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SUMMER SCHOOL AND CONFERENCE ON DYNAMICAL SYSTEMS

Polynomial Diffeomorphisms of C^2

(Lecture 4)

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We want to revisit potential theory in $\mathbb{C}$ paying particular attention to how things transform under holomorphic maps. As a consequence of this we will be able to formulate it in a way that makes sense for $\mathbb{C}^2$.

The Laplacian is not invariant under holomorphic coordinate changes. On the other hand the operator

$$f \mapsto \Delta f \, dx \wedge dy$$

from functions to two forms is invariant.

We will give an alternate description of this operator which makes the naturality clear.

The exterior derivative

$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy$$

which takes functions to 1-forms (and n-forms to (n+1)-forms) is natural with respect to smooth coordinate changes.

The operator J on one forms corresponds to rotating the cotangent space by i.

$$J: \begin{array}{l} dx \mapsto dy \\ dy \mapsto -dx \end{array}$$

This is natural with respect to holomorphic maps.

Define the twisted exterior derivative d^c as an operator from functions to 1-forms by

$$d^c = J \circ d$$

As an operator from n forms to n+1 forms we define it to be

$$d^c = J \circ d \circ J^{-1}.$$

This operator is holomorphically natural.

We compute

$$dd^c f = (dJd)f$$
$$= dJ\left(\frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy\right)$$
$$= d\left(\frac{\partial f}{\partial x}dy - \frac{\partial f}{\partial y}dx\right)$$
$$= \frac{\partial^2 f}{\partial x^2}dx\wedge dy + \frac{\partial^2 f}{\partial y\,\partial x}dy\wedge dy$$
$$- \frac{\partial^2 f}{\partial x\,\partial y}dx\wedge dx - \frac{\partial^2 f}{\partial y^2}dy\wedge dx$$
$$= \left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}\right)dx\wedge dy$$
$$= \Delta f\, dx\wedge dy.$$

The operator d^c is very convenient.

$g(z) = u(z) + i\,v(z)$ is holomorphic if and only if $d^c u = dv$.

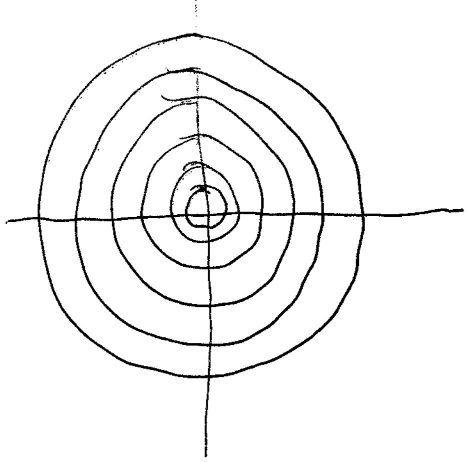
If we write

$$z = |z| \cdot e^{i \arg z}$$

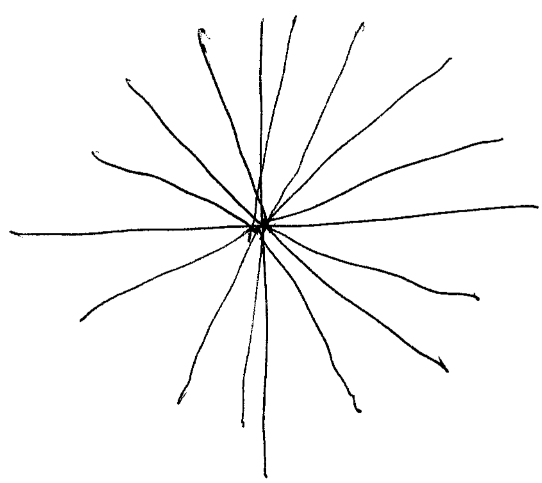
then $\log z = \log|z| + i \arg z$.

