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SUMMER SCHOOL AND CONFERENCE ON DYNAMICAL SYSTEMS

A Physical Anosov System

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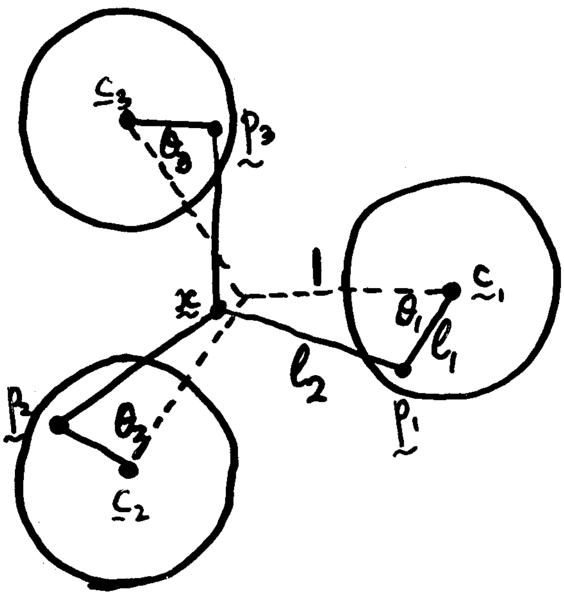
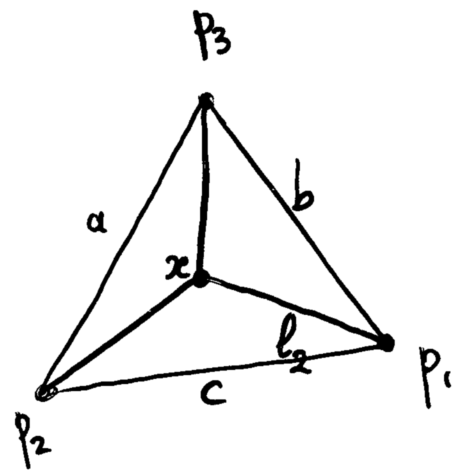
A physical Anosov system

11 mathematics lessons

based on T J Hunt & R S MacKay Nonlinearity 16 (2003) 1499



① Euclidean geometry



Configuration space Σ

$= \{ (\theta_1, \theta_2, \theta_3) \in \mathbb{T}^3 :$ circumradius of triangle $p_1 p_2 p_3 = \ell_2 \}$

i.e. $$\frac{abc}{\sqrt{(a+b+c)(a+b-c)(b+c-a)(c+a-b)}} = \ell_2$$

Can get a, b, c by Cartesian geometry

$$a^2 = 3 + 2\ell_1^2 + \ell_1 \left[\sqrt{3} \sin\theta_3 - 3\cos\theta_3 - \sqrt{3}\sin\theta_2 - 3\cos\theta_2 \right]$$

Hence equation for Σ

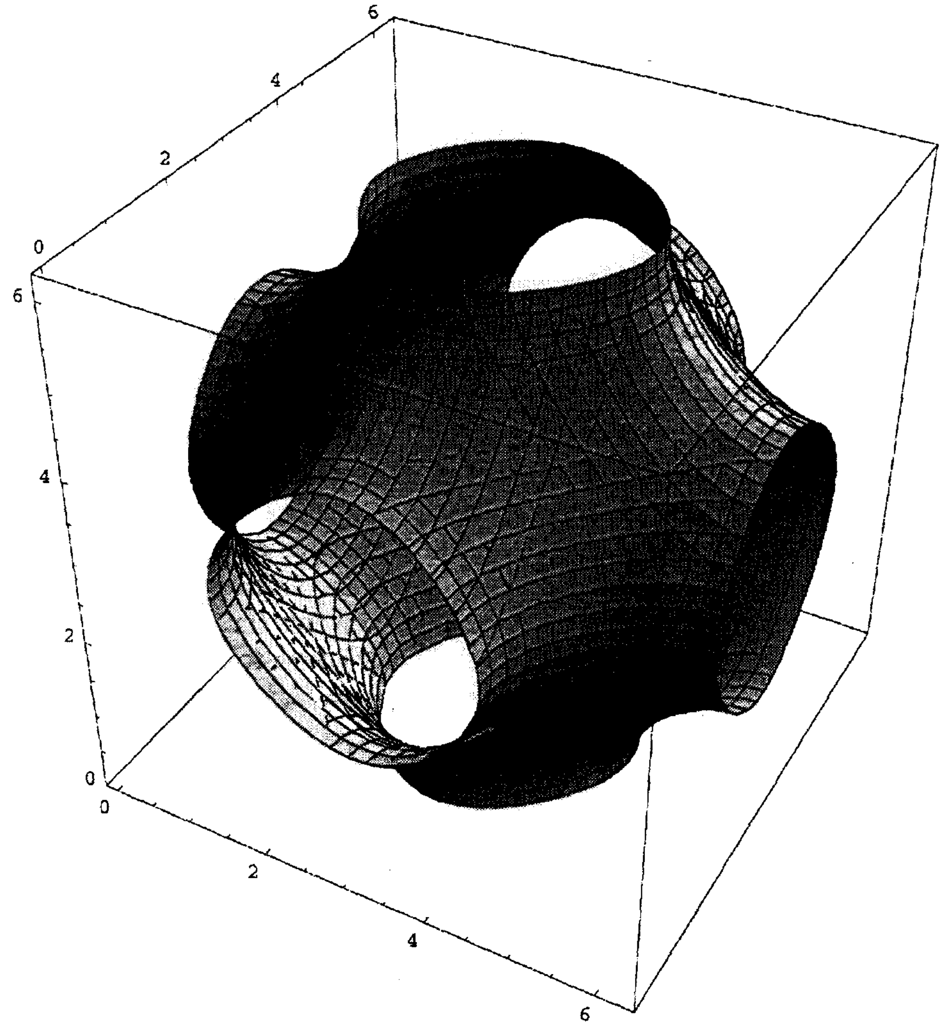


Figure 3. The configuration space as a subset of $\mathbb{T}^3$ for $l_1 = 19/180, l_2 = 181/180$.

