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SMR/1578 - 1

SCHOOL AND CONFERENCE ON FUNDAMENTAL ASPECTS OF COMPLEXITY

(6 - 10 September 2004)

" Clustering in complex networks "

presented by:

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Network Transitivity and Matrix Models

Phys. Rev. E 69, 026106 (2004), cond-mat/0310234;

Perturbing General Uncorrelated Networks

Phys. Rev. E 70, 026106 (2004), cond-mat/0401310;

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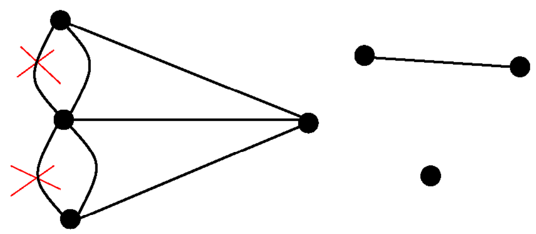
Plan of the talk

1. Introduction
   (a) Complex networks: from Internet to Biology;
   (b) Empirical observations: fat tails, small-world effect, clustering;
   (c) Statistical mechanics (Erdös-Rényi theory)
2. Statistical mechanics of adjacency matrix;
   (a) Matrix model of clustering;
   (b) Perturbative Erdös-Rényi phase;
   (c) Clustering of uncorrelated scale-free networks;
3. Summary

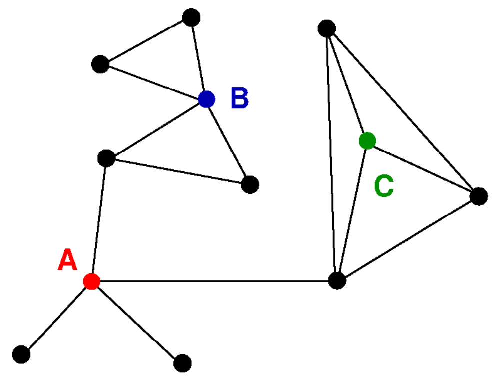
Complex networks

- Technological networks: Telecommunication; WWW; Internet; Transport;
- Abstract Networks: Sociology, Economy;
- Natural Networks: Epidemiology; Ecology; Biology;
- Complex Systems:
  Any definition of a complex system should reflect the fact that such a system consists of many mutually interacting components. These components are not identical. They interact with each other. This can be visualized as a graph whose nodes correspond to individual components and the edges to their mutual interactions. This graph is a backbone of the complex system along which various signals and perturbation may propagate.



