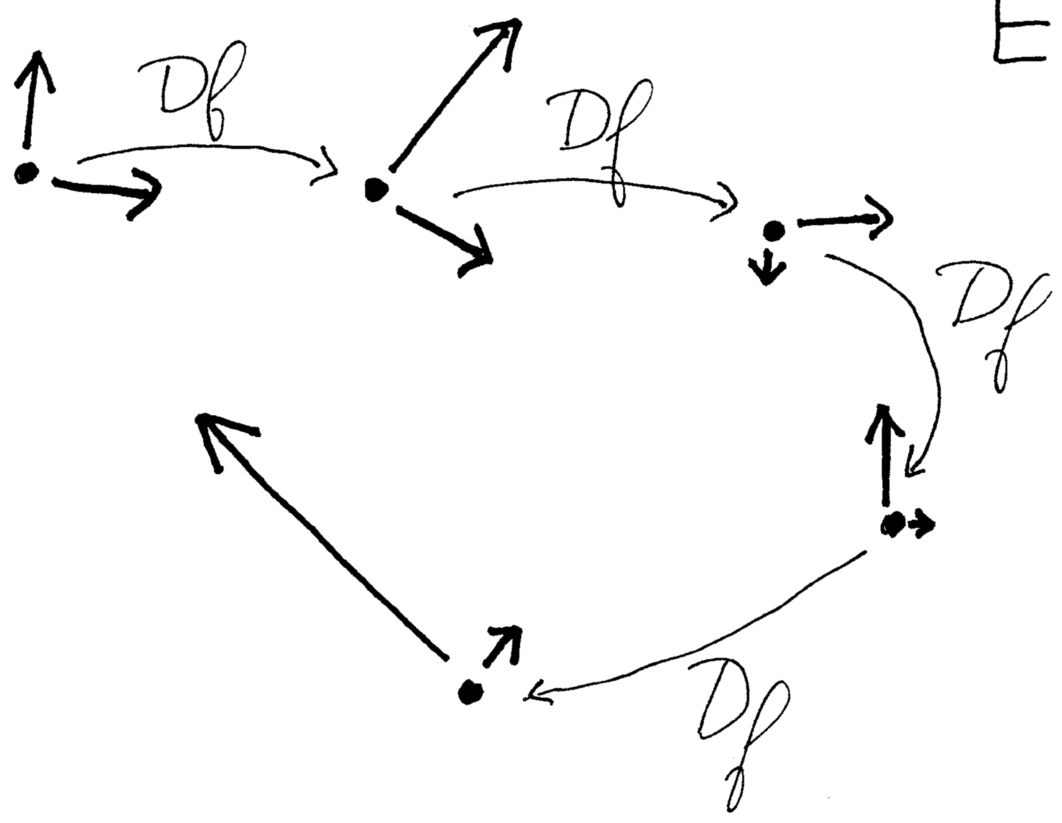
Dominated splitting:

$TM|_K = E \oplus F$, invariant by Df

$\exists C > 0 \ \exists 0 < \lambda < 1$

$\forall x \in K$ $\forall u \in E(x)$ unit vectors $\dfrac{\|Df^n(u)\|}{\|Df^n(v)\|} < C\lambda^n$
$\forall n > 0$ $\forall v \in F(x)$

$E <_{C,\lambda} F$



Exercizes:

1). γ periodic orbits. find a condition equivalent to "$\exists$ a dominated splitting on γ".

2) if K has a dominated splitting, isk it unique?

Properties

$K \subset M$ invariant. $TM|_K = E \oplus F$

- dim $E(x)$ constant on K.

  $\Rightarrow$ $x \mapsto E(x), F(x)$ continuous and extends to $\overline{K}$

- $f_n \xrightarrow{C^1} f$, $f_n(K_n) = K_n$

  $K \subset \limsup K_n$ (Hausdorff) $\quad TM|_{K_n} = E_n \oplus F_n$

  dim $E_n = d$

  $E_n <_{C\lambda} F_n \Rightarrow \exists\ TM|_K = E \oplus F$

  E / F

- K compact

$TM|_K = E \oplus F$, $E <_{c,\lambda} F$

$\forall c' > c$ $\quad$ $\exists\, \mathcal{U} \ni f$ open

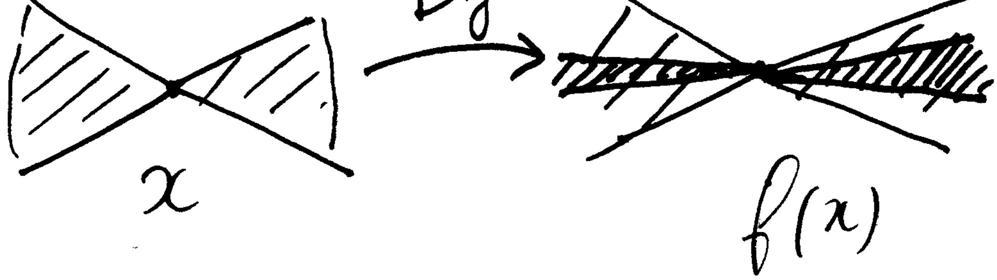
$\forall \lambda' \in]\lambda, 1[$ $\quad$ $\exists\, O \supset K$ open

$\forall g \in \mathcal{U}$ $\quad \bigcap_{n \in \mathbb{Z}} g^n(\bar{O})$ has

a (c', λ')-dominated splitting $E_g \oplus F_g$

$\dim E_g = \dim E$.

