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SCHOOL AND WORKSHOP ON QUANTUM ENTANGLEMENT, DECOHERENCE, INFORMATION, AND GEOMETRICAL PHASES IN COMPLEX SYSTEMS (1 November - 12 November 2004)

Entropy/information bounds and gravitation

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ICTP, Trieste
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Entropy/information bounds and gravitation

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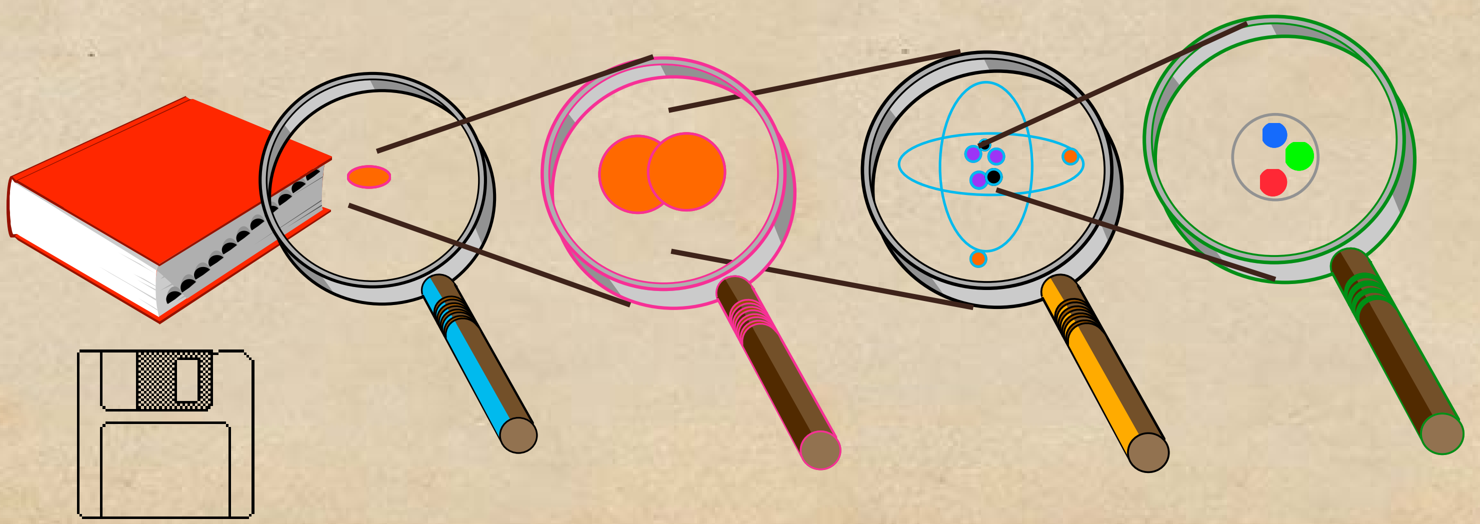
The problem

- What is the ultimate information storage capacity of a physical system ?
  - emphasis on storage, not communication
  - emphasis on physically significant parameters
  - emphasis on "ultimate"



Information storage capacity

Holevo's theorem for quantum communication channel

symbol i $\Longleftrightarrow \;\; {}_i$ $\qquad\qquad$ $\mathtt{S}(\;{}_i) = -\mathrm{Tr}\;\;{}_i \ln \;\;{}_i$

$$= \sum_i \mathtt{p}_i \;\; {}_i \qquad\qquad \mathtt{S}(\;\;) = -\mathrm{Tr}\;\;\ln$$

$$\mathtt{I} \leq \mathtt{S}(\;\;) - \sum_i \mathtt{p}_i \mathtt{S}(\;{}_i)$$

also bounds stored information

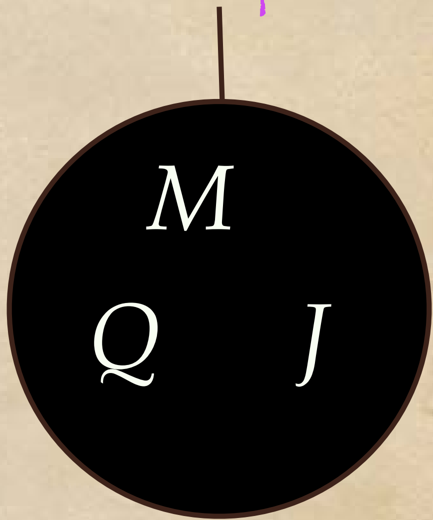
$$\mathtt{I} \leq \mathtt{S}(\;\;) \quad \text{any} \quad \text{subject to constraints}$$