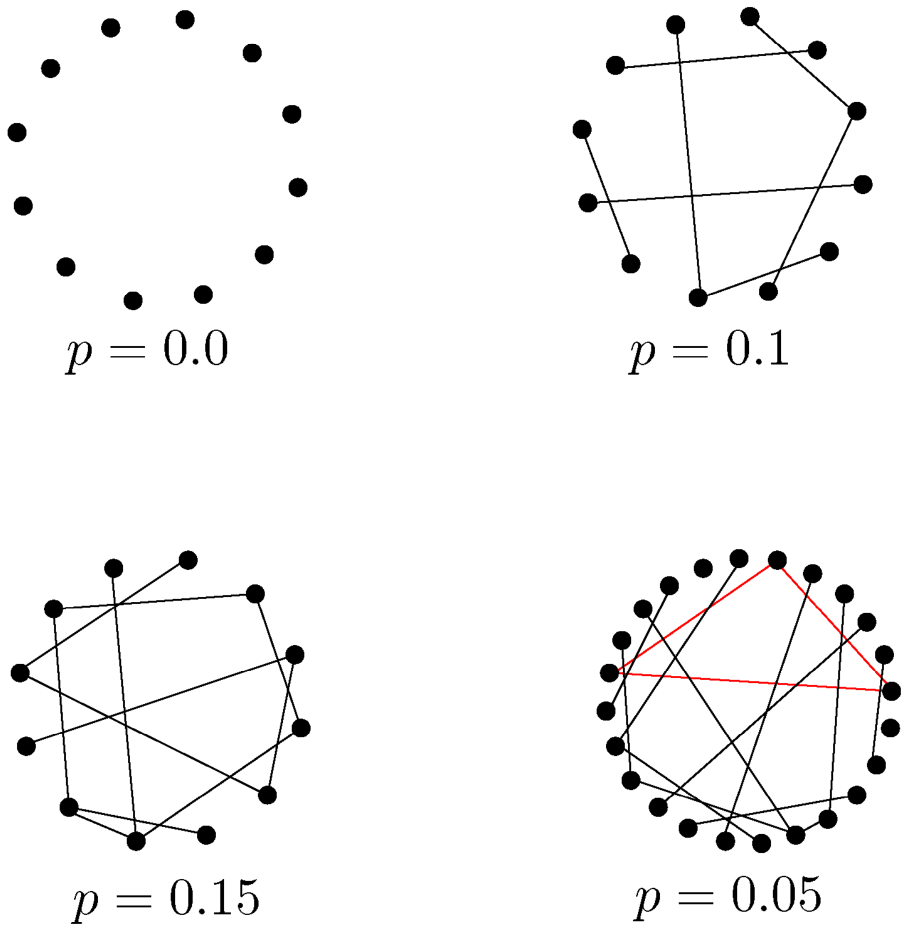
Empirical observations

- Fat tails in node degree distribution; ($p(q) \sim q^{-\gamma}$, Simon,Albert-Barabasi);
- Small world effect;
- **Large clustering**;



Clustering coefficient: $C_i = \frac{T_i}{\binom{q_i}{2}}$; $C_A = 0, \quad C_B = \frac{1}{3}, \quad C_C = 1;$

Erdös-Rényi graphs



Ensemble of graphs: $\{N, p\}$

$\binom{N}{2}$ pairs

Sparse graphs

$p = \frac{\alpha}{N} \to 0$

$\langle q \rangle \approx \alpha$

$\langle L \rangle = p\binom{N}{2} \approx \frac{1}{2}\alpha N$

$\langle T \rangle = p^3 \binom{N}{3} \approx \frac{1}{6}\alpha^3$

Adjacency matrix: $A_{ij} = A_{ji} = \begin{cases} 1 & \text{if } ij \text{ are neighbors} \\ 0 & \text{otherwise} \end{cases}$

$L = \frac{1}{2}\mathrm{Tr}A^2 = \frac{1}{2}\sum_{ij} A_{ij}A_{ji}\ , \quad T = \frac{1}{6}\mathrm{Tr}A^3 = \frac{1}{6}\sum_{ijk} A_{ij}A_{jk}A_{ki}$

Matrix model for E-R

$$Z_N = \sum_{\{A\}_N} p^L (1-p)^{\binom{N}{2}-L} \propto \sum_{\{A\}_N} \left(\frac{p}{1-p}\right)^L$$

