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SUMMER SCHOOL AND CONFERENCE ON DYNAMICAL SYSTEMS

Polynomial Diffeomorphisms of C^2

(Lecture 3)

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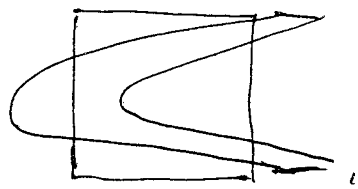
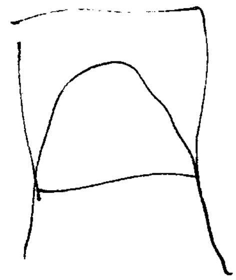
Recall that $f_{a,b}(x,y) = (-x^2+a-by, x)$ reduces to the one dimensional map $x \mapsto -x^2+a$ when $b=0$.

Since we will be using complex methods and complex dynamicists always use $z \mapsto z^2+c$ to write quadratic maps we will change our coordinate system to get

$$f_{a,b}(x,y) = (x^2+a-by, x)$$





Let $f_{a,b}: \mathbb{C}^2 \circlearrowleft$ be a complex Hénon diffeomorphism.

Let

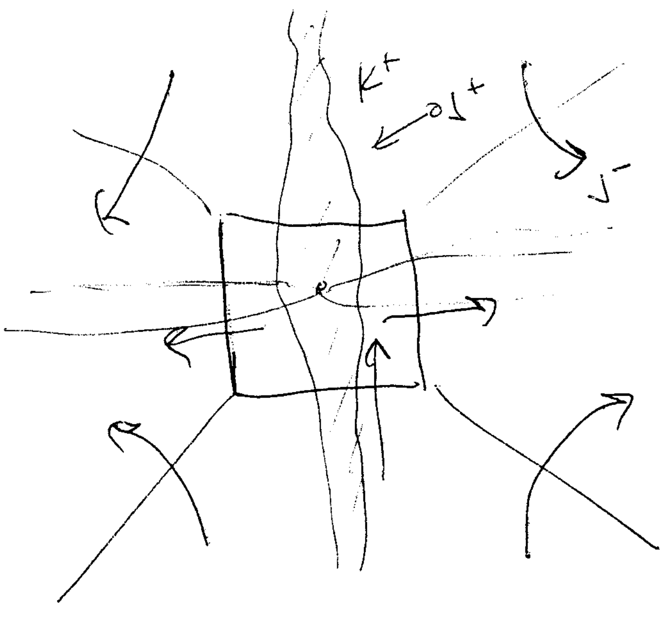
$$K^{\pm} = \{(x,y) : f^n(x,y) \not\to \infty \text{ as } n \to \pm\infty\}$$

Let $J^{\pm} = \partial K^{\pm}$.

Note that $\operatorname{int} K^{\pm}$ consists of points that are Lyapunov stable in forward/backward time.

Let $J = J^+ \cap J^-$.

We call J the Julia set of $f_{a,b}$.



The next result relates the Julia set $J_{a,b}$ to the Julia set J_a for certain special values of the parameters.

Since $f_{a,b}|J_{a,b}$ is invertible while $f_a|J_a$ is not invertible this might seem hard to do.

We take advantage of a standard construction which turns a map into an invertible map.

Given $f_a : J_a \circlearrowleft$ define

$$\check{J}_a = \{(\ldots z_{-1}, z_0, z_1 \ldots) : z_j \in J_a \text{ and } f_a(z_j) = z_{j+1}\}.$$

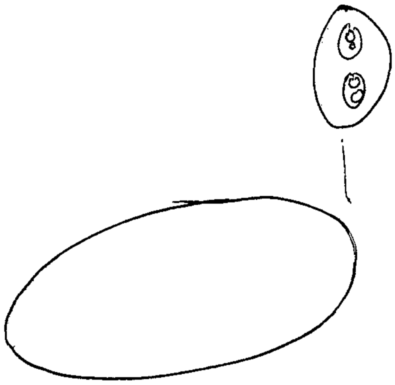
Since z_0 determines all the z_j with $j > 0$ a point in $\check{J}_a$ corresponds to a choice of a point $z_0 \in J_a$ and a sequence of inverse images.