<u>Proof</u> : cone fields :



The Mañé's ergodic closing lemma.

x is <u>well closable</u> means:

- $\forall \varepsilon > 0$ there is g ε-C^1-close to f
  $\exists\, n > 0$ $g^n(x) = x$ and
  $$d(g^i(x), f^i(x)) < \varepsilon \,,\quad i \in \{1, \dots, n\}$$

$$\mathcal{W}(f) = \{ x \in M \mid x \text{ is well closable} \}$$

Theorem (Mañé)

$$\left.\begin{array}{l} \forall \mu \text{ probability}, \\ f_*(\mu) = \mu \end{array}\right\} \Longrightarrow \mu(\mathcal{W}(f)) = 1$$

f C^1-generic: "any phenomenon seen by ergodic measures appears on a periodic orbit".

$\forall \mu$ ergodic $\exists\, \gamma_n \in Per(f)$ $\gamma_n \xrightarrow{\text{Hausdorff}} supp\, \mu$, $\gamma_n \xrightarrow{\text{weak}} \mu$

Frank's lemma.

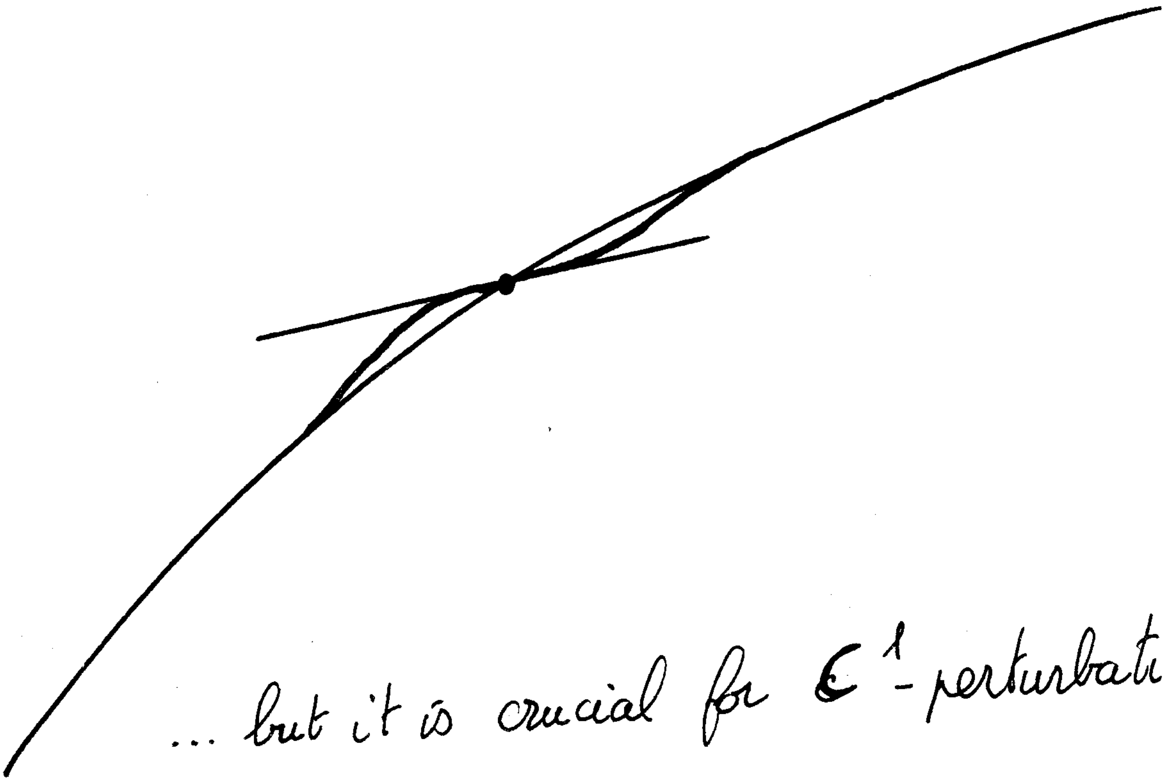
$\forall\, \mathcal{U}$ C^1-neighborhood of f,
$\exists\, \delta > 0$,
$\forall\, E \subset M$ finite $\forall\, V$ neighborhood of E
$\forall\, \{A(x),\ x \in E,\ A(x) : T_x M \to T_{f(x)} M\}$
such that $\| Df(x) - A(x) \| < \delta$

$\exists\, g \in \mathcal{U}$, $g \equiv f$ on $M \setminus V$

- $g = f$ on E
- $\forall\, x \in E$ $\quad Dg(x) = A(x)$

" one can realize a small perturbation of the differential of f by a small C^1-perturbation of f in an arbitrary neighborhood of E. ".

the proof is an exercise ...