$$Z_N \propto \sum_{\{A\}_N} \exp\left[-\frac{\ln N}{2}\mathrm{Tr}A^2\right]$$

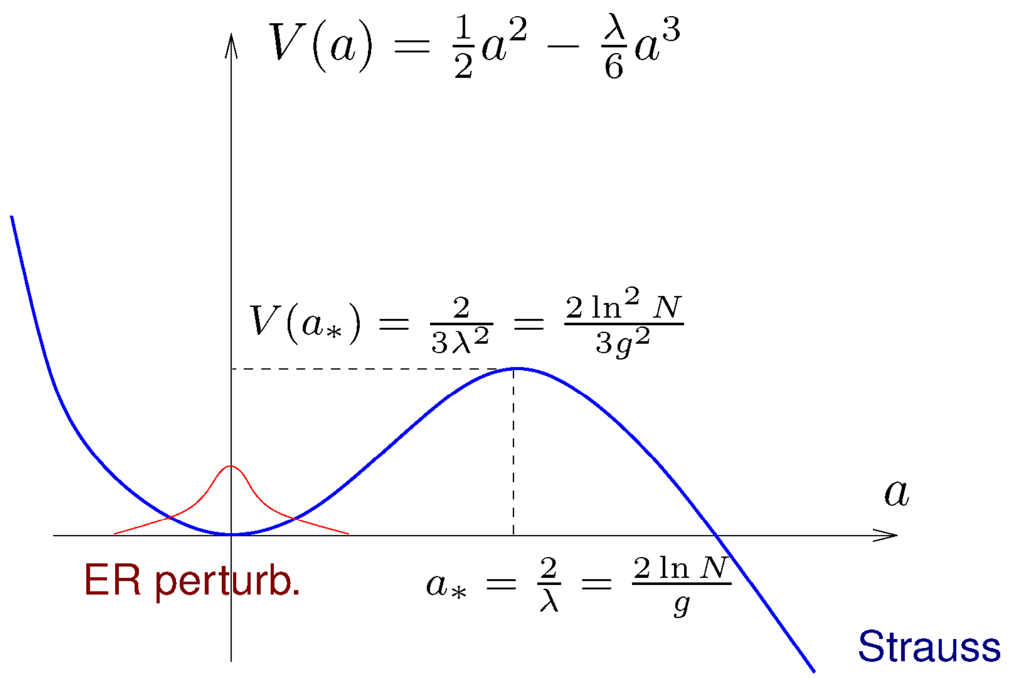
+ Clustering:

$$Z_N = \sum_{\{A\}_N} \exp\left[-\frac{\ln N}{2}\mathrm{Tr}A^2 + \frac{g}{6}\mathrm{Tr}A^3\right]$$

Perturbative phase and Strauss instability

$$Z_N = \sum_{\{A\}_N} \exp - \ln N \cdot \mathrm{Tr}\left(\frac{1}{2}A^2 - \frac{\lambda}{6}A^3\right)$$

where $\lambda = \frac{g}{\ln N}$ or $g = \lambda \ln N$.



Prob(ERP $\longrightarrow$ Strauss) $\sim \exp - \frac{2\ln^3 N}{3g^2}$

A - is a sparse matrix ($\#1 = 2L \propto N$ out of N^2 matrix elements)

Perturbative phase for E-R graphs

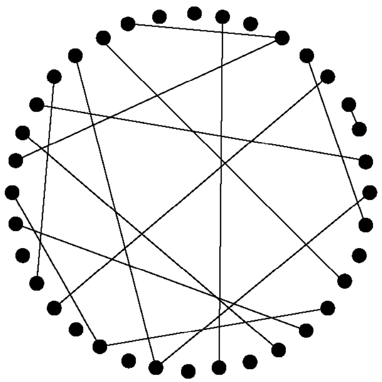
$$A = \left(\begin{array}{cccccccccc} 0 & 1 & 0 & 0 & 0 & 0 & 1 & \ldots & 1 & 0 \\ & 0 & 0 & 1 & 0 & 0 & 0 & \ldots & 0 & 1 \\ & & 0 & 0 & 0 & 0 & 0 & \ldots & 0 & 0 \\ & & & 0 & 0 & 0 & 0 & \ldots & 0 & 0 \\ & & & & 0 & 0 & 0 & \ldots & 0 & 1 \\ & & & & & 0 & 1 & \ldots & 0 & 0 \\ & & & & & & 0 & \ldots & 0 & 0 \\ & & & & & & & \ldots & 0 & 0 \\ & & & & & & & & 0 & 0 \\ & & & & & & & & & 0 \end{array} \right)$$