$$\check{J}_a = \{(\ldots z_{-1}, z_0, z_1 \ldots) : z_j \in J_a \text{ and } f(z_j) = z_{j+1}\}.$$

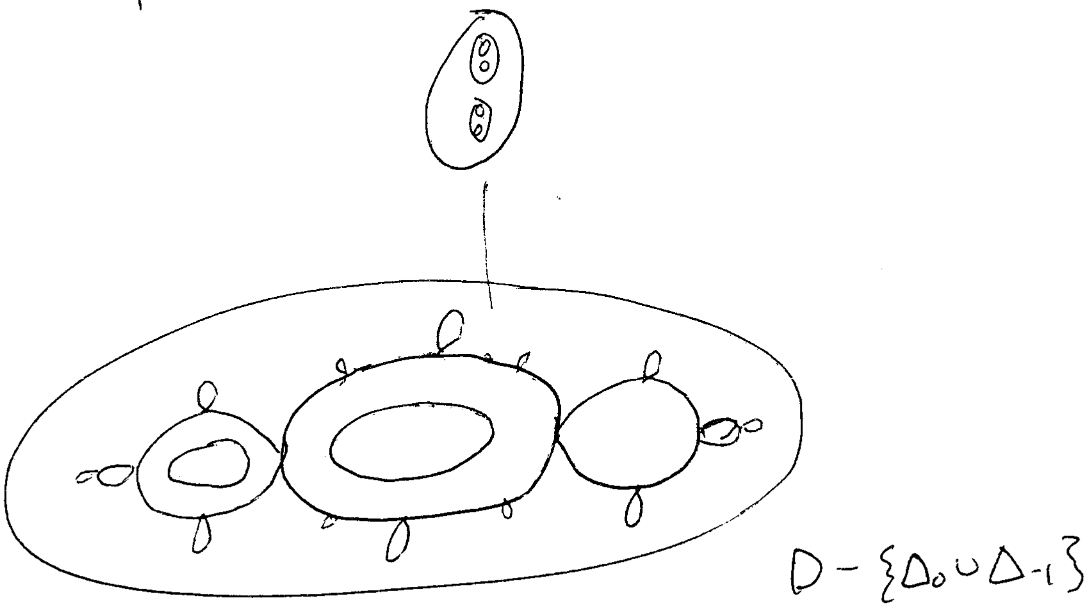
Let $\check{f}_a : \check{J}_a \circlearrowleft$ be the map obtained by applying f_a to each coordinate. Applying $\check{f}_a$ is equivalent to shifting the sequence to the left. The inverse of $\check{f}_a$ is obtained by shifting the sequence to the right.

Examples. If f_a is topologically equivalent to the one sided shift on two symbols then $\check{f}_a$ is the two sided shift on two symbols.

If f_a is the doubling map on the circle then $\check{J}_a$ is the solenoid.