... but it is crucial for C^1-perturbation.

- It is wrong in C^2-topology -

- The ergodic closing lemma transfers many problems seen by measures on the periodic orbits.
- The Franks's lemma transfers problems on the derivative of f in a problem of perturbation of products of matrices.

Dominated splitting / wild behavior

(∞ many
sinks or sources

Theorem (Mañé; Díaz, Pujals, Ures; B-, Díaz, Pujal; B-, Gourmelon, Vivier)

$\forall \mathcal{U} \ni f$ open

$\exists C > 0$, $0 < \lambda < 1$

$\exists N > 0$

$\left.\begin{array}{l} \forall p \in Per(f) \\ period(p) \geq N \end{array}\right\} \Rightarrow$

either $\exists g \in \mathcal{U}$, p sink or source

or $TM|_{orb(p)} = E \oplus F$

$E <_{C,\lambda} F$

(Abdenur, B-, Crovisier)

Corollary

$\left.\begin{array}{l} f \; C^1\text{-generic} \\ C \text{ chain rec. class} \end{array}\right\} \Rightarrow$

either $TM|_C = E \oplus F$

$E < F$

or. $\exists \gamma_n$ sinks or sources

$\gamma_n \xrightarrow[\text{Hausdorff}]{} C$

Corollary (Abdenur, B-, Crovisier).

f C^1-generic $\nearrow$ either $\#\{\text{sinks or sources}\} = \infty$

$\searrow$ or $\mathcal{R}_f = \Lambda_1 \cup \cdots \cup \Lambda_{d-1}$

- $TM|_{\Lambda_i} = E_i \oplus F_i$
- $E_i < F_i \qquad \dim E_i = i$
- $\Lambda_i \cap \Lambda_j = \emptyset, \; i \neq j.$

Dominated splitting / homoclinic tangencies.

Theorem (Pujals Sambarino, Wen, Gourmelon) $\forall \mathcal{U} \ni f$ open.

$\exists C>0 \ \exists N>0 \ \exists \lambda \in]0,1[$

$\forall p \in Per(f)$, $period(p) \geq N$ $\Rightarrow$

- either $E^s \oplus E^u$ is (C,λ) dominated on $orbit(p)$
- or $\exists g \in \mathcal{U}$, $W^s_{loc}(p)$ not $\pitchfork$ to $W^u_{loc}(p)$

<u>Corollary</u>

f C^1-generic, C isolated class $\Rightarrow$ $\exists$ $E^s \oplus E^c_1 \oplus \cdots \oplus E^c_k \oplus E^u$

- dominated,
- $\dim E^c_i = 1$
- E^s contracted
- E^u expanded

$\Rightarrow$ $\exists g$ C^1-close to f with a tangency in C.

<u>Conjecture</u> true without "isolated".

- what is the difficulty?:
- if E^s is not contracted uniformly, the defect may be in a small part of C.
- Using the ergodic closing lemma, the new orbit may not belong to C, and may be close to a small part of C.

Proof of the Theorem "Dom. Split / Sink or Source"

by induction on the dimension ..

- **dimension = 2 (Mañé).**

x_n sequence of periodic orbits

Π_n = period $\to \infty$.

and assume that (arguing by contradiction

1] none of x_n can be turned to be a $\left\{\begin{array}{l}\text{sink}\\ \text{source}\end{array}\right.$

2] there is no dominated splitting on $\bigcup \operatorname{orb}(x_n)$.

Remark | 1] $\Rightarrow$ $\exists\ \lambda \in]0,1[$ such that $\forall n$, $Df^{\Pi_n}(x_n)$ has 2 real eigenvalues $\left\{\begin{array}{l}\lambda_n^s < \lambda^{\Pi_n}\\ \lambda_n^u > \lambda^{-\Pi_n}\end{array}\right.$

Proof if not, multiply Df along the orbit by an homothety.

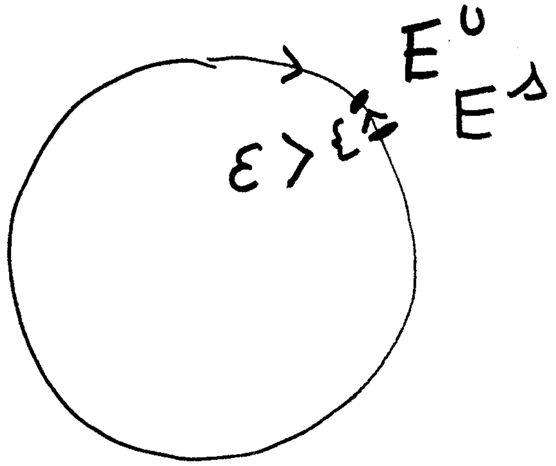
all x_n are saddles, $E^s \oplus E^u$ ...