Thus we have

$$d^c \log|z| = d \arg(z).$$



$d \log|z|$



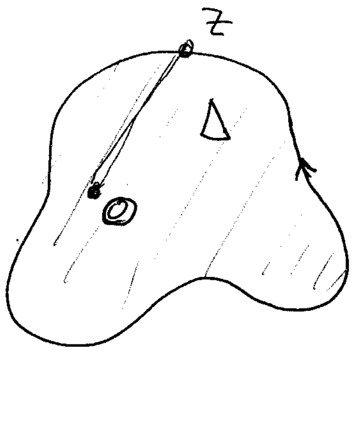
$d^c \log|z|$

We can show that $\frac{1}{2\pi}\log|z|$ is the potential function corresponding to δ_0.

$$\frac{1}{2\pi}\int_\Delta dd^c \log|z| = \frac{1}{2\pi}\int_{\partial\Delta} d^c \log|z|$$

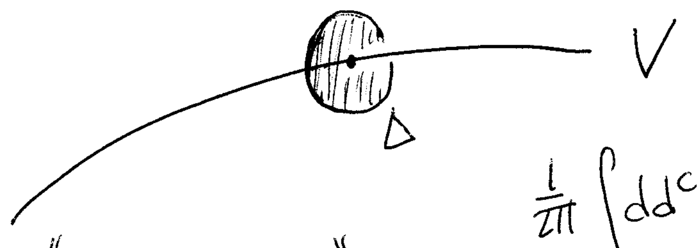
$$= \frac{1}{2\pi}\int_{\partial\Delta} d \arg(z)$$

$$= \begin{cases} 1 & \text{if } 0 \in \Delta \\ 0 & \text{if } 0 \notin \Delta \end{cases}$$



The operator dd^c makes sense in $\mathbb{C}^2$ and we can talk about potential functions in this setting.

Example. Let $V \subset \mathbb{C}^2$ be the variety defined by the equation $P(x,y)=0$ where $P: \mathbb{C}^2 \to \mathbb{C}$ is a polynomial. Then the current $\frac{1}{2\pi} dd^c \log|P|$ when evaluated on a disk Δ counts the intersection number of Δ with V.



$$\frac{1}{2\pi}\int_{\Delta} dd^c \log|P| = \frac{1}{2\pi}\int_{\partial\Delta} d \arg(P)$$

$d^c \log|P|$ is a "linking form".

Definition. A pluri-subharmonic function is an u.s.c. function whose restriction to any 1-dimensional complex variety is subharmonic.

Example: $\log|P|$

There are some dynamically important p.s.h. functions.

Definition. A pluri-harmonic function is a continuous function whose restriction to any 1-dimensional complex variety is harmonic.

If H is a plurisubharmonic function then $\frac{1}{2\pi} dd^c H$ is called a "positive closed $(1,1)$ current".

$\frac{1}{2\pi} dd^c H$ assigns to each complex disk Δ a measure μ_Δ supported on Δ.

We can compute this measure μ_Δ by restricting H to Δ and computing $\frac{1}{2\pi} dd^c (H|_\Delta)$.

In a region where H is pluriharmonic the measures μ_Δ are the zero measure.

Let $f_{a,b}: \mathbb{C}^2 \circlearrowleft$ be a Hénon diffeomorphism.

Define

$$G^+(p) = \lim_{n\to\infty} \frac{1}{2^n} \log^+ \|f^n(p)\|$$

$$G^-(p) = \lim_{n\to\infty} \frac{1}{2^n} \log^+ \|f^{-n}(p)\|.$$

$G^{+/-}$ are plurisubharmonic and pluriharmonic on $\mathbb{C}^2 - K^{+/-}$.

$G^{+/-}$ behave like Green functions for $K^{+/-}$.

Furthermore

$$G^+(f(p)) = 2\,G^+(p)$$

$$G^-(f(p)) = \frac{1}{2}\,G^-(p).$$

Define

$$\mu^+ = \frac{1}{2\pi} dd^c G^+$$

$$\mu^- = \frac{1}{2\pi} dd^c G^-.$$

μ^+ and μ^- are currents supported on J^+ and J^-.

They satisfy

$$f^*(\mu^+) = 2 \cdot \mu^+$$

$$f^*(\mu^-) = \frac{1}{2} \cdot \mu^-.$$

Theorem [Bedford-S, Fornaess-Sibony]

Let V be the x-axis in $\mathbb{C}^2$.