Example. $\check{f}_{-1} : \check{J}_{-1} \circlearrowleft$



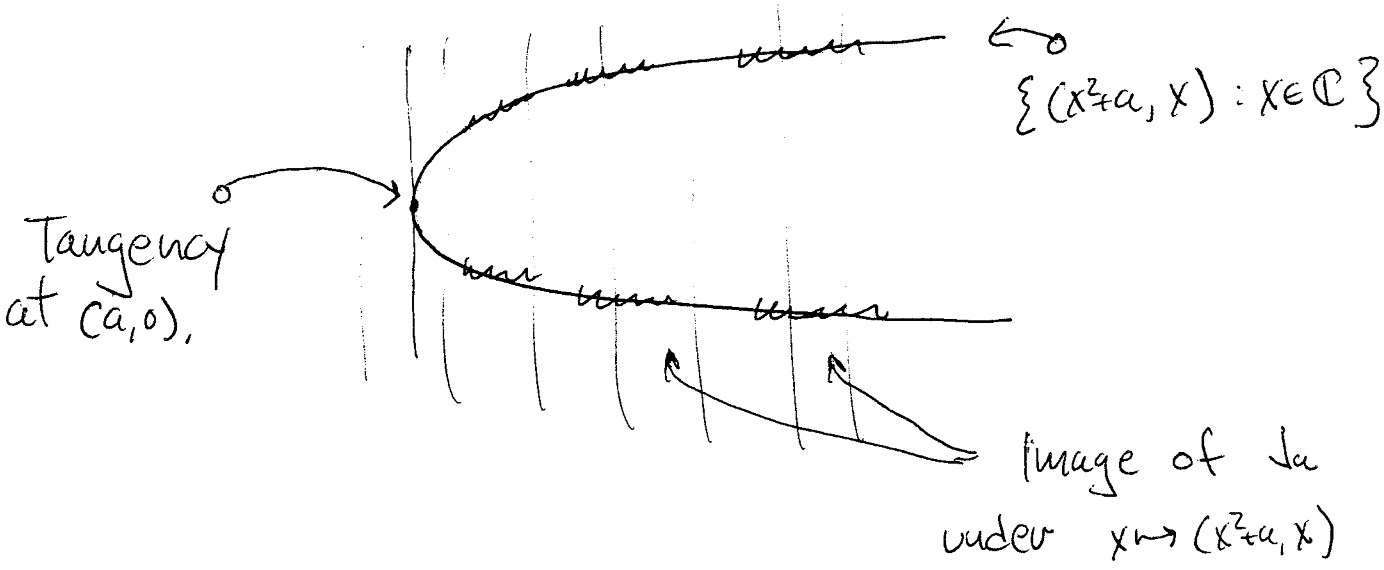
Advanced exercise: The kernel of the map from

$$\pi_1(D - \{\Delta_0 \cup \Delta_{-1}\}) \longrightarrow \mathrm{Aut}(\text{Cantor set})$$

is infinitely generated.

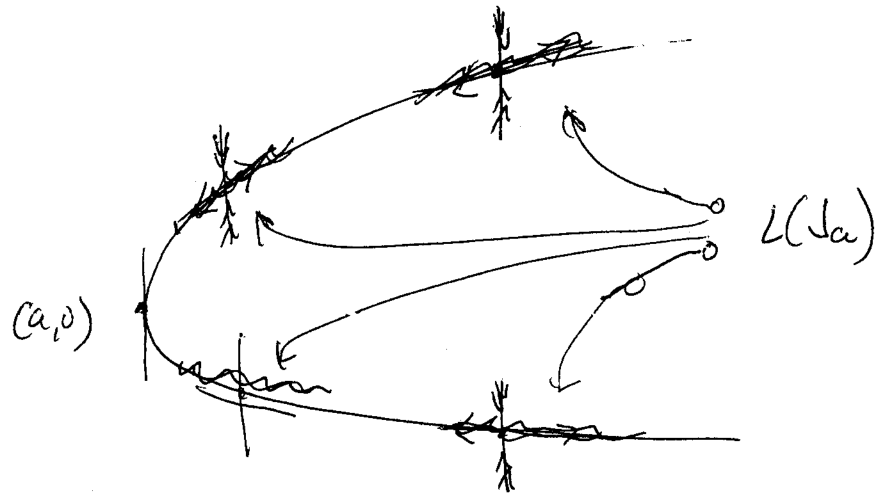
Theorem (Hubbard Oberste-Vorth)
If f_a is expanding on J_a then for $|b|$ sufficiently small $f_{a,b}$ is hyperbolic on $J_{a,b}$ and $f_{a,b}|J_{a,b}$ is topologically conjugate to $\check{f}_a : \check{J}_a \circlearrowleft$.

Proof. When $b=0$ we have the following picture



Note that the image of the Julia set is disjoint from the tangency.

Let $L(z) = (f_a(z), z)$ then $f_{a,0}(L(z)) = L(f_a(z))$ 7



To prove hyperbolicity we need expansion, contraction and transversality. When $b=0$ we get expansion along the image curve by assumption. We have contraction in the vertical direction and we have transversality because the critical point is not contained in J.