<u>lemma</u> the angle $\measuredangle(E^s, E^u)$ is greater than some constant $\varepsilon > 0$ over $\bigcup \text{orbit}(x_n)$.

<u>proof</u>, assume $\measuredangle(E^s(y_n), E^u(y_n)) \to 0$

$$y_n = f^{i_n}(x_n).$$

$Df^{\pi_n}(y_n)$ acts on the projective space $\mathbb{RP}^1 \simeq S^1$.



$\to \exists\, s \in [-1, 1]$

$$R_{s\varepsilon} \circ Df^{\pi_n}(y_n)$$

has a complex eigenvalue.

$\to$ sink or source.

$\square$.

→ up to a bounded change of coordinates on can assume that

$$\forall x \in \bigcup \operatorname{orb}(x_n) \quad Df(x) = \begin{pmatrix} a(x) & 0 \\ 0 & b(x) \end{pmatrix}$$

with

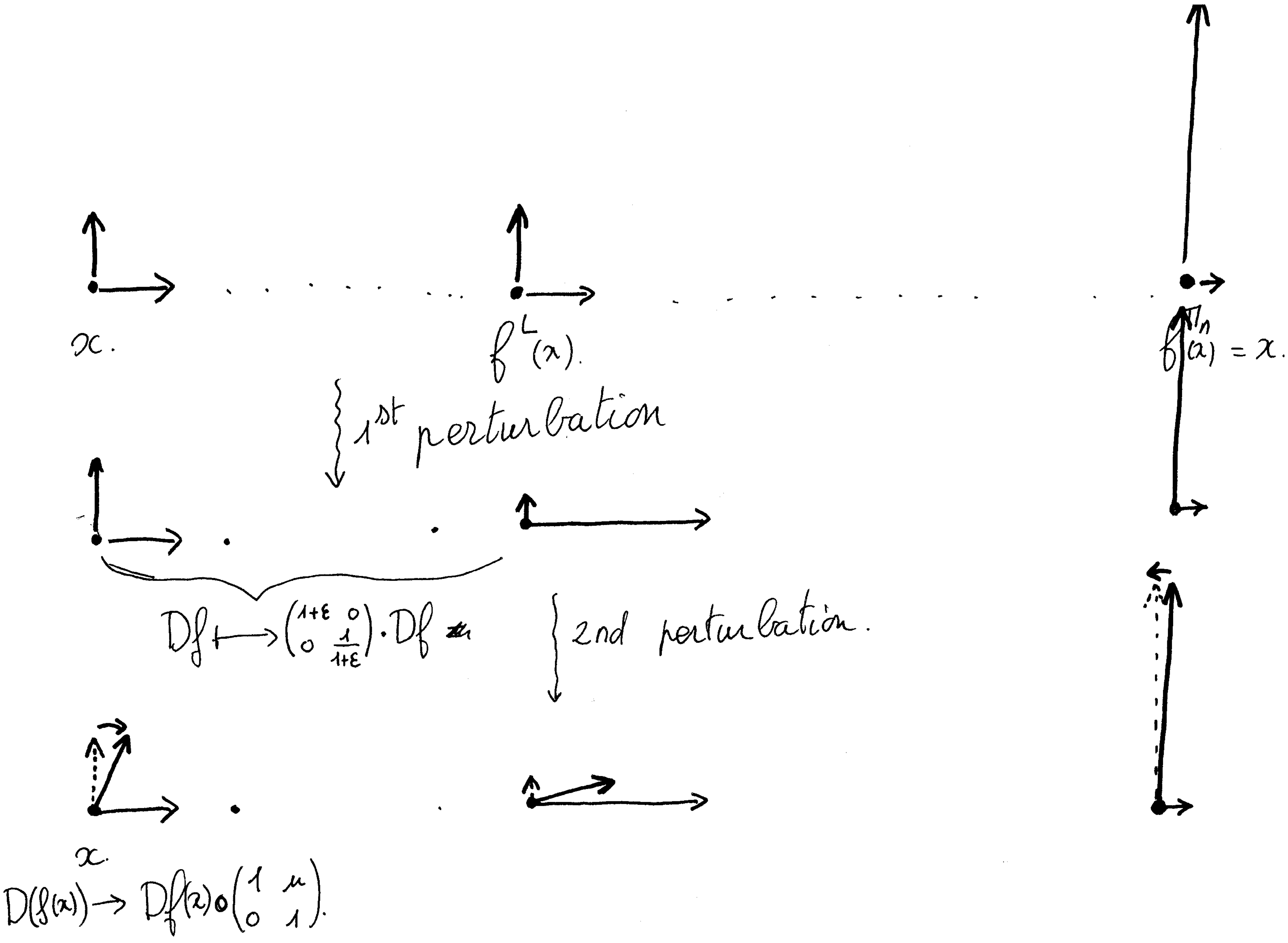
$$\begin{cases} \prod_{x \in \operatorname{orb}(x_n)} a(x) < \lambda^{\pi_n} \\ \prod_{x \in \operatorname{orb}(x_n)} b(x) > \lambda^{-\pi_n}. \end{cases}$$

No dominated splitting ?