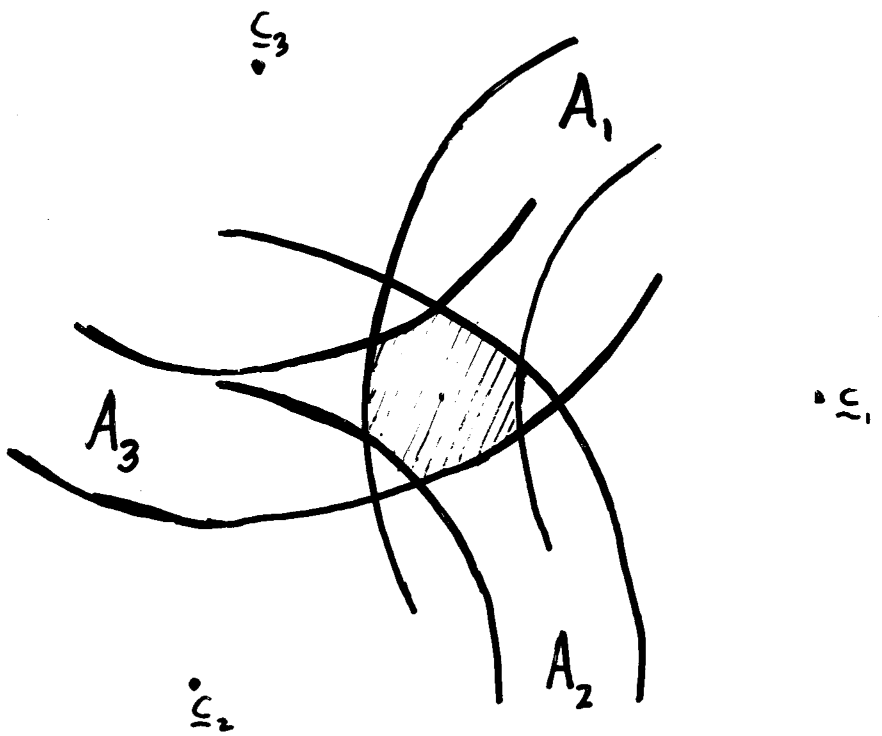
② Topology (cf. Thurston & Weeks)

{Allowed positions $\underline{x}$ of central pivot}

A = intersection of 3 annuli A_i, where

A_i is centred on centre of i^{th} disk, inner radius $|l_1 - l_2|$, outer radius $l_1 + l_2$

e.g. $l_1 = 0.2$, $l_2 = 1.0$



But to each $\underline{x}$ in the interior of A correspond 8 configurations, joining in pairs on the edges and in 4s at the vertices.

So Σ is made of $F = 8$ faces, meeting along $E = 24$ edges, and at $V = 12$ vertices. So Σ has Euler characteristic $\chi = F - E + V = -4$

Except for some special (ℓ_1, ℓ_2), Σ is an orientable compact surface, so homeomorphic to a disjoint union of X_g (surface of genus g) for some $g \geq 0$.
For $\ell_1 = 0.2$, $\ell_2 = 1.0$, Σ is connected, so just one X_g and then $\chi = 2 - 2g$ implies $g = 3$.

Can compute all the possibilities for the topological type of Σ as (ℓ_1, ℓ_2) vary.

Can also study the transitions between types ("cobordisms").

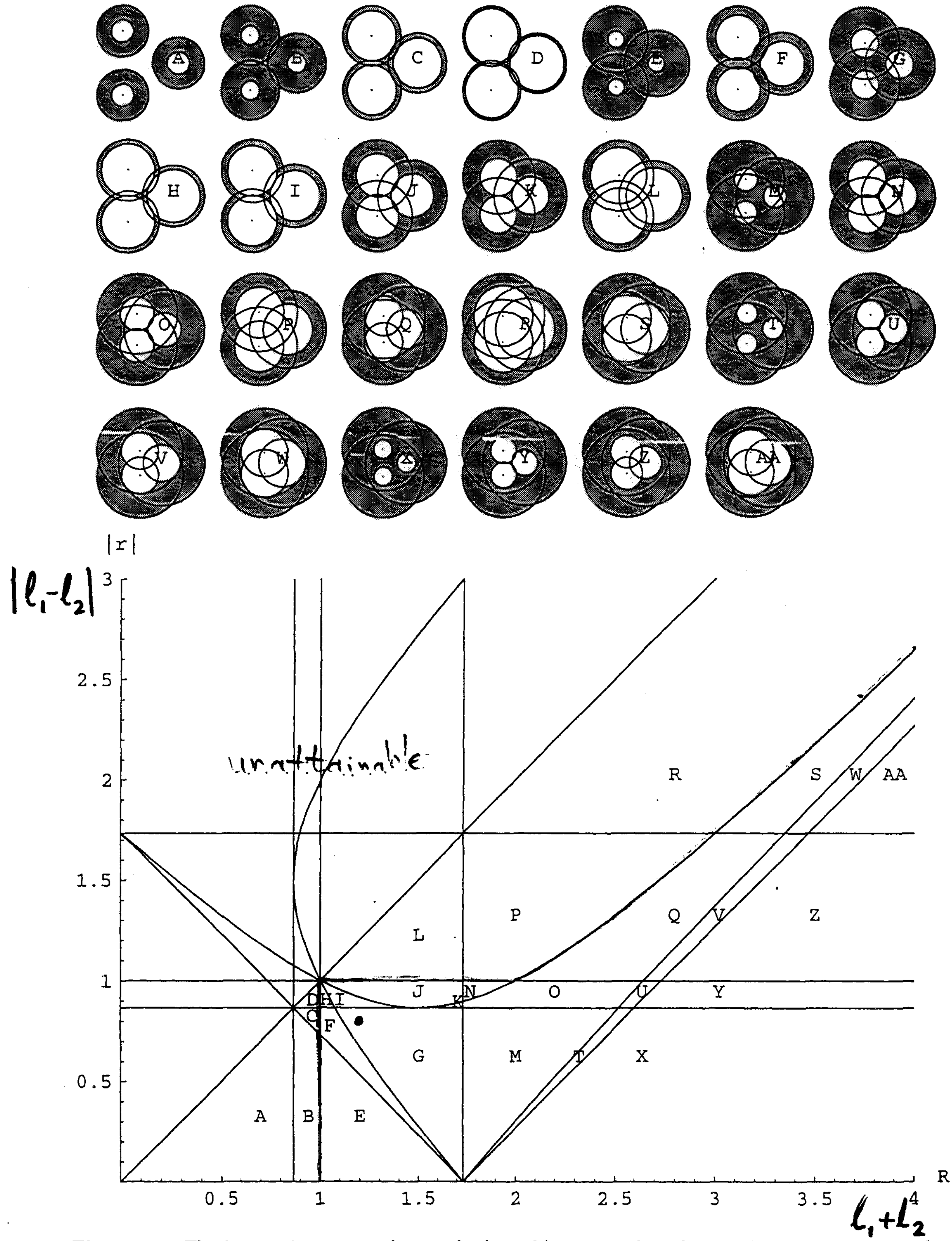


Figure 3.4: The 27 generic patterns that can be formed by the overlap of three identical annuli centred on the corners of an equilateral triangle and the regions of parameter space corresponding to each.