What is the connection between orbits for $f_{a,b}$ and those for f_a when b is small?

Say $(\cdots z_{-1}, z_0, z_1, z_2 \cdots) \in \check{J}$.
Consider the sequence $\cdots p_{-1}, p_0, p_1 \cdots$
where $p_j = L(z_j) = (\cancel{f}(z_j), z_j) = (z_{j+1}, z_j)$.

Then

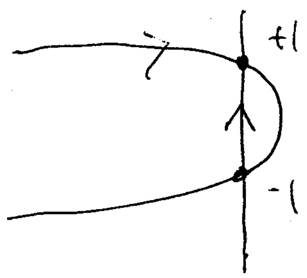
$$f_{a,b}(p_j) = f_{a,b}(z_{j+1}, z_j) = (f_a(z_{j+1}) - bz_j, z_{j+1})$$
$$= (z_{j+2} - bz_j, z_{j+1})$$
$$\approx$$
$$(z_{j+2}, z_{j+1}) = p_{j+1}.$$

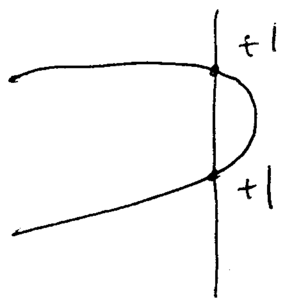
So an orbit for $\check{f}_a$ gives rise to an ε-pseudo-orbit for $f_{a,b}$. Similarly an orbit for $f_{a,b}$ gives rise to an ε-pseudo-orbit for $\check{f}_a$.

Remark: One advantage of working in $\mathbb{C}^2$ is that we have a well behaved theory of intersection multiplicities.

Real case

Complex case



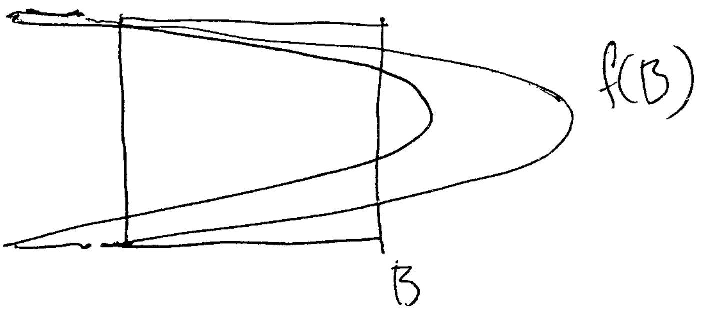


In both cases the sum of the intersection numbers remains the same as we perturb varieties. In the complex case all intersection numbers are positive.

Theorem. (Hubbard Oberste-Vorth)
If $|a| > (2+|b|)^2$ then $f_{a,b}$ is a horseshoe in $\mathbb{C}^2$.

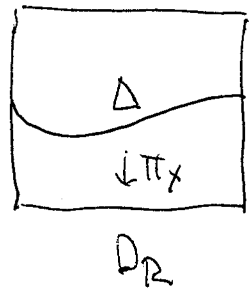
Proof. This numerical condition is the condition that $f(B) \cap B$ has two components.

$B = \{(x,y) : \begin{matrix} |x| < R \\ |y| < R \end{matrix}\}$

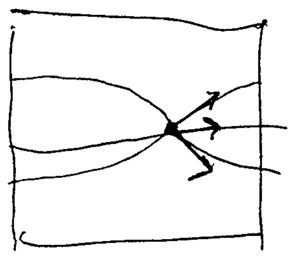


In order to construct E^u we need a cone field which is taken into itself and a metric on this cone field which is expanded.

We say that a complex disk $\Delta \subset B = D_R \times D_R$ has horizontal degree 1 if $\pi_x : \Delta \to D_R$ is a proper map of degree 1



At a point $p \in B$ let $C_p \subset T_p$ be the collection of tangent vectors to degree one disks Δ which pass through p.