Let $[V]$ be the dual current

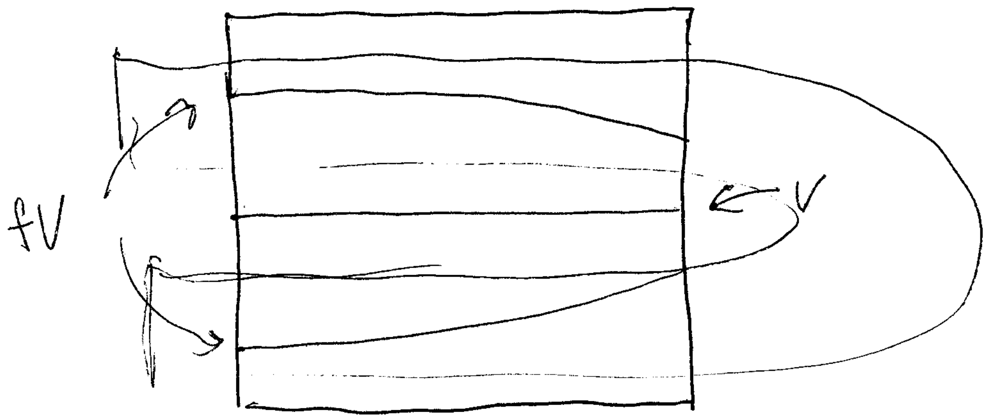
($[V]$ evaluated on Δ counts the intersections). Then

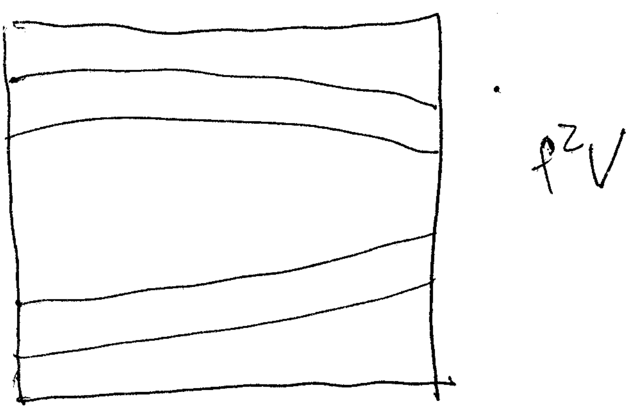
$$\lim_{n\to\infty} \frac{1}{2^n}\left[f^{-n}V\right] = \mu^+.$$

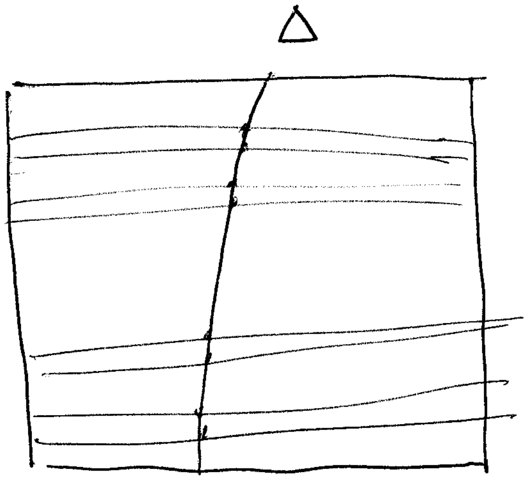
Remark. Some care needs to be taken with the precise definition of convergence of currents.

The proof follows the spirit of Brolin's argument.

Example. The horseshoe.







On any vertical transversal Δ the measure μ^+_Δ is the balanced measure on the Cantor set.

Interpretation: In the Axiom A setting this transverse measure μ^+_Δ is sometimes called the Margulis measure.

We have succeeded in extending part of the Axiom A theory to complex Hénon diffeomorphisms with no a priori restrictions on the parameters.

What else can we do?

Define $\mu = \mu^+ \wedge \mu^-$.

