SUMMER SCHOOL AND CONFERENCE
ON DYNAMICAL SYSTEMS

Polynomial Diffeomorphisms of $\mathbb{C}^2$
(Lecture 3)

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These are preliminary lecture notes, intended only for distribution to participants
Recall that $f_{a,b}(x,y) = (-x^2 + a - by, x)$ reduces to the one dimensional map $x \mapsto -x^2 + a$ when $b = 0$.

Since we will be using complex methods and complex dynamicalists always use $z \mapsto z^2 + c$ to write quadratic maps we will change our coordinate system to get

$$f_{a,b}(x,y) = (x^2 + a - by, x)$$
Let $f_{a,b} : \mathbb{C}^2 \rightarrow \mathbb{C}^2$ be a complex Hénon diffeomorphism.

Let
$$K^\pm = \{ (x, y) : f^n(x, y) \rightarrow \infty \text{ as } n \rightarrow \pm \infty \}.$$ 

Let $J^\pm = \partial K^\pm$.

Note that $\partial K^\pm$ consists of points that are Lyapunov stable in forward/backward time.

Let $J = J^+ \cap J^{-}$.

We call $J$ the Julia set of $f_{a,b}$. 
The next result relates the Julia set $J_{a,b}$ to the Julia set $J_a$ for certain special values of the parameters.

Since $f_{a,b}$ is invertible while $f_a$ is not invertible this might seem hard to do.

We take advantage of a standard construction which turns a map into an invertible map.

Given $f_a: J_a \rightarrow \mathbb{C}$ define

$$J_a = \{ \ldots, z_1, z_0, z_1, \ldots \}: z_i \in J_a \text{ and } f_a(z_i) = z_{i+1}.$$  

Since $z_0$ determines all the $z_i$ with $i > 0$ a point in $J_a$ corresponds to a choice of a point $z_0 \in J_a$ and a sequence of inverse images.
\[ f_a = \varepsilon (\ldots z_{-1}, z_0, z_1, \ldots) : z_j \neq \lambda \text{ and } f(z_j) = z_{j+1}. \]

Let $f_a: \Lambda^\mathbb{Z}$ be the map obtained by applying $f_a$ to each coordinate. Applying $f_a$ is equivalent to shifting the sequence to the left. The inverse of $f_a$ is obtained by shifting the sequence to the right.

Examples: If $f_a$ is topologically equivalent to the one-sided shift on two symbols then $f_a$ is the two-sided shift on two symbols.

If $f_a$ is the doubling map on the circle then $f_a$ is the solenoid.
Example. $\tilde{f} : \tilde{Y} \rightarrow \tilde{X}$

Advanced exercise: The kernel of the map from

$$\pi_1(D - \Delta_0 \cup \Delta_{-1}) \to \text{Aut}(\text{Cantor set})$$

is infinitely generated.
Theorem (Hubbard- Cassette-Vouleth)

If \( f_a \) is expanding on \( \mathbb{D} \) then for \( |a| \) sufficiently small \( f_a, b \) is hyperbolic on \( \mathbb{D} \) and \( f_a | \mathbb{D} \), is topologically conjugate to \( f_{\lambda} | \mathbb{D} \).

Proof. When \( b = 0 \) we have the following picture

Note that the image of the Julia set is disjoint from the tangency.
Let \( L(z) = (f_a(z), z) \) then \( f_{a_0}(L(z)) = L(f_a(z)) \).

To prove hyperbolicity we need expansion, contraction and transversality. When \( b = 0 \) we get expansion along the image curve by assumption. We have contraction in the vertical direction and we have transversality because the critical point is not contained in \( J \).
What is the connection between orbits for \( f_{a,b} \) and those for \( f_a \) when \( b \) is small?

Say \((\ldots, z_{-1}, z_0, z_1, z_2, \ldots) \in \mathcal{I}\).

Consider the sequence \( p_{-1}, p_0, p_1, \ldots \) where \( p_j = L(z_j) = (\mathring{f}(z_j), z_j) = (z_j+1, z_j) \).

Then
\[
+f_{a,b}(p_j) = f_{a,b}(z_{j+1}, z_j) = (fa(z_{j+1}) - b z_j, z_{j+1})
= (z_{j+2} - b z_j, z_{j+1})
\]
so
\[
(z_{j+2}, z_{j+1}) = p_{j+1}.
\]

So an orbit for \( f_a \) gives rise to an \( \varepsilon \)-pseudo-orbit for \( f_{a,b} \). Similarly an orbit for \( f_{a,b} \) gives rise to an \( \varepsilon \)-pseudo-orbit for \( f_a \).
Remark: One advantage of working in \( \mathbb{C}^2 \) is that we have a well-behaved theory of intersection multiplicities.

Real case

\[ \begin{array}{c}
\bullet \\
\bullet \quad +1 \\
-1 \quad +1
\end{array} \]

Complex case

\[ \begin{array}{c}
\bullet \\
\bullet \quad +1 \\
-1 \quad +1
\end{array} \]

In both cases the sum of the intersection numbers remains the same as we perturb varieties.

In the complex case all intersection numbers are positive.
Theorem. (Hubbard Oberste-Vorth)

If $|a| > (2+|a|)^2$ then $f_{a,b}$ is a horseshoe in $C^2$.