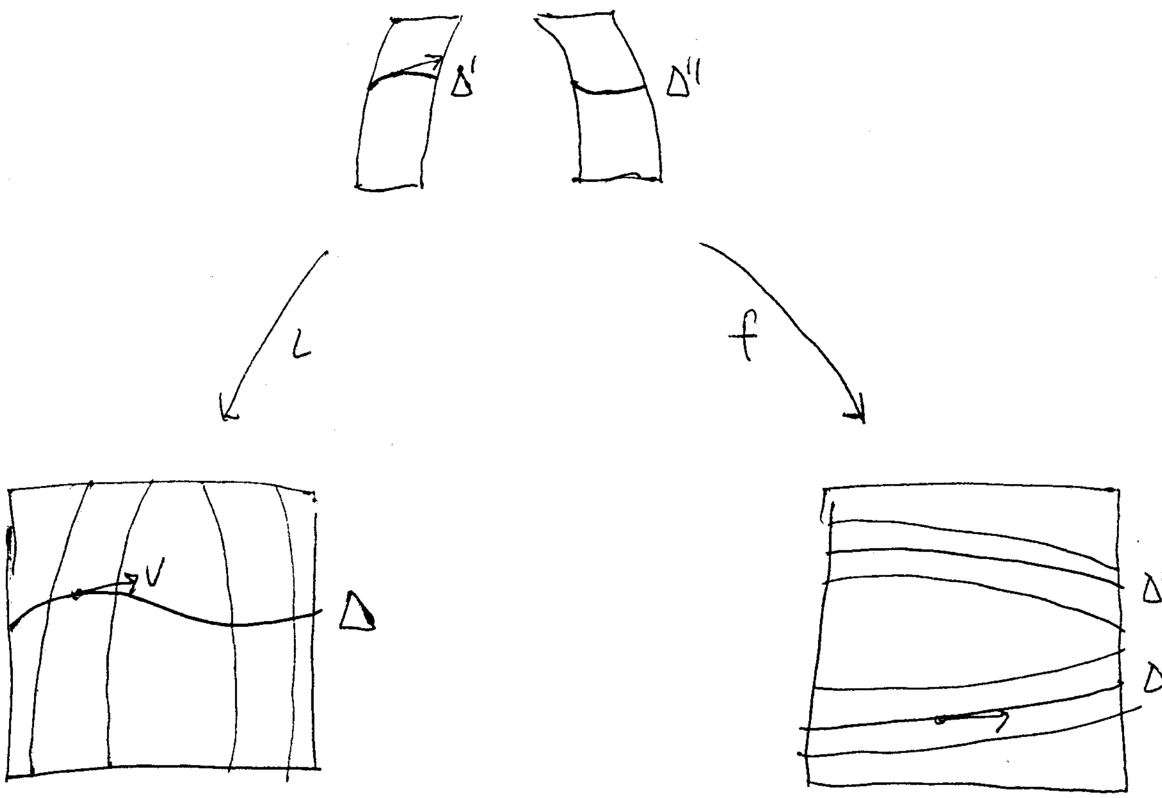


The cone C_p consists of vectors v based at p for which $|\pi_x(v)| \geq |\pi_y(v)|$ where both lengths are measured with respect to the Poincaré metric on D_R.

This follows since Δ is the graph of a holomorphic function $h: D_R \to D_R$.
$\Delta = \{(x, h(x))\}$.

If v is tangent to Δ then $Dh(\pi_x(v)) = \pi_y(v)$ so by the Schwarz-Pick Lemma: $|\pi_y(v)| \leq |\pi_x(v)|$.

We define a metric $|\cdot|^*$ on C_p by taking $|v|^*$ to be the length of v with respect to the Poincaré metric on Δ or, equivalently, to be $|\pi_X v|$.



Df preserves lengths.

$D\iota$ decreases lengths.

$D(f \circ \iota^{-1})$ increases lengths.