I, G, M : X_3 H, F, E, J, N : X_0 X : X_{12}

Z, AA : X_7 Q, V, S : $3X_0$ Y : $X_7 \cup X_0$

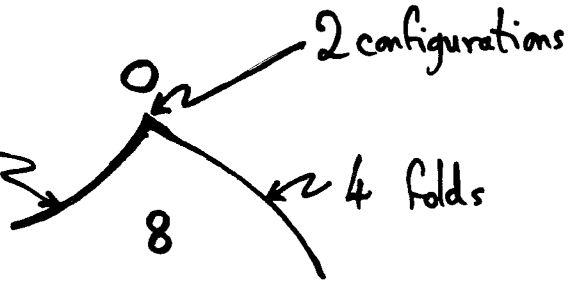
W : $6X_0$ K, O : $4X_0$ U : $7X_0$ T : $X_3 \cup 3X_0$

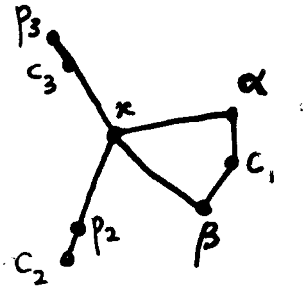
③ Singularity Theory

For generic (l_1, l_2), Σ is a smooth surface.

Hexagon A is an exceptional image of a smooth X_3.

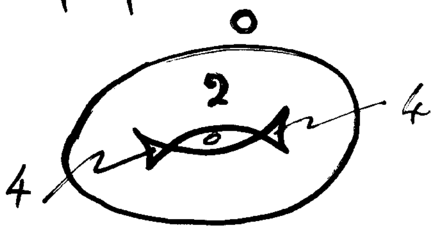
cf. perfectly folded towel



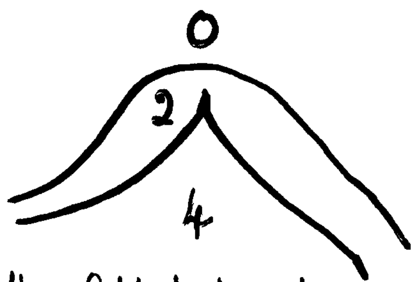


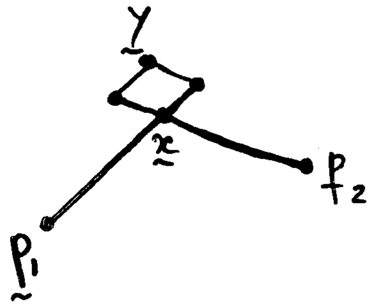
Generic image of a smooth surface X under a smooth map $f : X \to \mathbb{R}^2$ is a region bounded by pieces of fold curves, which can intersect transversely or meet at cusp points

e.g. torus $X_1 \xrightarrow{f}$



So if instead of mapping $\underset{\sim}{\theta}$ to $\underset{\sim}{x}$, we use a generic mapping to say $\underset{\sim}{y}$ e.g. add a pantograph, then will see two copies of:





generically folded towel

"hyperbolic umbilic" singularity

[Levine] A global result: For generic $f : \Sigma \to \mathbb{R}^2$, for each component c of the singular set in Σ, define $r(c) = \#\ \frac{1}{2}$-revolutions made by unoriented tangent to its image, following c in the direction keeping extra preimages on left: $\sum r(c) = \chi(\Sigma)$

④ Group Theory

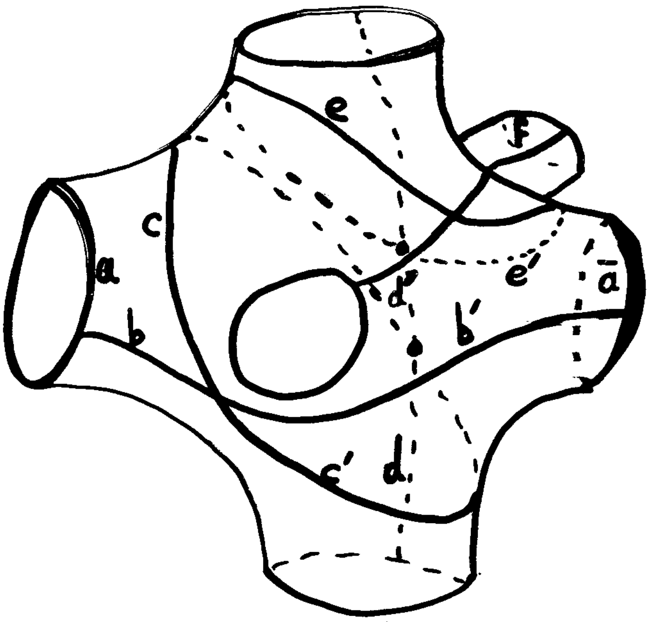
Take Σ connected (or take one connected component)

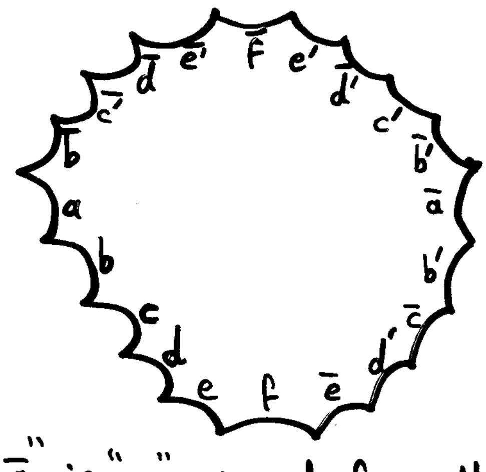
Choose a point $\underline{\theta}^\circ \in \Sigma$. Say 2 (oriented) curves on Σ from $\underline{\theta}^\circ$ to $\underline{\theta}^\circ$ are homotopic if they can be continuously deformed into each other, keeping the end points fixed.

Let $\Gamma = \{$ homotopy classes $\}$

It is a group under concatenation, and get isomorphic groups for different $\underline{\theta}^\circ$: "fundamental group" $\pi_1(\Sigma)$

Can describe Γ by generators and relations, e.g. cut X_3 along 6 simple closed curves as shown, to obtain a 20-gon. [Koebe]





"$\bar{a}$" is "a" viewed from other side, etc

Given $\underline{\theta}^\circ \in$ interior of 20-gon, can label any curve from θ° to θ° by its "cutting sequence". Every sequence can occur. 2 curves are homotopic iff their cutting sequences can be obtained from each other by inserting or deleting words of the form $x\bar{x}$ or $wxyz$ for going round any of the 5 vertices.