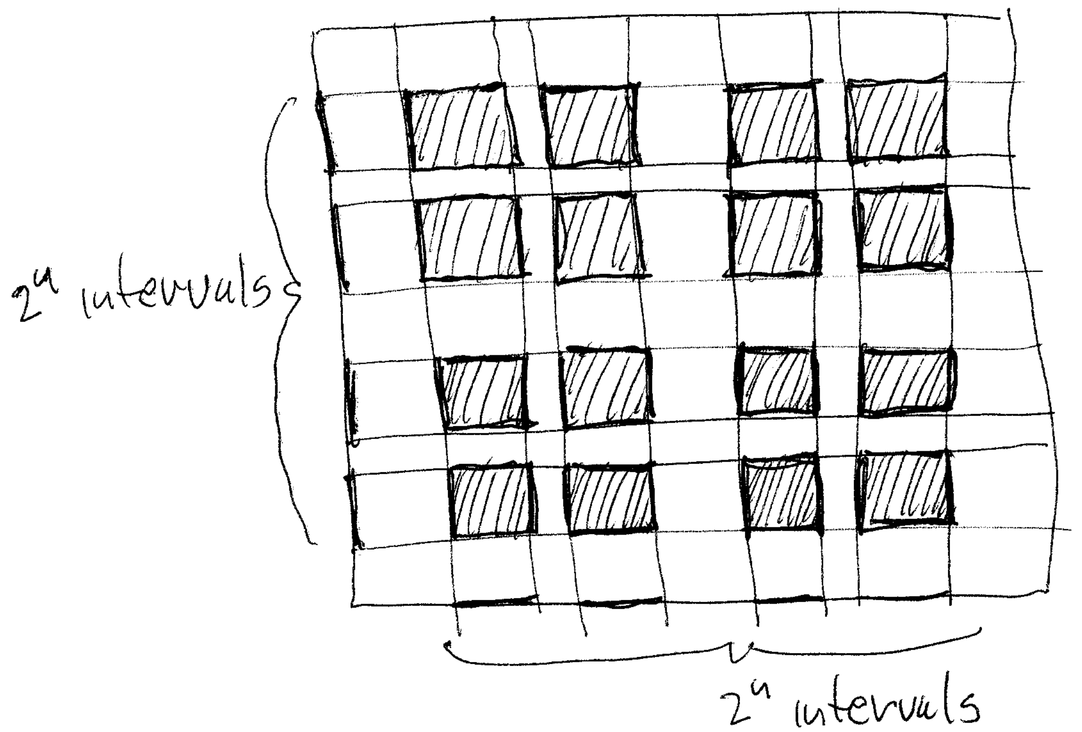
Note: Attention must be paid to the regularity of μ^+ and μ^- to make sure that this construction works.

If we apply f we get

$$f^*(\mu) = f^*(\mu^+) \wedge f^*(\mu^-)$$
$$= 2\mu^+ \wedge \tfrac{1}{2}\mu^-$$
$$= \mu^+ \wedge \mu^-$$
$$= \mu.$$

μ is an invariant probability measure.

Example: The horseshoe.



μ assigns to each of the 2^{2n} boxes at level n the measure $\frac{1}{2^{2n}}$.

In the Axiom A setting this measure is called the Bowen measure.

Theorem. [Bedford-Lyubich-S]

Let f be any complex Hénon diffeomorphism. Let

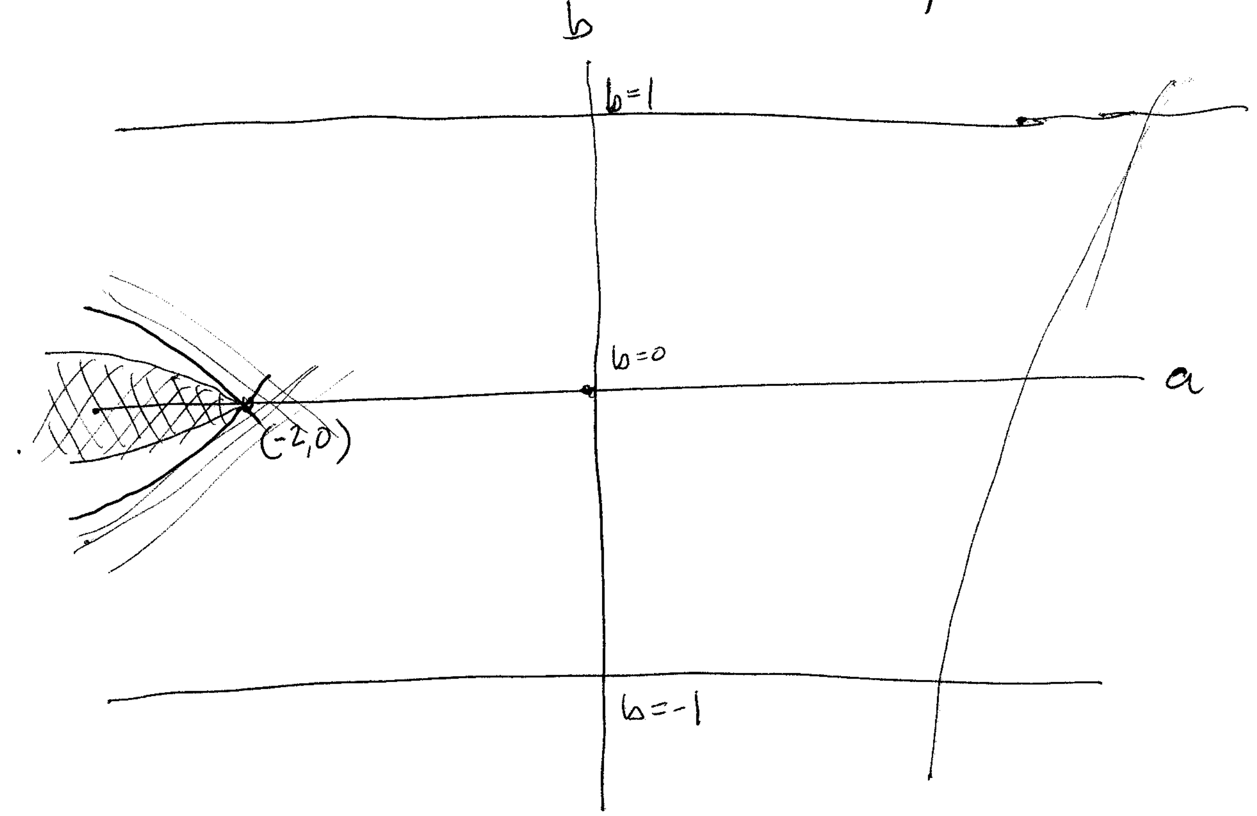
$$\mu_n = \frac{1}{2^n} \sum_{f^n(p)=p} \delta_p .$$