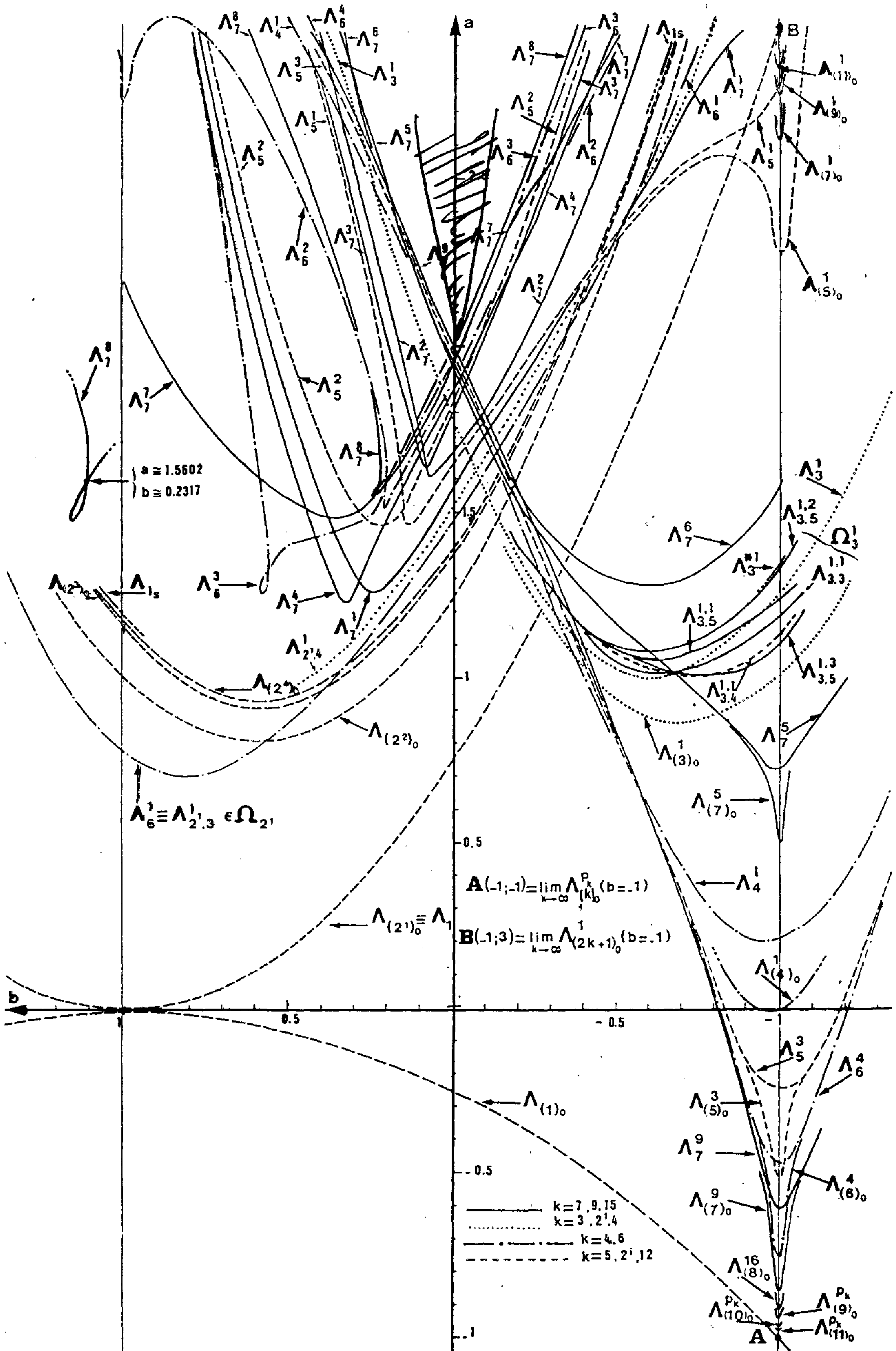