Thus $\Gamma \cong \langle a, b, c, d, e, f, \bar{e}, d' \ldots\ldots \mid a\bar{a}, \ldots, cb'c'\bar{b}, \ldots \rangle$

Can deduce that # homotopy classes represented by a sequence of length $\leq N$ grows exponentially with N, $\sim (18.\ldots)^N$

"Free" homotopy classes of closed curves, where $\underline{\theta}^\circ$ is allowed to vary, correspond to conjugacy classes in Γ

$$C_\gamma = \{ g^{-1} \gamma g : g \in \Gamma \}$$

The number of conjugacy classes represented by a sequence of length $\leq N$ also grows exponentially with N

⑤ Mechanics

If we ignore friction, the motion is given by Lagrange's equations
$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) = \frac{\partial L}{\partial q_j} \quad j=1,2 \quad \text{for any local}$$
coordinate system q_1, q_2, where $L = T - V$,

$T(q,\dot{q})$ = kinetic energy
$V(q)$ = potential energy

e.g. if centre of mass $\underset{\sim}{c}$ of each rod is on the line joining its pivots, then
$$T = \frac{1}{2}\left(\mu_1 |\dot{\underset{\sim}{x}}|^2 + \mu_2\, \dot{\underset{\sim}{x}} \cdot \sum \dot{\underset{\sim}{p}}_i + \mu_3 \sum |\dot{\underset{\sim}{p}}_i|^2\right)$$
with
$$\mu_1 = \frac{3}{\ell_2^2}\left(I_c + m\,|\underset{\sim}{p}-\underset{\sim}{c}|^2\right)$$
$$\mu_2 = -\frac{2}{\ell_2^2}\left(I_c + m\,(\underset{\sim}{x}-\underset{\sim}{c})\cdot(\underset{\sim}{p}-\underset{\sim}{c})\right)$$
$$\mu_3 = \frac{I}{\ell_1^2} + \frac{1}{\ell_2^2}\left(I_c + m\,|\underset{\sim}{x}-\underset{\sim}{c}|^2\right)$$
where m = mass of rod
I_c = moment of inertia of rod about its centre of mass
I = moment of inertia of disk

No obvious coordinate systems (though could use redundant set $\underset{\sim}{\theta}$).

Instead, note that energy $E = T+V$ is conserved, and up to a reparametrisation of time, the motion at energy E is equivalent to the geodesic flow of

Riemannian metric

$$ds = 2\sqrt{(E - V(q))\, T_q(dq)} \qquad \text{[Jacobi \& Maupertuis]}$$

on the part of Σ with $V(q) \leq E$,

i.e. paths q making $\displaystyle\int_{\tau_1}^{\tau_2} 2\sqrt{(E - V(q(\tau)))\, T_{q(\tau)}(\dot{q}(\tau))}\, d\tau$

stationary with respect to variations fixing $q(\tau_1), q(\tau_2)$

(and recover true time-parametrisation t by $\dfrac{dt}{d\tau} = \dfrac{1}{2\sqrt{E-V}}$).

Focus on $E > \max_{q \in \Sigma} V(q)$

⑥ Hyperbolic Geometry

One can view X_3 as $\mathring{D}/\Gamma$ where

$\mathring{D} = \{ z \in \mathbb{C} : |z| < 1 \}$ with Poincaré metric P: $ds = \dfrac{2|dz|}{1-|z|^2}$,

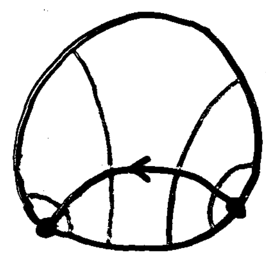
and Γ acts on $\mathring{D}$ by a certain isomorphic group of isometries of $\mathring{D}$ of the form

$$z \mapsto \frac{\alpha z + \beta}{\bar{\beta} z + \bar{\alpha}} \quad \text{with } |\alpha|^2 - |\beta|^2 = 1 \text{ and } \mathrm{Re}\,\alpha > 1$$

(except for the identity: $\alpha = 1, \beta = 0$)

These are "translations", moving everything in $\mathring{D}$ from one fixed point on $\partial\mathring{D}$ to another

The straight lines in $\mathring{D}$ are arcs of circles perpendicular to the boundary