Then

$$\lim_{n\to\infty} \mu_n = \mu .$$

(μ describes the distribution of periodic points.)

Recall the parameter space for the real Hénon family.

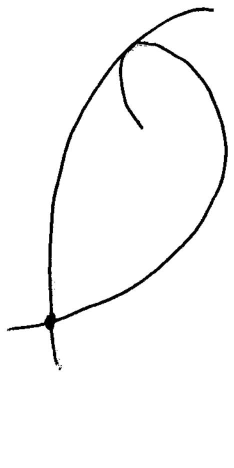


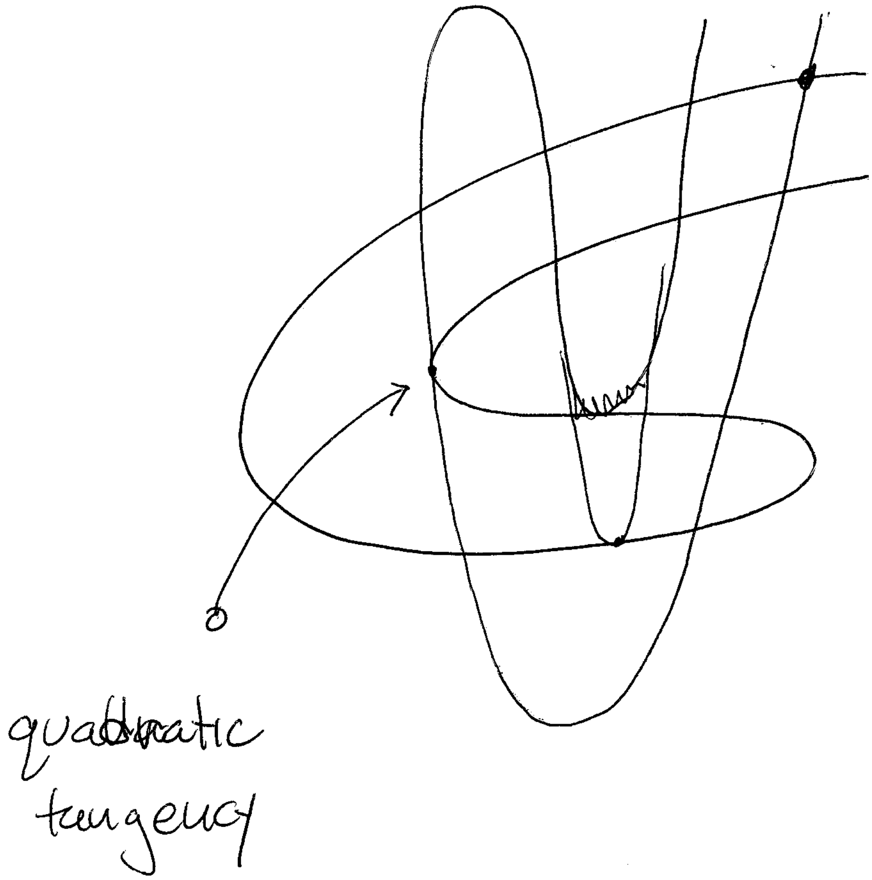
Let $\mathcal{HA}$ be the set of (a,b) for which $f_{a,b}$ is Axiom A and $f_{a,b}$ is topologically conjugate to the full 2-shift on its chain recurrent set.