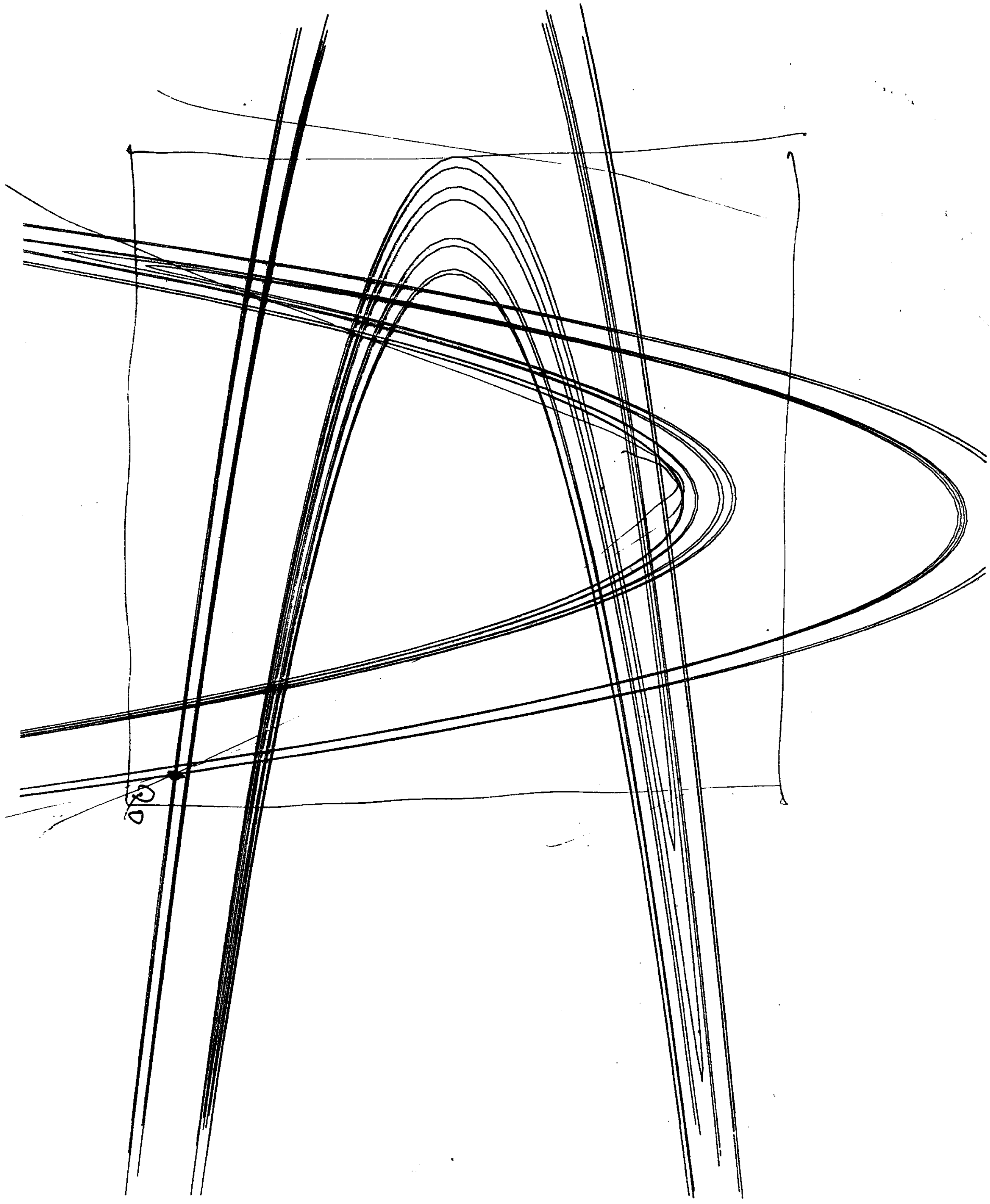