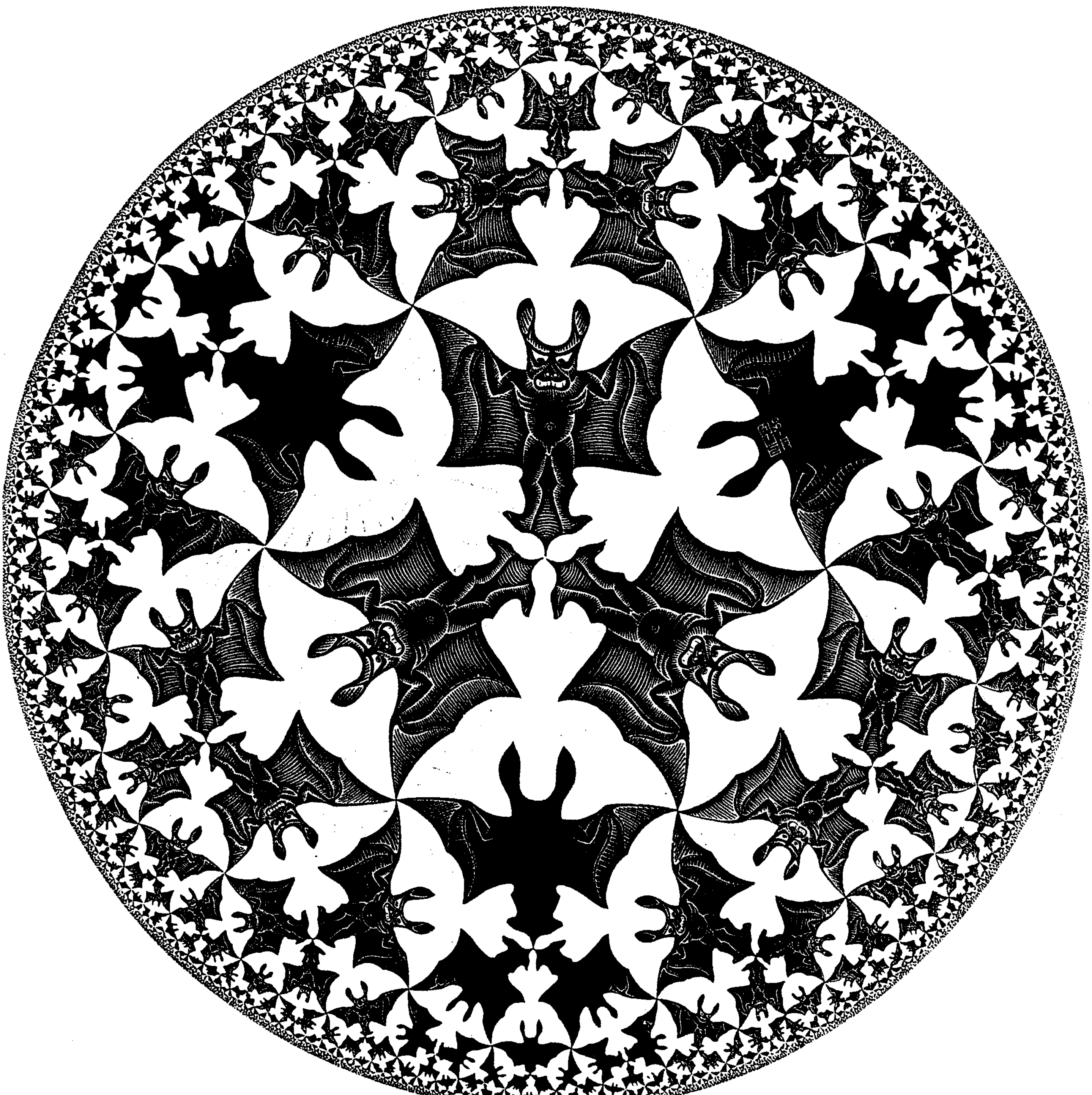


Can choose a 20-gon in $\mathring{D}$ whose sides are straight lines, and a translation for each generator x of Γ that takes side $\bar{x}$ to side x (in fact, there is a lot of freedom of choice: Teichmüller theory)

cf. Escher for X_2

The geodesics for the Poincaré metric on $\mathring{D}/\Gamma$ are the quotient of the straight lines by Γ

e.g. γ/Γ is a closed geodesic on $\mathring{D}/\Gamma$ iff γ is the axis of a (non-identity) element of Γ, so there is precisely one closed geodesic for each (non-identity) conjugacy class in Γ.



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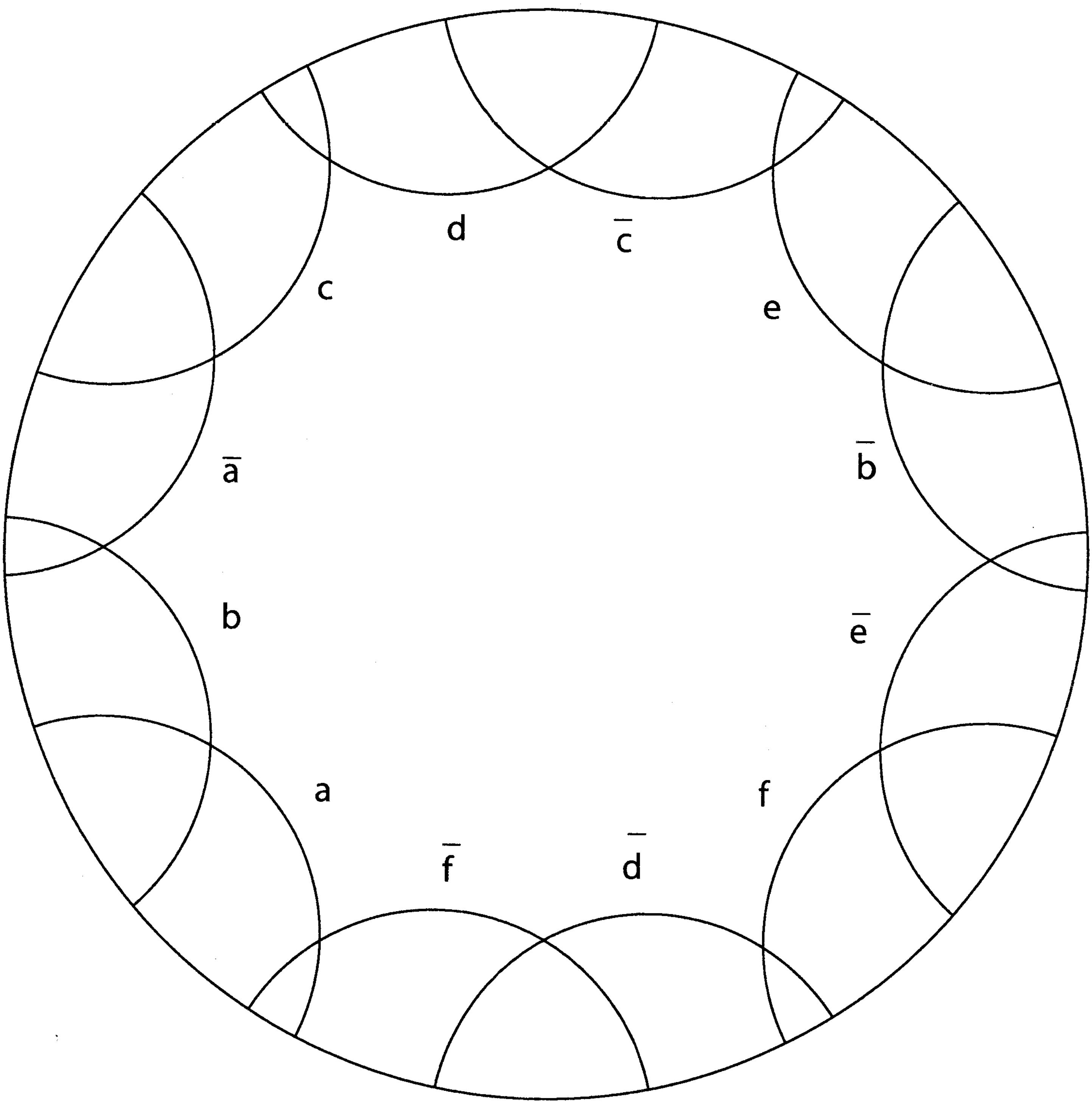
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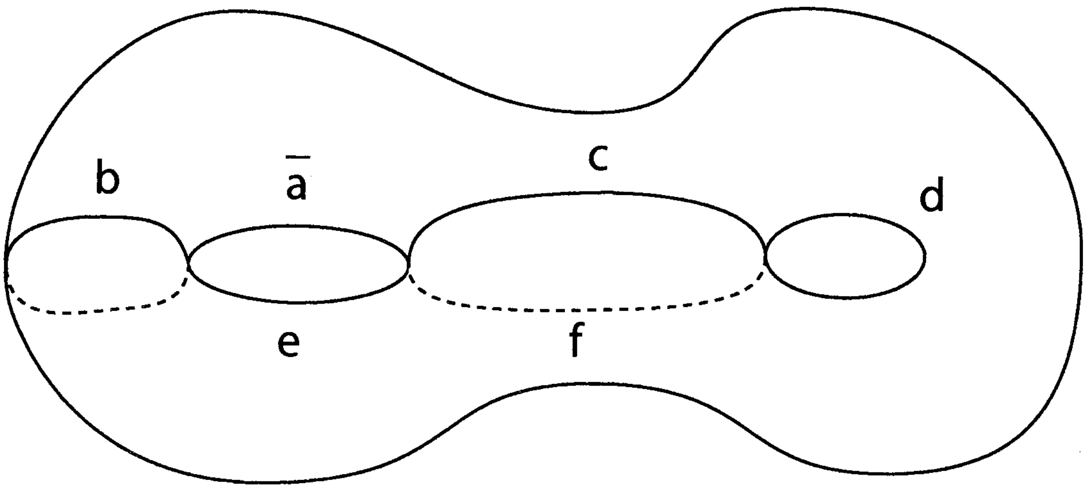
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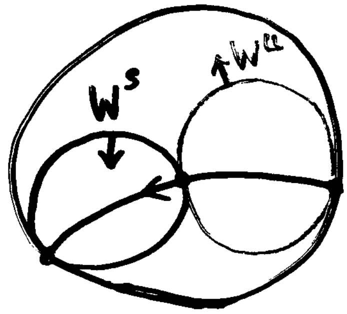
More generally, there is precisely one geodesic for each equivalence class (under the relations) of doubly infinite sequences of generators of Γ : "symbolic dynamics"