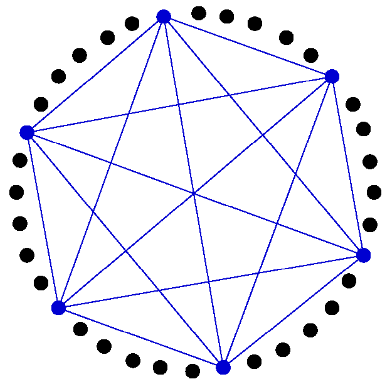
Strauss phase

$$A = \left(\begin{array}{cccccccccc} 0 & 1 & 1 & 1 & 1 & 1 & 1 & \ldots & 0 & 0 \\ & 0 & 1 & 1 & 1 & 1 & 1 & \ldots & 0 & 0 \\ & & 0 & 1 & 1 & 1 & 1 & \ldots & 0 & 0 \\ & & & 0 & 1 & 1 & 1 & \ldots & 0 & 0 \\ & & & & 0 & 1 & 1 & \ldots & 0 & 0 \\ & & & & & 0 & 1 & \ldots & 0 & 0 \\ & & & & & & 0 & \ldots & 0 & 0 \\ & & & & & & & \ldots & 0 & 0 \\ & & & & & & & & 0 & 0 \\ & & & & & & & & & 0 \end{array} \right)$$

complete graph of size $\propto \sqrt{N}$ + empty vertices

Tr$A^3 \propto N^{3/2}$ dominates over $N \ln N$





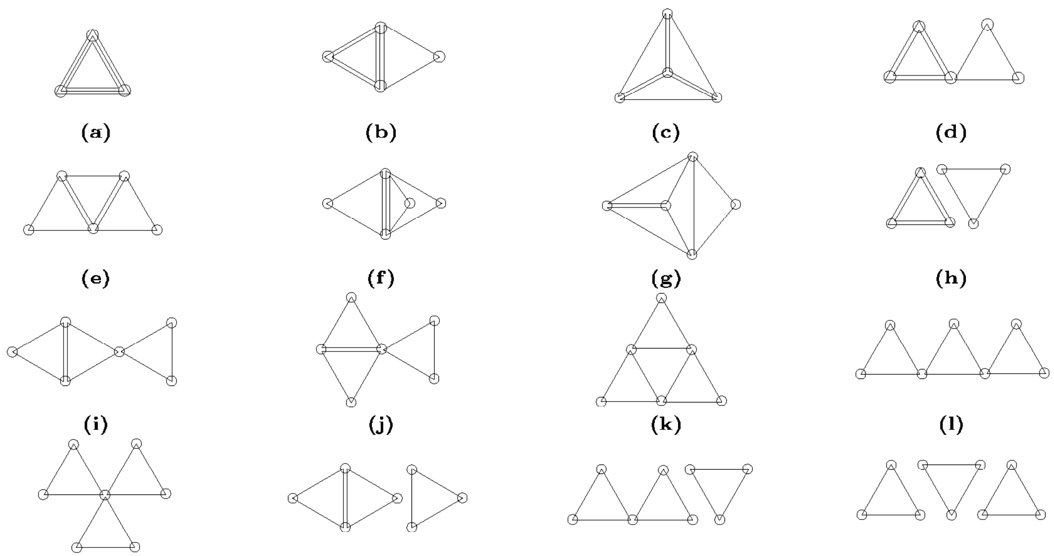
Perturbation theory:

$$Z_N = \left\langle \exp \frac{g}{6} \mathrm{Tr} A^3 \right\rangle_{ER} = \sum_k \frac{g^k}{6^k k!} \left\langle \left(\mathrm{Tr} A^3\right)^k \right\rangle_{ER}$$

2^{nd} order $\frac{g^2}{6^2 2!} 6 p^3 N^{\underline{3}}$ $\frac{g^2}{6^2 2!} 18 p^5 N^{\underline{4}}$ $\frac{g^2}{6^2 2!} 9 p^6 N^{\underline{5}}$ $\frac{g^2}{6^2 2!} p^6 N^{\underline{6}}$

(a) (b) (c) (d)