→ $\forall L > 0 \ \exists n$ such that

$\exists x \in \operatorname{orb}(x_n)$ wich do not see

any domination of $E^s(x)$ by $E^u(x)$ before the time L.


x.
$f^L(x)$.
$f^n(x) = x$.
1st perturbation
$Df \longmapsto \begin{pmatrix} 1+\varepsilon & 0 \\ 0 & \frac{1}{1+\varepsilon} \end{pmatrix} \cdot Df$
2nd perturbation.
x.
$D(f(x)) \to Df(x) \circ \begin{pmatrix} 1 & \mu \\ 0 & 1 \end{pmatrix}$.

- So we created a small angle between the eigenspaces at $f^L(x)$.
- a new perturbation creates a complex eigenvalues.

indeed, our theorem is:

$\forall \varepsilon \ \exists N, \ \pi_n > N \Rightarrow$ there is an ε-perturbation of Df along the orbit such that $\delta(x_n) = 0$.

<u>idea 0</u>: make a sequence of perturbation decreasing $\delta(x_n)$..

<u>idea 1</u>. look all ~~the~~ plan~~s~~ ~~containing the eigen vectors associated to $\lambda^s(x_n)$ and $\lambda^u(x_n)$~~ $E^s(x_n) \oplus E^u(x)$

→ we have a product of n matrices 2×2 with norm bounded by $\|Df\|$ (and inverse bounded by $\|Df^{-1}\|$).

assume that $E^s(x) \not\perp E^u(x)$

⇝ Mañé's argument in this plane, but stoped just before making a complex eigenvalue ⇝

$$\tilde{\delta}(x_n) \leq \frac{1}{2}\,\delta(x_n)$$

(one can realize the perturbation on $E^s \oplus E^u$ without changing the eigenvalue λ^c, but this changes $E^c \dots$).

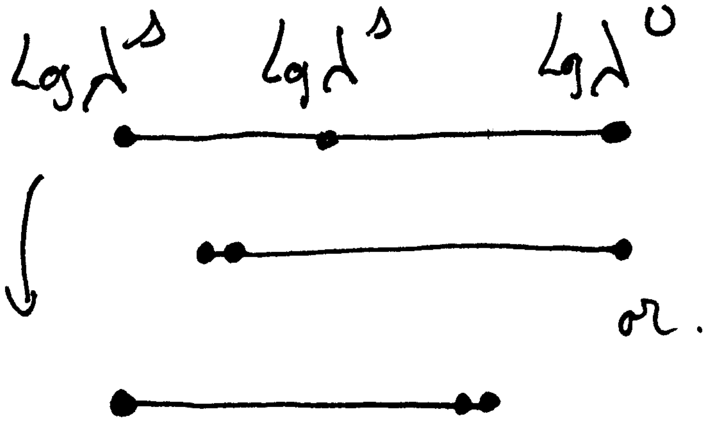
so assume $E^s < E^u$

- We look at the plans $\mathbb{R}^3/_{E^s}$ and $\mathbb{R}^3/_{E^u}$ with the metrics of $(E^s)^\perp$ and $(E^u)^\perp$.

the qotient system are also bounded by $\|Df\|$ and $\|Df^{-1}\|$.

<u>claim</u> if both quotient system are not dominated, $\exists\varepsilon$ perturbation

$$\tilde{\delta}(x_n) \leq \frac{3}{4}\,\delta(x_n).$$



so assume that on the quotient system on $\mathbb{R}^3/E^s$ ~~has a~~ the projection $E^c_{/E^s} \prec E^u_{/E^s}$

Lemma (B., Díaz, Pujals)

$$\left.\begin{array}{l} E^s \prec E^u \\ E^c_{/E^s} \prec E^u_{/E^s} \end{array}\right\} \Rightarrow (E^s \oplus E^c) \prec E^u$$

so, if Df has no dominated splitting it is possible to decrease $\delta(x) \to \leq \frac{3}{4}\delta(x)$

- in dimension 3.

lemma 1 (B-, Crovisier) $\forall \varepsilon, \exists N \; \forall x \in \mathrm{Per}(f)$ with $\pi(x) > N$, $\exists$ ε-perturbation of Df along $\mathrm{orb}(x)$ such that all eigenvalues (at the period) are real and with multiplicity 1.