Every geodesic of the Poincaré metric has Lyapunov spectrum $\{+1,-1\}$.

The stable and unstable manifolds of a unit tangent vector are the sets of unit normals to the "horocycles"



<u>Morse</u>: For any Riemannian metric R on $\Sigma = \mathring{D}/\Gamma$ there is a closed subset Y of $T_1\Sigma$ and continuous surjection $h: Y \to T_1\Sigma$ carrying geodesics on Σ_R to geoderics on Σ_P

(generalised by Denvir & MacKay to surfaces with $\chi<0$ and geodesically convex boundary)

Thus the triple linkage at $E > \max V$ on any component of the configuration space with genus ≥ 2 has at least one motion which does qualitatively the same as any Poincaré geodesic.

⑦ Riemannian Geometry

The linearised equations about a geodesic ($\Rightarrow$ Lyapunov exponents)

are $\frac{d^2}{ds^2}\xi_\perp + \kappa\,\xi_\perp = 0$, where $\kappa(q)$ = Gaussian curvature at q

and $\frac{d^2}{ds^2}\xi_\parallel = 0$

Curvature measures deviation from flatness locally, e.g.

Length of set of points at distance r from $q \sim 2\pi\left(r - \frac{\kappa}{6}r^3\right)$

area of set within distance $r \sim \pi\left(r^2 - \frac{\kappa}{12}r^4\right)$

e.g. Poincaré metric has $\kappa = -1$ everywhere

Can compute by Brioschi formula: if $ds^2 = [du\ dv]\begin{bmatrix} E & F \\ F & G \end{bmatrix}\begin{bmatrix} du \\ dv \end{bmatrix}$

in local coordinates u, v, then

$$\kappa = \frac{1}{4(EG-F^2)^2}\left(\det\begin{bmatrix} 2E_{vv}+4F_{uv}-2G_{uu} & E_u & 2F_u - E_v \\ 2F_v - G_u & E & F \\ G_v & F & G \end{bmatrix} - \det\begin{bmatrix} 0 & E_v & G_u \\ E_v & E & F \\ G_u & F & G \end{bmatrix}\right)$$

Gauss-Bonnet: $\int_\Sigma \kappa\, dS = 2\pi\,\chi(\Sigma)$

So $\chi = -4 \Rightarrow \kappa < 0$ on average

If $\kappa < 0$ everywhere then Jacobi equation implies exponential separation of most geodesics and strong chaotic properties (like for the Poincaré metric)

So Tim searched parameter space $(l_1, l_2, \mu_1, \mu_2, \mu_3)$ for a case with $\kappa < 0$ everywhere, and found one:

$$l_1 = \frac{7}{40},\ l_2 = \frac{41}{40},\ \mu_1 = \frac{11}{50},\ \mu_2 = \frac{3}{100},\ \mu_3 = \frac{23}{100}$$

(taking $V = 0$)