Theorem. [Bedford-S]

If $(a,b) \in \partial \mathcal{H}$ then there is a quadratic tangency between stable and unstable manifolds of periodic points.

Remark. In other families of smooth diffeomorphisms horseshoes can degenerate in different ways.



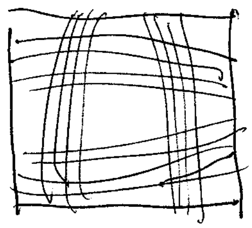


The statement of the Theorem is consistent with the pictures drawn by computers.

We work with the complex extension of $f = f_{a,b}$.

$$\begin{array}{ccc} f: \mathbb{R}^2 & \longrightarrow & \mathbb{R}^2 \\ \cap & & \cap \\ f: \mathbb{C}^2 & \longrightarrow & \mathbb{C}^2. \end{array}$$

A priori the Julia set is a subset of $\mathbb{C}^2$. For the horseshoe the Julia set J is a subset of $\mathbb{R}^2$.

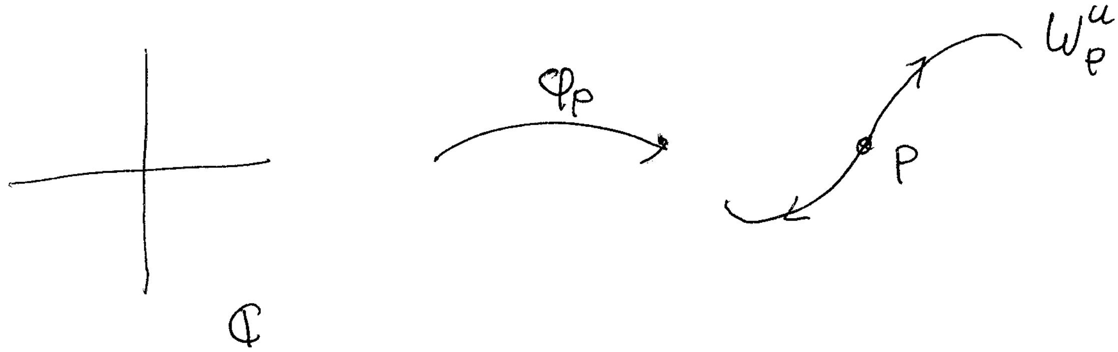


The property that $J_{a,b} \subset \mathbb{R}^2$ is a closed property in parameter space so it remains true for $(a,b) \in \overline{\mathcal{HA}}$.

Strategy of proof. We will attempt to find invariant line fields and metrics which are expanded and contracted.

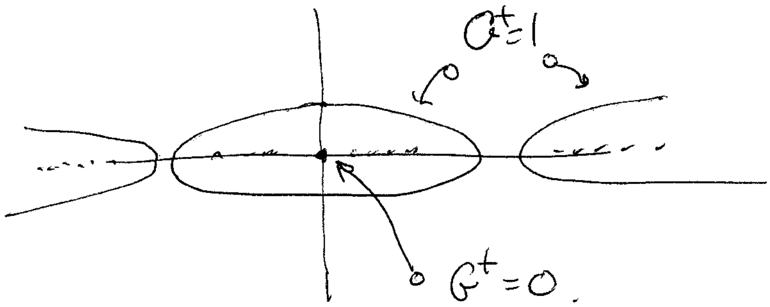
From general principles we know that periodic saddle points are dense in J.

Let $\{p_1, p_2 \dots p_n\}$ be a periodic saddle orbit. We will start by constructing a metric on the space $E^u_{p_j}$.



