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SMR/1578 - 5

SCHOOL AND CONFERENCE ON FUNDAMENTAL ASPECTS OF COMPLEXITY

(6 - 10 September 2004)

" Universal phase transition in combinatorial optimization "

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Universal Phase Transition in Combinatorial Optimization

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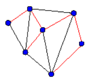
Optimization Problems

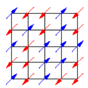
Minimize $H(\sigma) = \sum_i a_i \sigma_i$ σ = feasible solution

- graph problems (MST, TSP, ...)



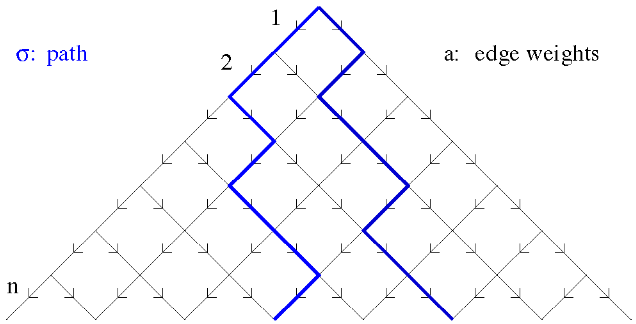
a_i: edge weights, $\sigma_i \in \{0, 1\}$ edge selector

- Ising-Spin-Glasses (EA, SK)



$J_{ij} \mapsto a_i$ couplings
$\mathrm{sign}(J_{ij}) s_i s_j \mapsto \sigma_i \in \{\pm 1\}$

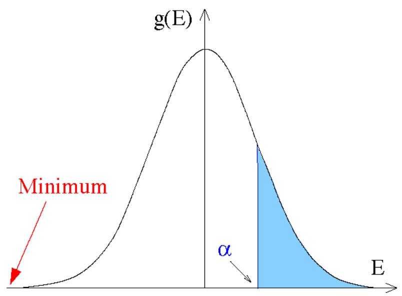
DPRM



Directed Polymer in Random Media

- single-source shortest-path problem
- solvable in polynomial time (Bellman-Ford)

Constrained DPRM



Find shortest path among all paths with length $\geq \alpha$.

- cannot be easier than unconstrained case ($\alpha = -\infty$)
- is NP-hard

NP Hardness

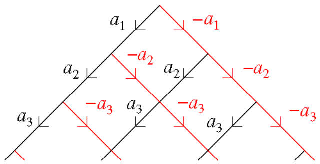
NP-hard reference problem: Subset Sum

Given n numbers $a_i \in \mathbb{Z}$ and a threshold $\alpha \in \mathbb{Z}$, minimize

$$H(\sigma) = \sum_{i=1}^{N} a_i \sigma_i \qquad \sigma \in \{\pm 1\}^n$$

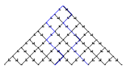
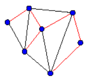
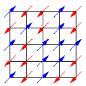
subject to the constraint $H(\sigma) \geq \alpha$.

Subset-Sum $\Rightarrow$ DPRM:



More Examples

Minimizing $H(\sigma)$ is polynomial for

- DPRM, shortest path 
- Minimum Spanning Tree 
- $1d$ and $2d$ Spin-Glasses 

Minimizing s.t. $H(\sigma) \geq \alpha$ is NP-hard for all these problems.

Random Instances

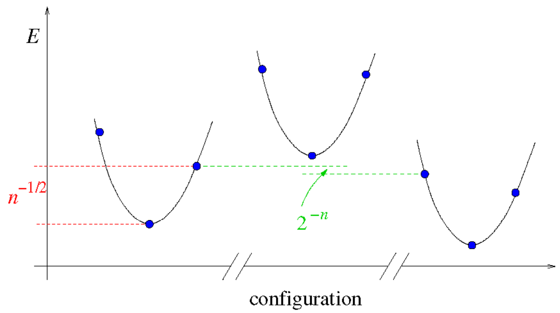
DPRM with i.i.d. random weights $a_i \in \mathbb{R}$, $a_i \in \mathcal{N}(0, n^{-1/2})$.