- Then one may assume $\forall x \in \bigcup_n \mathrm{orb}(x_n)$, Df is a triangular matrix $\begin{pmatrix} a(x) & ? & ? \\ 0 & b(x) & ? \\ 0 & 0 & c(x) \end{pmatrix}$ with

$$\Big|\prod_{x\in \mathrm{orb}(x_n)} a(x)\Big| < \Big|\prod_{x\in \mathrm{orb}(x_n)} b(x)\Big| < \Big|\prod_{x\in \mathrm{orb}(x_n)} c(x)\Big|$$

$$\lambda^s(x) \qquad \lambda^c(x) \qquad \lambda^u(x)$$

$$E^s(x) \qquad E^c(x) \qquad E^u(x)$$

- Lijapunov diameter of x_n:

$$\delta(x_n) = \frac{\mathrm{Log}\,\lambda^u(x_n) - \mathrm{Log}\,\lambda^s(x_n)}{\pi_n}$$

5 Hyperbolic properties of the chain recurrence classes

5.1 The ergodic closing lemma

A point $x \in M$ is called *well closable* if for any $\varepsilon > 0$ there is g ε-C^1-close to f and $n > 0$ such that $g^n(x) = x$ and $d(f^i(x), g^i(x)) < \varepsilon$ for every $i \in \{0, \dots, n\}$. Let $\mathcal{W}$ be the set of well closable points.

Mañé proved:

Theorem 5.1. *For any invariant probability μ one has $\mu(\mathcal{W}) = 1$.*

5.2 Perturbations of the differential along periodic orbits

Lemma 5.1. *(Franks's lemma) Let M be a compact manifold and f a C^1 diffeomorphism of M. Let $E \subset M$ be a finite set, and $\varepsilon > 0$. For any $x \in E$ let $A_x \colon T_xM \to T_{f(x)}M$ be a linear map such that $\|A_x - D_xf\| < \varepsilon$. Then, for any neihborhood U of E, there is g such that:*

1. $\|f - g\|_1 < \varepsilon$
2. *$f = g$ out of U and on E*
3. *for any $x \in E$ one has $D_xg = A_x$*

This lemma has been used by Mañé for perturbing the differential of f along periodic orbits allowing to turn a saddle in a sink or a source: the fact that it is possible or not just depends of the differential of f allon the orbit. Mañé discovered that the main restrictions on what we get after pertubating the differential comes from the dominated splitting carried by the orbit.

Definition 5.2. *Let K be a f-invariant set, and $T_KM = E \oplus F$ be an invariant splititng over K (that is, $E(f(x)) = D_xf(E(x))$ for x in K). This splitting is* dominated *if there is $C > 0$ and $\lambda < 1$ such that for any $x \in K$ and any unit vectors $u \in E(x)$ and $v \in F(x)$ ($\|u\| = \|v\| = 1$), for any $n > 0$ one has:*

$$\|D_xf^n(u)\| < C \cdot \lambda^n \|D_xf^n(v)\|$$

We denote $E \prec F$. Sometimes we emphathize the strenght of the domination: the splitting is (C, λ)-dominated and we denote $E \prec_{C,\lambda} F$.

First Mañé in dimension 2 and then [DPU] (in dimension 3, with dynamical assumption), [BDP](in any dimension but with dynamical assumption) and now [B-, Gourmelon, Vivier]

Theorem 5.2. *Let M be a compact riemannian manifold, and $k > 0$ some positive number. Then for any $\varepsilon > 0$ there is $n > 0$, $C > 0$ and $\lambda \in]0, 1[$ such that, for any diffeomorphism f verifying $max\{\|D_xf\| \|D_x(f^{-1})\| x \in M\} < k$, for any periodic point x with period $p(x) \geq n$ one of the two following properties holds:*

1. *either f admits a (C, λ)-dominated splitting on the orbit of x*
2. *or, for any neighborhood U of the orbit of x, there is g , ε-C^1-close to f, coinciding with f out of U and on the orbit of x, and such that the differential $Dg^{p(x)}(x)$ has all eigenvalues real and of same modulus.*

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Pujals and Sambarino in dimension 2, then Wen in any dimension(but with a weaker statement), and finally Gourmelon in any dimension proved:

Theorem 5.3. *Let M be a compact riemannian manifold, and $k > 0$ some positive number. Then for any $\varepsilon > 0$ there is $n > 0$, $m > 0$, $C > 0$, and $\lambda \in]0, 1[$ such that, for any diffeomorphism f verifying $max\{\|D_x f\| \|D_x(f^{-1})\| x \in M\} < k$, for any periodic point x with period $p(x) \geq n$ one of the two following properties holds:*

1. *either f admits a (C, λ)-dominated splitting $E^s \oplus E_1^c \oplus \cdots \oplus E_k^c \oplus E^u$ on the orbit of x such that*
   (a) *$dim E_i^c = 1$*
   (b)
   $$\Pi_{i=0}^{p(x)-1} \|Df^m(f^i(x))|_{E^s(f^i(x))}\| < \lambda^{p(x)} \quad and \quad \Pi_{i=0}^{p(x)-1} \|Df^{-m}(f^i(x))|_{E^u(f^i(x))}\| < \lambda^{p(x)}$$
2. *or, for any neighborhood U of the orbit of x, there is g , ε-C^1-close to f, coinciding with f out of U and on the orbit of x, and such that the (local)invariant manifolds of x present a homoclinic tangency.*

6 Solution of the exercises

1. Rotations of varying angles on a continuous family of circles (for instance on the torus or on the sphere S^2), or the shift on the Cantor set $\{0,1\}^{\mathbb{Z}}$.