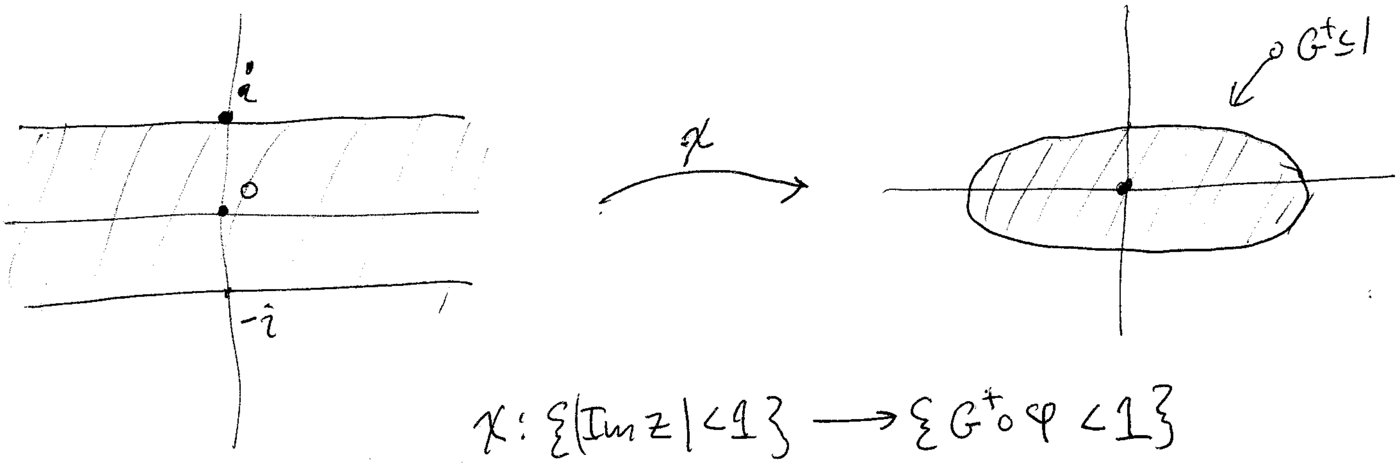
Consider the function $z \mapsto G^+(\varphi_p(z))$ on $\mathbb{C}$.

This function is subharmonic and harmonic outside of the real axis.



$W^u_p \subset J^-$. G^+ is pluriharmonic outside of J^+. $G^+ \circ \varphi_p$ is harmonic outside of $\varphi_p^{-1}(J^- \cap J^+)$ but $J^- \cap J^+ = J \subset \mathbb{R}^2$.

We use $G^+ \circ \varphi$ to establish a "scale" on W^u_p.