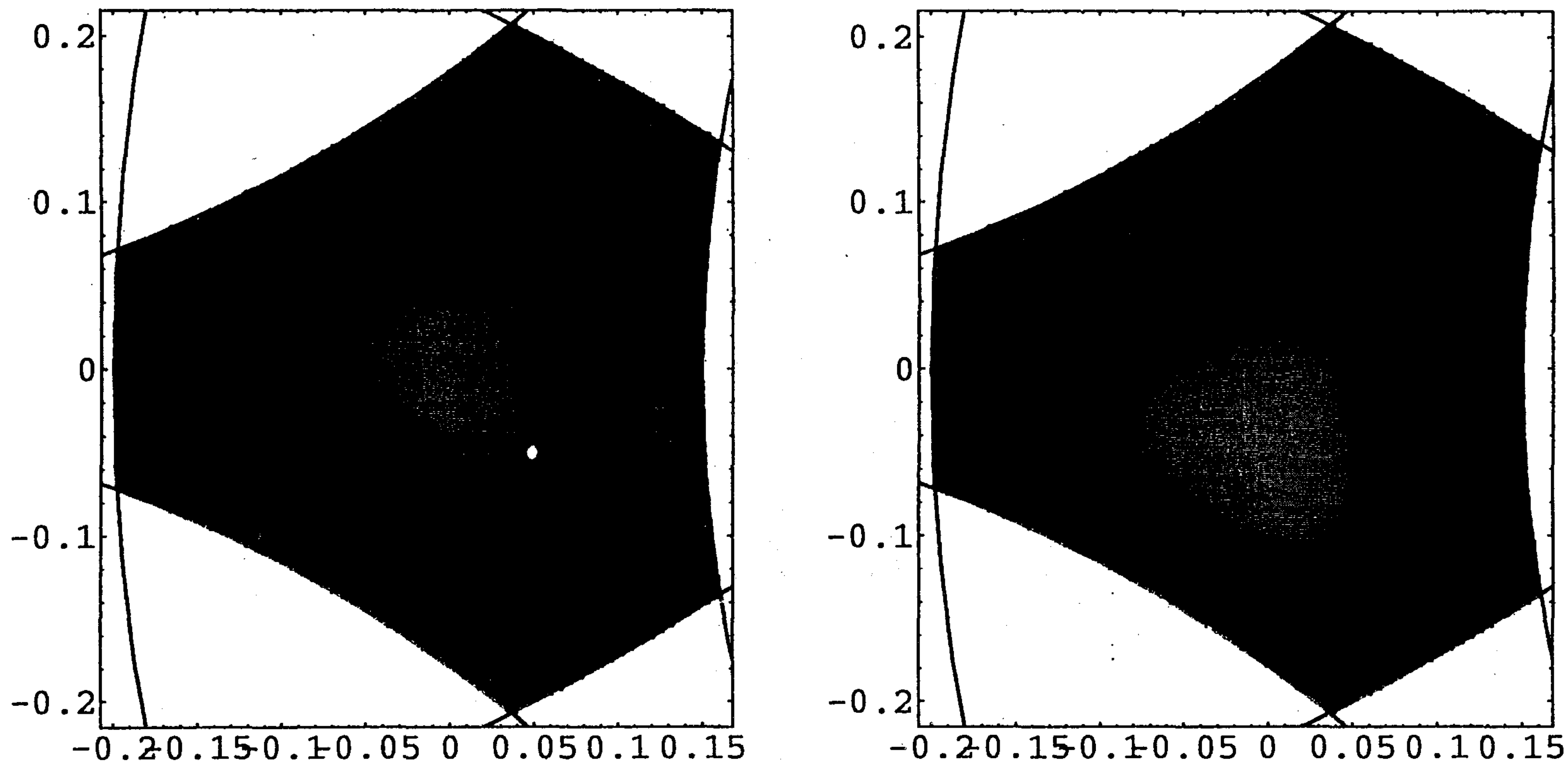


Figure 3.24: A plot of curvature over the whole of the configuration space at $(\mu_1, \mu_2, \mu_3) = (\frac{11}{50}, \frac{3}{100}, \frac{23}{100})$, $(l_1, l_2) = (\frac{7}{40}, \frac{41}{40})$

The main component of the configuration space always contains the type I fixed points. Figures 3.12 to 3.15 show that it is difficult to get negative curvature at these points. In fact it seems to be necessary to be near the bottom corner of the mass parameter space where most of mass is in the μ_3

This suggested studying the limiting cases

$$\ell_2 = 1 + b\,\ell_1 \;,\; \text{some } b \;,\; \ell_1 \to 0,$$

$$\mu_1, \mu_2 \to 0 \;,\; \mu_3 = 1/\ell_1^2$$

Then Σ has formula $\sum \cos\theta_i = -3b$,

and the kinetic energy is just $T = \frac{1}{2}\sum \dot{\theta}_i^2$.

Then κ is just the product of the principal curvatures of Σ as a surface embedded in Euclidean $\mathbb{T}^3$, which evaluates to

$$\kappa = \left(\frac{9}{2}b^2 + 3b\prod\cos\theta_i - \frac{1}{2}\sum\cos^2\theta_i\right) \Big/ \sum \sin^2\theta_i$$

For $|b| < \frac{1}{3}$, $\Sigma \cong X_3$; and for $b = 0$, $\kappa \leq 0$ with equality only if all $\theta_i \in \{\pm\frac{\pi}{2}\}$.

The case $b=0$ is the Schwarz P-surface from the theory of minimal surfaces (soap films), i.e.

Sum of principal curvatures $\equiv 0$.

It is also the Fermi surface for a half-filled simple cubic tight-binding model.

⑧ Anosov systems

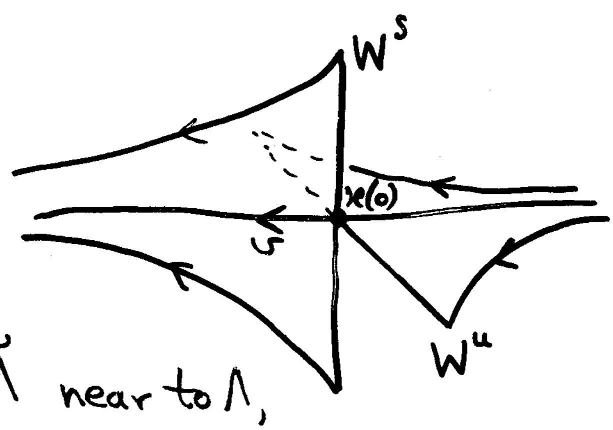
An invariant set Λ for an autonomous system
$\dot{\underline{x}} = \underline{v}(\underline{x})$ on a manifold M is uniformly hyperbolic

if $\exists K$ s.t. the linearised equations for the $\perp$ component $\xi_\perp$ of a displacement from any orbit x in Λ under any bounded forcing function $f_\perp$

$$\dot{\xi}_\perp = P_\perp Dv_{x(t)} \xi_\perp + f_\perp(t)$$

have a unique bounded solution $\xi_\perp$ and $\|\xi_\perp\| \leq K \|f_\perp\|$.

It follows that the set of points whose forward orbit converges with $x(t)$ is a smooth submanifold $W^s(x(0))$, and the same for backwards: $W^u(x(0))$, and the orbits converge together at least like $C e^{-\lambda|t|}$ for some C, λ.



Also if $\tilde{v}$ is a small perturbation of v then $\exists \tilde{\Lambda}$ near to Λ, invariant and u. hyp. under $\tilde{v}$, and with topologically equivalent dynamics.

The case where the whole of M is uniformly hyperbolic is called Anosov.

For geodesic flow on a surface, the uniform hyperbolicity condition is equivalent to $\exists K$ s.t.

$$\ddot{\xi}_\perp + \kappa(q(t))\xi_\perp = f_\perp(t)$$

has a unique bounded solution $\xi_\perp$, and $\|\xi_\perp\| \le K\|f_\perp\|$.

If $\kappa < 0$ everywhere, we see that the geodesic flow is Anosov (because the Green function decays exponentially both ways). The same is true if $\kappa \le 0$ and $= 0$ at only a finite set of points.

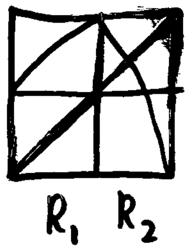
Thus the geodesic flow on the Schwartz P-surface is Anosov.

But then all nearby vector fields on $T_1\Sigma$ are also Anosov and topologically equivalent.

This includes the geodesic flow for the triple linkage for all ℓ_1, b, m small enough, for variations in V sufficiently small compared with E. It also includes breaking the D_6 symmetry weakly. Also Anosov and topological equivalence are preserved under the time reparametrisation, so the true motion is Anosov on each (large enough) energy level.

Anosov systems have other wonderful properties e.g. a "Markov partition": a finite partition of M into regions $(R_i)_{i=1}^N$ such that (except for ambiguities on the boundaries) there is a 1-1 correspondence between orbits and doubly infinite paths in an associated graph G with nodes $1, \dots N$ and the graph has a node with at least 2 paths back to it.