Distribution of energies:

$$g(E) = \frac{1}{2^n} \sum_{\sigma} \delta\left(E - H(\sigma)\right) \simeq \mathcal{N}(0,1)$$

- gap between adjacent energy levels: $\propto 2^{-n}$
- energy difference of overlapping paths: $\propto 1/\sqrt{n}$

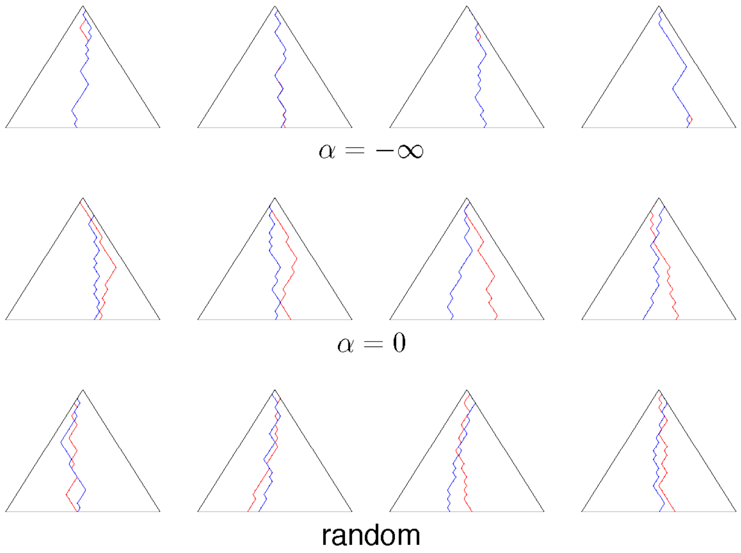
Energy Landscape



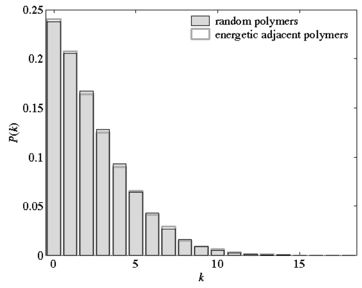
configs.: close in energy, close in config space

choose one

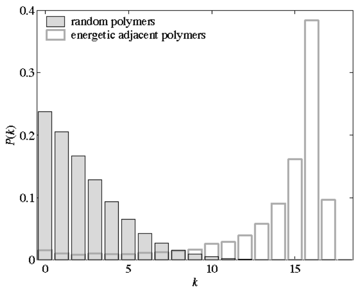
Energetically Adjacent Paths



Overlap of Configurations

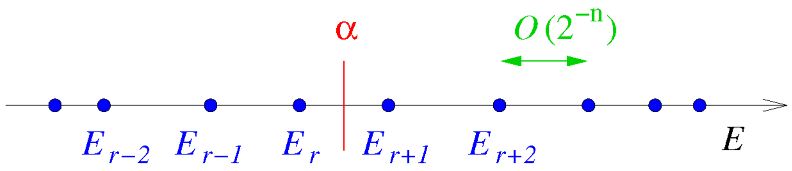


$\alpha = 0$



$\alpha = -\infty$

Local REM Hypothesis

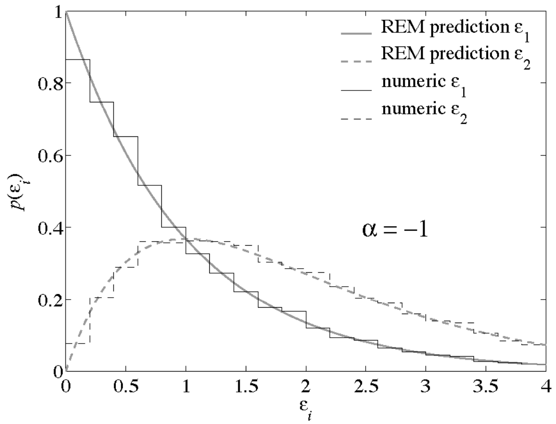


$$2^n\, g(\alpha)\, \left[(E_{r+1}, E_{r+2}, \ldots, E_{r+\ell}) - \alpha\right] \xrightarrow[n\to\infty]{\mathcal{D}} (W_1, W_1 + W_2, \ldots, W_1 + W_2 + \ldots + W_\ell)$$

with W_i iid, exp(1).

Configurations $\sigma_r, \sigma_{r+1}, \ldots, \sigma_{r+\ell}$ are uncorrelated.

Local REM: Energies



$\varepsilon_1 = W_1 \qquad \varepsilon_2 = W_1 + W_2$

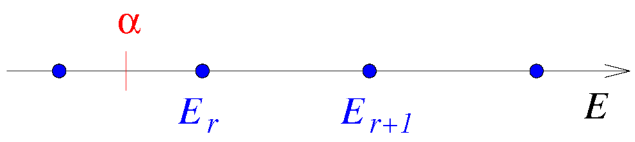
Heuristic explanation

Let a_i's be B-bit random numbers, $\pm a_i \in \{0, 1, \ldots, 2^B - 1\}$.

$$H(\sigma) = \sum_i a_i \sigma_i \in [-n2^B, n2^B]$$

- $B < n$: $\Delta E_r = E_{r+1} - E_r = 2$
- $B > n$: $\Delta E_r \propto 2^{B-n}$
  - ΔE_r determined by $B - n$ low order bits of a
  - σ controls only n high order bits J
- local REM should
  - depend on the bit-entropy of the disorder
  - be universal