2. A compact orbit is either periodic or has no isolated point. However any countable compact set has isolated points, contradiction.

3. On the torus $T^2 = \mathbb{T}^2/\mathbb{Z}^2$, consider the vector field $Z(x,y) = \left((\sin(\pi x))^2 + (\sin(\pi y))^2\right)\frac{\partial}{\partial x}$. It has a unique zero at $(0,0)$. All the trajectories of the vectorfield are periodic except on the circle $y = 0$, where all the orbits tends to the fixed point $(0,0)$. Let f be the time one map of Z. Then $\mathrm{Rec}(f)$ is not compact.

4. Let f be the shift on $\{0,1\}^{\mathbb{Z}}$.

   Consider the point

   $\cdots,0,0,0,\cdots,0\,,1\,,0,0\,,0,1\,,1,0\,,1,1\,,0,0,0\,,0,0,1\,,0,1,0\,,0,1,1\,,1,0,0\,,1,0,1\,,1,1,0\,,1,1,1\,,0,0,0,0\,,$

   Its negative orbit tends to $\cdots 000000000000000000000000000000000000000000\cdots$ and so it is not α-recurrent. Its positive orbits is dense in $\{0,1\}^{\mathbb{Z}}$ so that is is ω-recurrent.

5. (a) Irrational rotations shows that $\overline{\mathrm{Per}(f)}$ may be strictly contained in $\mathrm{Rec}(f)$.

   (b) Consider the vectorfield

   $$Z(x,y) = \left((\sin(\pi x))^2 + (\sin(\pi y))^2\right)\left(\frac{\partial}{\partial x} + \sin(\pi y)^2\frac{\partial}{\partial y}\right)$$

   Then, for every point $p = (x,y)$ with $y \neq 0$ the trajectory of p acumulates on the whole circle $y = 0$. Let f be the time one map of Z. Then $\mathrm{Rec}(f)$ is the fixed point $(0,0)$ but $\mathrm{Lim}(f)$ is the circle $y = 0$.

   (c) On the Mœbius band $\mathbb{M} = \mathbb{R}\times[-1,1]/(x,y) \sim (x+1,-y)$ one consider a function $\varphi\colon \mathbb{M} \to [0,+\infty[$ vanishing exactly on the segment $\{0\}\times[0,1]$. One denote by f the time 1 map of the vector field $Y = \varphi(x,y)\frac{\partial}{\partial x}$. Then the limit set of f is precisely the segment $\{0\}\times[0,1]$ but the non-wandering set is the union of this segment and of the circle $S^1\times\{0\} \subset \mathbb{M}$.

   (d) The time one map of $\sin(k\pi x)^2\frac{\partial}{\partial x}$ on the circle S^1 is a simple example where $\Omega(f) \neq \mathcal{R}(f)$.

6. Let σ denote the shift on $\Sigma = \{0,1\}^{\mathbb{Z}}$.

   We will exhibit two numbers n, m and two invariant disjoint compact sets Σ_n and Σ_m such that f^n admits an invariant compact set in Σ_0 on which f^n is conjugated to the shift f, and f^m admits an invariant compact set in Σ_1 on which f^m is conjugated to f.

   Let denote $a = (0,1)$, $b = (1,0)$ and $c_n = (00\cdots 00110011\cdots 11)$ begininig by n letters "0" and finishing by n letters "1".

   Let $S_n \subset \Sigma$ be the set of infinite words generated by the finite words $a_n = c_{n-2}a$ and $b_n = c_{n-2}b$, that is words obtained by (infinite) concatenation of words of length n each of them being a_n or b_n. Let $\Sigma_n = S_n \cup \sigma(S_n) \cup \cdots \cup \sigma^{n-1}(S_n)$. Then for any n, m

greater than 5 the sets Σ_n are pairwize disjoint invariant compact sets. Furthermore, $S_n, \sigma(S_n), \dots, \sigma^{n-1}(S_n)$ are pairwize disjont and the restriction of f^n to S_n is conjugated to the shift σ

Then each of these shifts contains two disjoint invariant compact sets on which some iterates induces a shift... one build a family of decreasing invariant compact sets parametrized by an infinite regular tree, and each end of this tree corresponds to an invariant compact set: then one get a family of pairwize disjoint compact sets parametrized by a Cantor set. Finally each of these compact sets contains a minimal set. Indeed, it is not hard to verify that the compact sets we build are minimal and uniquely ergonic and even conjugated to some adding machine.

7. Just consider any homeomorphism on the circle having a unique fixed point. Then the non-wandering set is reduced to the fixed point, but the union of the positive (resp. negative) iterates of any neighborhood of this fixed point is the whole circle.