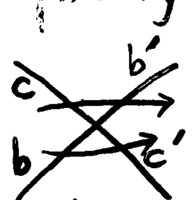
e.g.



$1 \rightleftarrows 2 \circlearrowleft$

"chaos" , "symbolic dynamics"

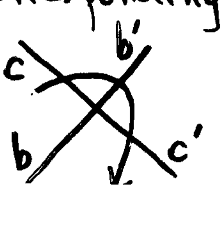
We can make a concrete Markov partition for any Anosov geodesic flow on Σ: take the 6 curves a, b, c, d, e, f to have minimal length in their free homotopy classes, then for each pair of the 20 symbols $x = a, \bar{a}, b, b', \bar{b}, \bar{b}'$ etc., let R_{xy} be the set of unit tangents whose geodesic has crossing sequence beginning xy. Pairs of the form $x\bar{x}$ don't occur, and also pairs corresponding to passing opposite sides of a vertex e.g.



$cb' = b\bar{c}'$

are regarded as equivalent. So we have 360 regions.

Put an edge from xy to yz for all x, y, z except those corresponding to going round a vertex more than 180° e.g.



$cb' \to b'c'$ is forbidden.

⑨ Probability Theory

What do typical orbits do?

For a Hamiltonian system H, Liouville measure

$\delta(H-E)\prod_i dp_i dq_i$ is invariant on energy level $H^{-1}(E)$.

For a geodesic flow on a surface this reduces to $dS\,d\varphi$

$\uparrow$ surface area $\quad \uparrow$ direction

For an Anosov energy level, Liouville measure is <u>ergodic</u>, meaning there is no decomposition into 2 invariant sets of positive measure.

It follows that for almost every orbit the fraction of time it spends in any subset is proportional to its measure.

Actually, for any transitive Anosov system there is a (unique) invariant (ergodic) probability measure μ^+ (called SRB) s.t. almost every orbit wrt any Borel measure spends fraction $\mu^+(A)$ of forward time in A. It can be obtained as the unique Gibbs state for symbol sequences with energy function $\int \lambda^u(\gamma(t))\,dt$, where $\lambda^u(\gamma(t))$ is the expansion rate along $W^u(\gamma(t))$ and γ is the orbit corresponding to the given symbol sequence.

Note that in general μ^+ is <u>not</u> Markov.

Also backwards SRB $\mu^- \neq \mu^+$ in general (but both = Liouville for Hamiltonian systems)

Actually, the triple linkage is not just ergodic (on energy levels), but also <u>mixing</u>: $\forall$ measurable subsets A, B,

$$\mu(\varphi_t A \cap B) \xrightarrow{t \to +\infty} \mu(A)\,\mu(B) \qquad (\text{normalising } \mu(M)=1)$$

For transitive Anosov flows, either there is a surface of section with constant return time, or it is mixing.
All Anosov geodesic flows are mixing, and so are all flows near an Anosov geodesic flow. Hence the triple linkage in our parameter regime. This includes allowing a potential with small variation compared to E because then the time-rescaling rate is close to constant.

Under certain conditions on a mixing Anosov flow (which hold for Anosov geodesic flows on surfaces and near-constant time-rescaling rates), the correlations of Hölder continuous functions decay exponentially:

$$\int f(\varphi_t x)\, g(x)\, d\mu(x) \leq C_{fg}\, e^{-\mu t} \qquad [\text{Dolgopyat}]$$

In particular the velocity-autocorrelations in any Abelian cover are integrable: $C_{ij}(t) = \int h_i(v(\varphi_t x))\, h_j(v(x))\, d\mu(x)$,
$D_{ij} = \int_0^\infty C_{ij}(t)\, dt < \infty$, and D is positive-definite.

So the motion in the Abelian cover is "diffusive" on a large scale.

Note: $B_1(T_1 X_3) = 7$ so there are k additional Abelian directions

⑩ PDE

Given a kinetic energy function T on a surface Σ, can you design a potential V to make the Jacobi metric for given energy have constant negative curvature?

"Yamabe problem"

Yes, for $g \geq 2$:
$$\Delta \phi - e^{2\phi} = \kappa$$
has a unique solution $\phi : \Sigma \to \mathbb{R}$,

where κ and Δ are the curvature and Laplacian of T.

Then choose $V = E - \frac{1}{2} e^{2\phi}$

Hence energy level $H^{-1}(E)$ (and all nearby ones) are Anosov.

(11) Normally hyperbolic theory

Real system suffers from friction.

Can we add a driving force to obtain a (non-trivial) uniformly hyperbolic attractor?

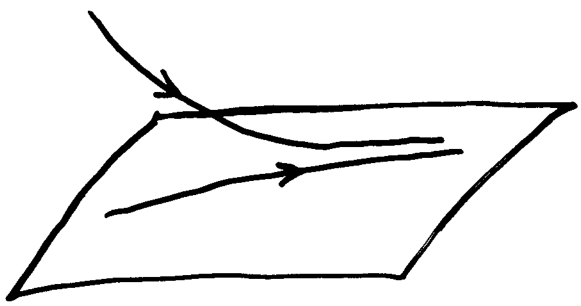
Yes, choose E sufficiently $> V_{max}$,

let $\tilde{K} = \frac{1}{2} \sum I \dot{\theta}_j^2$ and apply feedback torques

$\Gamma_j = -\gamma (\tilde{K} - E) \dot{\theta}_j$ for some $\gamma > \sqrt{\frac{I}{2E}}$

(or any feedback law close on a neighbourhood of $\tilde{K}^{-1}(E)$).

Then $\tilde{K}^{-1}(E)$ is "normally hyperbolic" and attracting for the limit system, and the motion on it is unchanged. So for all small smooth perturbations it persists to an invariant attracting normally hyperbolic submanifold, and the dynamics on it is a small perturbation of the frictionless case, so is topologically equivalent and Anosov.



$T_1 \Sigma$

(12) Partially hyperbolic theory

If instead of the high gain feedback required for normal hyperbolicity, we can use only low gain feedback, what can we get? (or even just a constant or time-periodi driving force)

The unperturbed system is "partially hyperbolic" on the whole of $T\Sigma$:

$$T_\gamma(T\Sigma) = E^+_\gamma \oplus E^-_\gamma \oplus \underbrace{\mathbb{R} X_H(\gamma) \oplus \mathbb{R} \nabla H(\gamma)}$$

"centre subspace" with weak (actually only linear) expansion or contraction.

Perturbation keeps partial hyperbolicity

$$T(T\Sigma) = \tilde{E}^+ \oplus \tilde{E}^- \oplus \tilde{E}^c$$

There are robustly transitive partially hyperbolic attractors. Can we make one using friction and a weak driving force?