Maximum Overlap



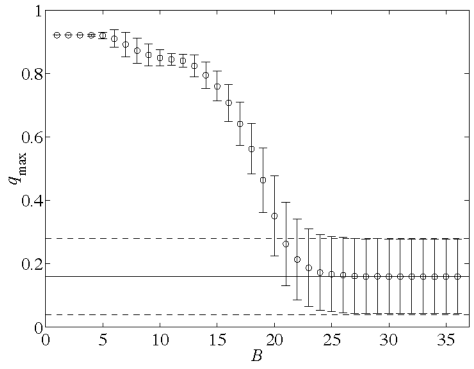
- $B \ll n$: levels are exponentially degenerated

$$q_{\max} \simeq 1 - \frac{2}{n}$$

- $B \gg n$: levels are non-degenerated

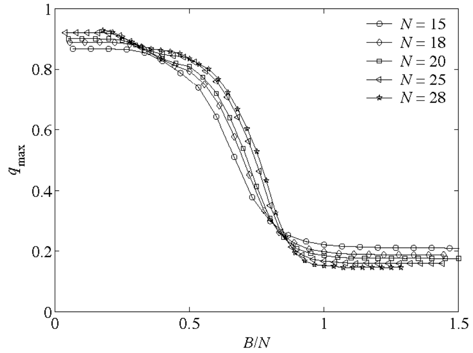
$$q_{\max} = q_{\text{random}}$$

Maximum Overlap



Max. overlap of two energetically adjacent configurations.
5×5 EA-model, $\alpha = 0$. Horizontal lines: REM prediction.

Phase Transition



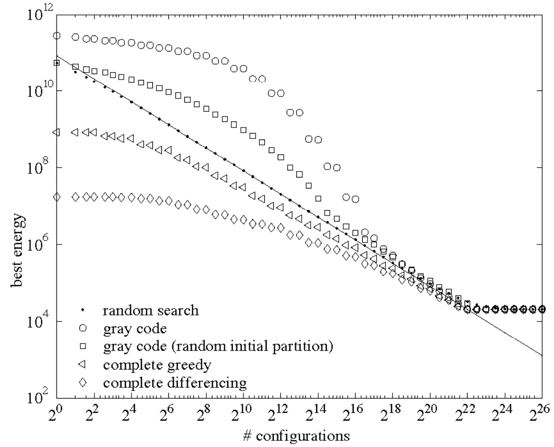
Maximum overlap between energetically adjacent configurations.
EA-model in $2d$.

Universality

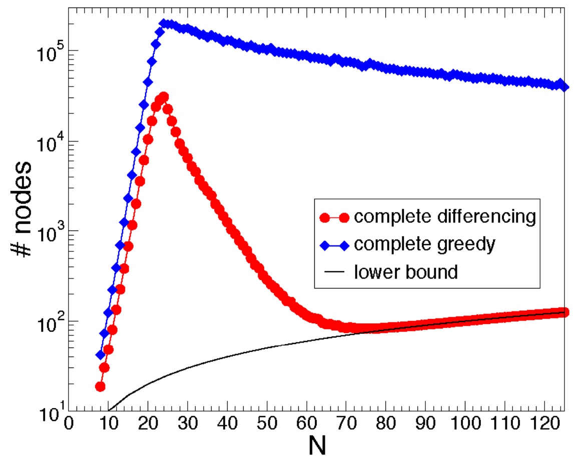
Local REM property found in

- Spin-Glasses: $1d$, $2d$, $3d$, SK, Potts
- Optimization: MST, DPRM, TSP, Subset-Sum
- no counter example so far
- rigorously established for Subset-Sum at $\alpha = 0$
  C.Borgs, J.Chayes, B.Pittel, *Rand. Struct. & Alg.* **19**, 247 (2001)

Lost in Space



Algorithms: Running Times



Subset-Sum with 20-bit numbers

Summary

Local REM:

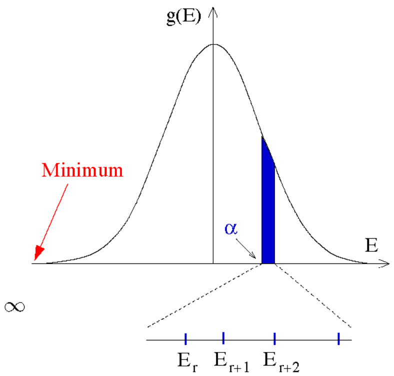
$\Delta E_r := E_{r+1} - E_r$ are iid

$\sigma(E_r), \sigma(E_{r+1})$ uncorrelated

universal, fools algorithms

requires $B > n$

transition at $B = n$ gets sharp as $n \to \infty$



H.Bauke, S.M., *Phys. Rev. E* **70**, 025102(R) (2004)
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