8. On a manifold, given any two disjoint compact sets K_0 and K_1, there is a smooth function ψ with value on $[0,1]$ such that $\psi^{-1}(0) = K_0$ and $\psi^{-1}(1) = K_1$. So we chose ψ_n equal to 1 exactly on $M \setminus f^n(U)$ and equal to 1 exactly on $f^{n+1}(\bar{U})$.

   Now one choose $a_i \in]0, \frac{1}{i^2}[$ such that (in a choice of chart covering the manifold M) all the derivatives of order les than i of $a_i\psi_i$ are bounded by $\frac{1}{i^2}$. Now the announced function is $\frac{1}{\sum_i a_i} \sum_i a_i \psi_i$.

9. if $x \dashv y_n$ and $y_n \to y$ then $x \dashv y$: for y_n close enough to y and $x = x_0, x_1, \dots, x_k = y_n$ an $\frac{1}{2}\varepsilon$-pseudo orbit joining x to y_n then $x = x_0, x_1, \dots, x'_k = y$ is a ε-pseudo orbit joining x to y. In the same way, if $z_n \dashv x$ and $z_n \to z$ then $z \dashv x$. So the clas of x is the intersection of two compact sets.

   The invariant follows from the easy remark that $x \in \mathcal{R}(f)$ implies that $x \dashv f^i(x)$ for all $i \in \mathbb{Z}$.

10. for any x consider the set $W^u_\varepsilon(x) = \{y \mid x \dashv_\varepsilon y\}$. One will show that $W^u_\varepsilon(x)$ is open and close and no empty, so that it coincides with X by connexity of X. This set is open by definition of ε-pseudo-orbits. We will show that it contains its $\varepsilon/2$-neighborhood, implying that it has no boundary, and so is compact. Let $y \in W^u_\varepsilon(x)$. Then the ε-open ball around $f(y)$ is also contained in $W^u_\varepsilon(x)$. On the other hand, as $\Omega(f) = X$, there are points z, arbitrarily close to $f(y)$, such that z has positive iterate $f^k(z)$ arbitrarly close to y. Then the open ε-ball around $f^k(z)$ is contained in $W^u_\varepsilon(x)$. In particular the $\varepsilon/2$-ball around y is contained in $W^u_\varepsilon(x)$.

11. if not, the Lyapunov fnction of the Theorem decrease strictly along an orbit, so that this orbit has its α and ω limits in different classes.

12. Just notice that any continuous function is a Lyapunov function for the identity map, which has a unique chain recurrent class if the space is connected.

    Consider now an irrational rotation around the vertical axe of the sphere S^2 (considered as being the unit sphere of $\mathbb{R}^3$). In that case, there is a unique chain recurrence class but the Lyapunov function are precisely the continuous functions depending only of the vertcal coordinate z.

13. Is is just the compacity of $\mathcal{R}(f)$.

14. $E^s(x)$ (resp. $E^u(x)$) is the set of vectors whose positive (resp. negative) iterates tends in norm to 0. If $u \in T_xM$ is a limit vector of a sequence $u_n \in E^s(x_n)$ with $x_n \to x$, the the uniform contraction of u_n by forward iterates implies the uniform contraction of u by forward iterates, and the uniform expansion of u_n by negative iterates implies the expansion of u. The same holds for the limit vectors of the unstable direction. This implies directly the continuity of the bundles.

15. I found the following example, obtained as the time one of a vector field on S^2. Just consider two saddles p, q of a vector field having two common separatrices γ_1 and γ_2 (one stable and one unstable for each one); the union of this two common separatrices is a topological circle separating S^2 is two disks D_1 and D_2; furthermore we assume that the two other separatrices of p are in D_1, and those of q are in D_2.

    It is easy two complete the picture by one sink and one source in each disk D_i, such that every non-singular orbit has its α andω-limits on two distinct singularity. Let f be the time one of this flow.

    Then $\Omega(f)$ is reduced to the fixed points, all hyperbolic, but $\mathcal{R}(f)$ is the sinks the sources and the circle $\gamma_1 \cup \gamma_2$ (containing the two saddles), and is not hyperbolic.

16. (a) Approximate (for the Hausdorff metric) the closure of the set of hyperbolic periodic points by a finite set of it. Now each of this points varies locally continuously (because they are hyperbolic) for the C^r-topology, $r \geq 1$: so evry g C^r close to f has the closure of its hyperbolic periodic points containing a subset Hausdorff-close to those of f.

    (b) If g_n converges to f (for the C^r-topology), $r \geq 0$, and $x_n \to x$, $x_n \in \mathcal{R}(g_n)$, then $x \in \mathcal{R}(f)$. For seing that just notice that the sequences $x, y_1^n, \dots, y_{k-1}^n, x$, where $x_n = y_0^n, y_1^n, \dots, y_{k-1}^n, x_n = y_k^n$ is a ε-pseudo-orbit for g_n, is a 2ε-pseudo-orbit for f for n large enough.

17. For f generic $W^u_{\dashv}(x, f) = \{y \mid x \dashv y\}$ and $\overline{W^u(x, f)}$ is Lyapunov stable.

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