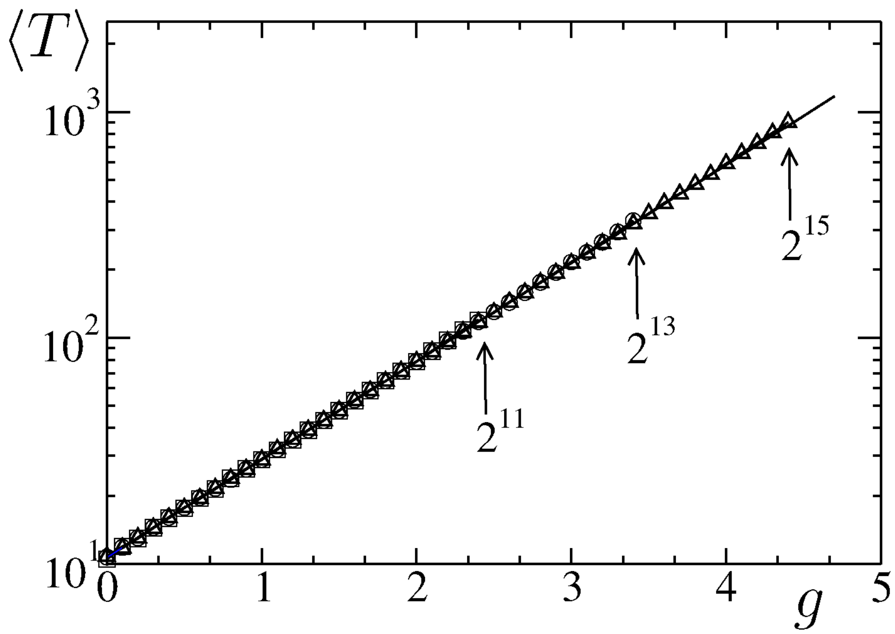
3^{rd} order



(a) (b) (c) (d) (e) (f) (g) (h) (i) (j) (k) (l) (m) (n) (o) (p)

Result

$$F_N(g) = \ln Z_N(g) \quad , \quad \langle T \rangle = \frac{dF_N}{dg}(g) = \frac{\alpha^3}{6} e^g$$



$$g = \lambda \ln N$$

$\langle T \rangle = \frac{\alpha^3}{6} e^g = \frac{\alpha^3}{6} N^\lambda$ for $\lambda < \lambda_*$, e.g. $\lambda_* \approx 0.7$ for $\alpha = 4$

Uncorrelated weighted graphs

$$Z_{NL} \sim \sum_{A} w(q_1) \dots w(q_N)$$

Erdös-Rényi $\quad w(q) = 1 \ \rightarrow \ p(q) = e^{-\alpha} \frac{\alpha^q}{q!}$

Scale free graphs: $\exists w(q) : \quad p(q) \sim q^{-\gamma}$

Number of triangles $\langle T \rangle = \frac{1}{6} \left(\frac{\langle q(q-1) \rangle}{\langle q \rangle} \right)^3$

$$\langle T \rangle \sim \begin{cases} \text{const} & \gamma > 3 \\ \ln^3 N & \gamma = 3 \\ N^{\frac{3}{2}(3-\gamma)} & 2 < \gamma < 3 \end{cases} \qquad \text{for} \quad N \rightarrow \infty$$

Summary

Clustering / the number of triangles (three-cycles)

- Statistical mechanics approach to random graphs;
- Matrix model: Erdös-Rényi graphs + perturbation e^{gT};
- Perturbative phase: $\langle T \rangle = \frac{\alpha^3}{6} N^{\lambda}$
- For uncorrelated networks: $\langle T \rangle = \frac{1}{6} \left(\frac{\langle q(q-1) \rangle}{\langle q \rangle} \right)^3$
- For uncorrelated scale-free networks for $2 < \gamma < 3$: $\langle T \rangle = N^{\frac{3}{2}(\gamma - 2)}$