a
b
B
A
0.5
- 0.5
1
1.5
-0.5
-1
a ≅ 1.5602
b ≅ 0.2317
$\Lambda_6^1 \equiv \Lambda^1_{2^1,3} \in \Omega_{2^1}$
$\Lambda_{(2^1)_0} \equiv \Lambda_1$
$\Lambda_{(1)_0}$
$\Lambda_{(2^2)_0}$
Λ_{1s}
Ω_3^1
$\mathbf{A}(-1;-1) = \lim_{k\to\infty} \Lambda^{p_k}_{(k)_0}(b=-1)$
$\mathbf{B}(-1;3) = \lim_{k\to\infty} \Lambda^{1}_{(2k+1)_0}(b=-1)$
k = 7, 9, 15
k = 3, 2^1.4
k = 4.6
k = 5, 2^i, 12

FIG. 1



Potential Theory

Consider two electrons in $\mathbb{R}^d$ which repel each other with a force proportional to $\frac{1}{r^{d-1}}$ where r is the distance between them. Fix one electron at the origin. The work in moving the second electron from position z_0 to position z_1 is given by $P(z_1) - P(z_0)$. where P is a potential function

$$P(x) = |x| \quad \text{if } d=1$$

$$P(x) = \log\|x\| \quad \text{if } d=2$$

$$P(x) = \frac{-1}{\|x\|^{d-2}} \quad \text{if } d \geq 3.$$

If we want to consider a charge distribution rather than a single electron we can represent this by a measure μ. The potential function is given by a convolution:

$$P_\mu(z) = \int P(z-w)\,d\mu(w).$$

If we a function which we know to be a potential function of some measure μ is it possible to recover μ from the function?

We can recover μ from P_μ by taking the Laplacian:

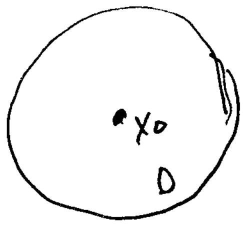
$$\Delta P_\mu = c \cdot \mu.$$

In general taking the Laplacian of a function which is not smooth yields a distribution. In this particular case the distribution is a positive distribution and is represented by a measure.

Example. In one dimension potential functions are convex. In this case the Laplacian is just the 2nd derivative operator

$$f \mapsto \frac{d^2}{dx^2} f.$$

In two dimensions potential functions are <u>subharmonic</u>: They are upper semi-continuous and have the property that the value at the center of a disk is bounded above by the average value on the boundary.



$f(x_0) \le$ avg val on ∂D.

Polynomials on $\mathbb{C}$ give rise to potential functions in a natural way.

Let $Q(z) = z^n + a_{n-1} z^{n-1} + \cdots a_0$.

Then $P(z) = \log |Q(z)|$ is a potential function.

We compute:

$$\begin{aligned} \Delta P(z) &= \Delta \log |Q(z)| \\ &= \Delta \log \left| \prod_j (z - z_j) \right| \\ &= \sum_j \Delta \log |z - z_j| \\ &= \sum_j \delta_{z_j}. \end{aligned}$$

This construction is useful in the following setting. We have some sequence of polynomials of increasing degree. We want to describe some asymptotic behavior. The polynomials do not converge but in some situations the potential functions (when suitably normalized) may converge.

Let f be a monic polynomial of degree d. Define

$$G(z) = \lim_{n \to \infty} \frac{1}{d^n} \log^+ |f^n(z)|.$$

$$\log^+(x) = \max(\log x, 0).$$

The function G measures the rate of escape. G is also a potential function.

G is subharmonic

G is harmonic outside of K

G is non-negative and equal to 0 exactly on the set K.

G is the Green function of K and $\mu_K = \Delta G$ is the equilibrium measure for K.

Theorem (Brolin 1965). Let f be a monic polynomial of degree d and let $c \in \mathbb{C}$ be a nonexceptional point. Then

$$\lim_{n\to\infty} \frac{1}{d^n} \sum_{\{z : f^n(z) = c\}} \delta_z = \mu_K .$$

Let $\mu_n = \frac{1}{d^n} \sum_{f^n(z)=c} \delta_z$.

Let G_n be the potential function for μ_n. μ_n corresponds to the roots of the polynomial

$$f^n(z) - c = 0$$

so

$$G_n = \frac{1}{d^n} \log |f^n(z) - c|.$$

Want to show that $G_n \to G$.

$$\lim_{n\to\infty} \frac{1}{d^n} \log |f^n(z) - c| = \lim_{n\to\infty} \frac{1}{d^n} \log^+ |f^n(z)|$$

For $z \notin K$ this is clear. The analysis of convergence on K requires a little potential theory.