Proof. This numerical condition is the condition that $f(B) \cap B$ has two components.

\[ B = \{(x,y) : |x| < R \land |y| < R \} \]

\[ f(B) \]

In order to construct $E^u$ we need a cone field which is taken into itself and a metric on this cone field which is expanded.
we say that a complex disk $\Delta \subset \mathbb{B} = \mathbb{D}_R \times \mathbb{D}_R$ has horizontal degree 1 if $\pi_x: \Delta \to \mathbb{D}_R$ is a proper map of degree 1.

At a point $p \in \mathbb{B}$ let $C_p \subset T_p$ be the collection of tangent vectors to degree one disks $\Delta$ which pass through $p$. 
The cone $C_p$ consists of vectors $v$ based at $p$ for which $|\pi_x(v)| \geq |\pi_y(v)|$ where both lengths are measured with respect to the Poincaré metric on $D_R$.

This follows since $\Delta$ is the graph of a holomorphic function $h: D_R \rightarrow D_R$, $\Delta = \{(x, h(x)) : x \in D_R\}$.

If $v$ is tangent to $\Delta$ then $Dh(\pi_x(v)) = \pi_y(v)$ so by the Schwarz-Pick Lemma: $|\pi_y(v)| \leq |\pi_x(v)|$. 
We define a metric \( l_1 \) on \( C_p \) by taking \( l_1^* \) to be the length of \( v \) with respect to the Poincaré metric on \( \Delta \) or, equivalently, to be \( l_1 x_v l_1 \).

\[ \xymatrix{ \Delta' & \Delta'' \ar[l]_{\ell} \ar[r]^{f} & \Delta' \ar[u]^{\Delta} } \]

\[ Df \text{ preserves lengths.} \]
\[ Dl_1 \text{ decreases lengths.} \]
\[ D(f \circ l_1^*) \text{ increases lengths.} \]
Potential Theory

Consider two electrons in \( \mathbb{R}^d \) which repel each other with a force proportional to \( \frac{1}{r^d} \), where \( r \) is the distance between them. Fix one electron at the origin. The work in moving the second electron from position \( z_0 \) to position \( z_1 \) is given by \( P(z_1) - P(z_0) \), where \( P \) is a potential function.

\[
P(x) = \begin{cases} 
|x| & \text{if } d = 1 \\
\log ||x|| & \text{if } d = 2 \\
\frac{-1}{||x||^{d-2}} & \text{if } d \geq 3.
\end{cases}
\]
If we want to consider a charge distribution rather than a single electron we can represent this by a measure $\mu$. The potential function is given by a convolution:

$$P\mu(z) = \int P(z-w) d\mu(w).$$

If we a function which we know to be a potential function of some measure $\mu$ is it possible to recover $\mu$ from the function?
We can recover \( \mu \) from \( P\mu \) by taking the Laplacian:

\[
\Delta P\mu = c \cdot \mu.
\]

In general taking the Laplacian of a function which is not smooth yields a distribution. In this particular case the distribution is a positive distribution and is represented by a measure.

Example. In one dimension potential functions are convex. In this case the Laplacian is just the 2nd derivative operator

\[
f \mapsto \frac{d^2}{dx^2} f.
\]
In two dimensions potential functions are subharmonic. They are upper semi-continuous and have the property that the value at the center of a disk is bounded above by the average value on the boundary.
Polynomials on $\mathbb{C}$ give rise to potential functions in a natural way.

Let $Q(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_0$.

Then $P(z) = \log |Q(z)|$ is a potential function.

We compute:

$$\Delta P(z) = \Delta \log |Q(z)|$$

$$= \Delta \log \left| \prod_j (z - z_j) \right|$$

$$= \sum_j \Delta \log |z - z_j|$$

$$= \sum_j s_{z_j}.$$
This construction is useful in the following setting. We have some sequence of polynomials of increasing degree. We want to describe some asymptotic behavior.

The polynomials do not converge but in some situations the potential functions (when suitably normalized) may converge.
Let $f$ be a monic polynomial of degree $d$. Define

$$G(z) = \lim_{n \to \infty} \frac{1}{n} \log^+ |f^n(z)|.$$ 

$\log^+ (x) = \max (\log x, 0)$.

The function $G$ measures the rate of escape. $G$ is also a potential function.

- $G$ is subharmonic
- $G$ is harmonic outside of $K$
- $G$ is non-negative and equal to 0 exactly on the set $K$. 
$G$ is the Green function of $K$ and $\mu_k = \Delta G$ is the equilibrium measure for $K$.

Theorem (Booijin 1965). Let $f$ be a monic polynomial of degree $d$ and let $c \in \mathbb{C}$ be a nonexceptional point. Then

$$\lim_{n \to \infty} \frac{1}{d^n} \sum_{z \in \mathbb{Z} : f^n(z) = c^3} S_z = \mu_k.$$
Let $\mu_n = \frac{1}{f_n} \sum_{f_n(z) = c} z$.

Let $G_n$ be the potential function for $\mu_n$. $\mu_n$ corresponds to the roots of the polynomial $f_n(z) - c = 0$

so $G_n = \frac{1}{f_n} \log |f_n(z) - c|$.

Want to show that $G_n \to G$.

$$\lim_{n \to \infty} \frac{1}{f_n} \log |f_n(z) - c| = \lim_{n \to \infty} \frac{1}{f_n} \log |f_n(z)|$$

For $z \neq K$ this is clear. The analysis of convergence on $K$ requires a little potential theory.