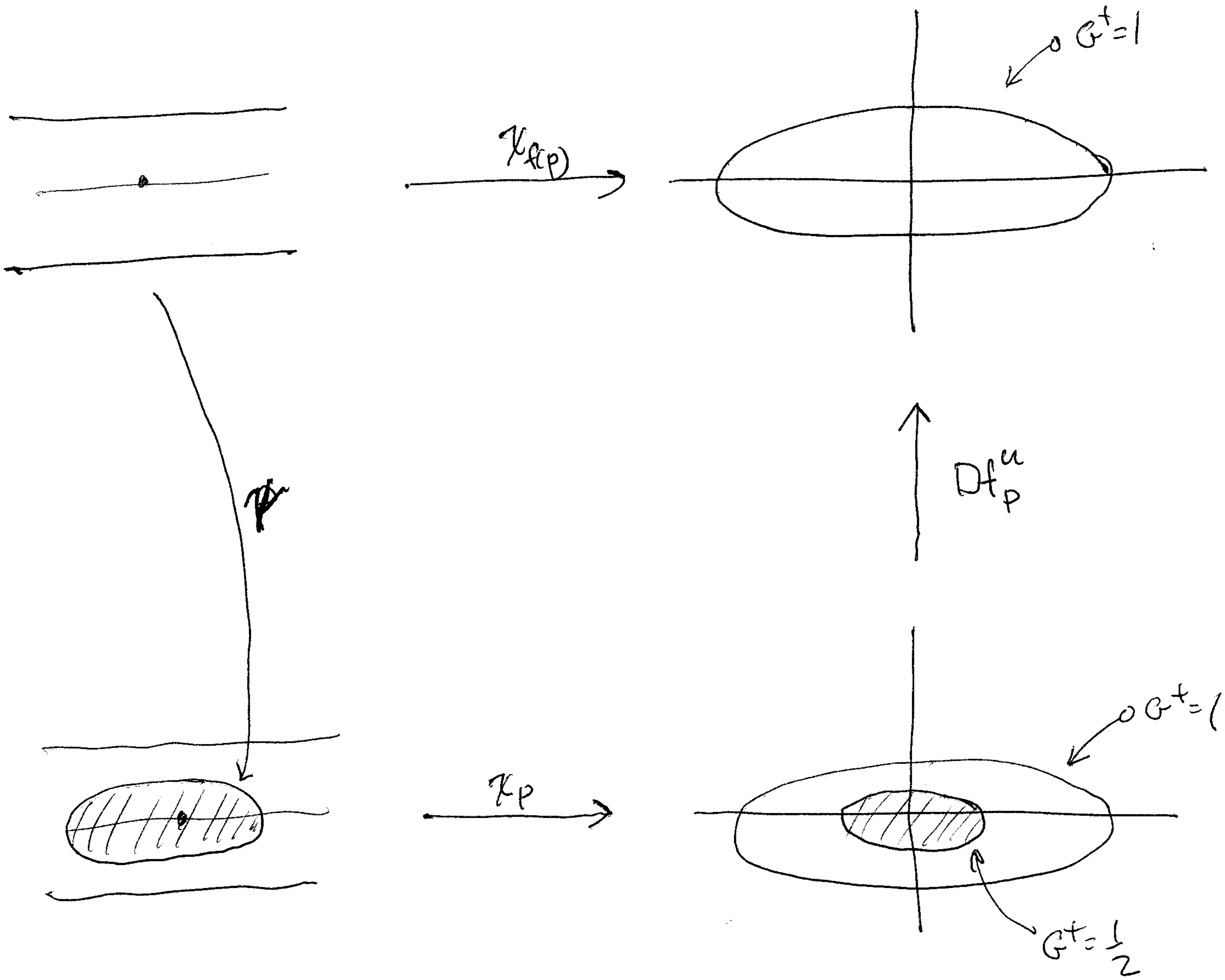


$$\chi : \{|\mathrm{Im}\, z| < 1\} \longrightarrow \{G^+ \circ \varphi < 1\}$$

Let χ be a parametrization which is real and takes 0 to 0.

Define a metric on E^u_p so that $D\chi(0)$ ~~has~~ is an isometry.

Lemma. Df expands this metric by a factor greater or equal to 2.



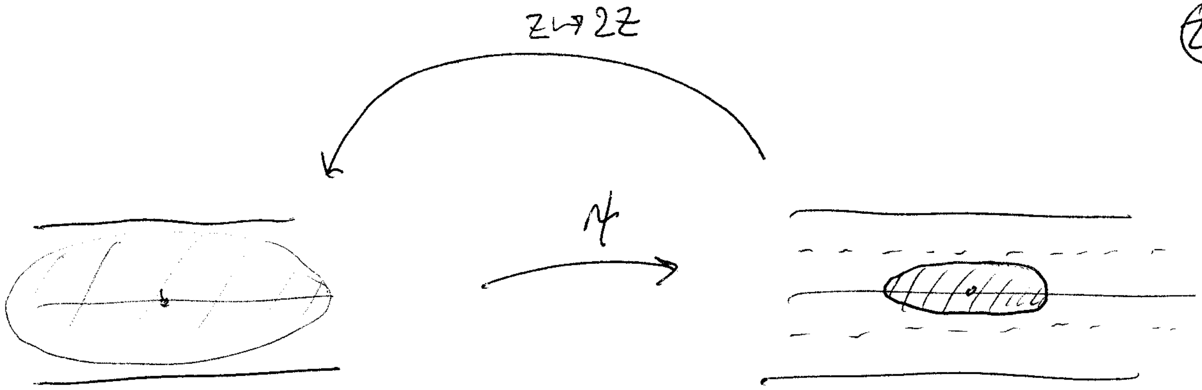
Define $\psi = \chi_p^{-1} \circ (Df^u_p)^{-1} \circ \chi_{f(p)}$.

Claim that $\operatorname{im} \psi \subset \{ |\operatorname{Im} z| < \frac{1}{2} \}$.

Compare two functions on the strip:

$|\operatorname{Im} z|$ and $G^+ \circ \chi_p$. Conclude $G^+ \circ \chi_p \geq |\operatorname{Im} z|$.

Thus $\{ G^+ \circ \chi_p < \frac{1}{2} \} \subset \{ |\operatorname{Im} z| < \frac{1}{2} \}$.



Schwartz Lemma shows

$\|D\, 2\psi(0)\| \le 1$ so $2\|D\psi(0)\| \le 1$ $\|D\psi(0)\| \le \frac{1}{2}$.

But $\|D\psi\| = \|D\chi_p^{-1} \circ (Df_p^u)^{-1} \circ D\chi_{f(p)}\|$

$= \|Df_p^u\|^{-1}$.

Thus $\|Df_p^u\|^{-1} \le \frac{1}{2}$ and $\|Df_p^u\| \ge 2$.