Black holes in natural settings

- Collapse of star at end of its active life
  - stellar mass black hole; 2-20 $M_{\odot}$
  - powerhouses of some of the galactic X-ray sources
  -
- Aggregation of smaller black holes in galaxy's core
  - supermassive black hole; 10^{6}-10^{9} $M_{\odot}$
  - powerhouses of quasars
- Collapse of overdense regions early in universe
  - primordial black holes 10^{12}-10^{30} Kg
  - hypothetical; may lurk everywhere

Black hole varieties

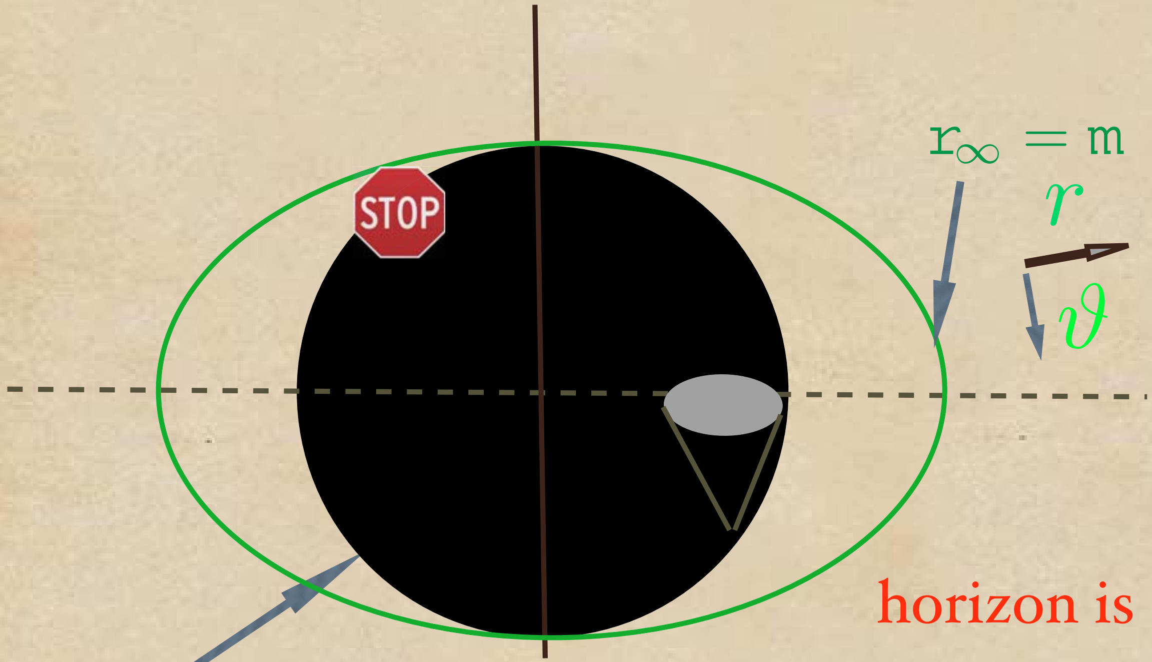
- Mathematically black holes are peculiar solutions of Einstein's gravity equations - (Einstein did not accept such objects)
  - they behave like particles
  - also show up in competing theories
- Simple objects - few parameters
  - Schwarzschild (1916) - mass M
  - Reissner-Nordstrom (1918) - mass and charge Q
  - Kerr (1962) - mass and angular momentum J
  - Newman (1965) -M,Q,J



The black hole horizon

$$m = GM/c^2 \qquad q = \sqrt{G}\,Q/c^2 \qquad a = J/mc$$

$$r_\infty = m + \sqrt{m^2 - a^2 \cos^2}$$



horizon is one-way membrane

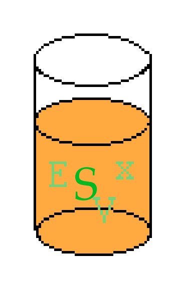
$$r_h = m + \sqrt{m^2 - a^2}$$

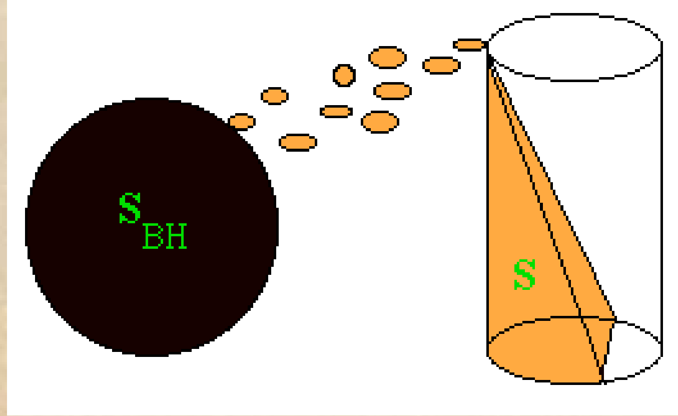
“Black holes have no hair” (Wheeler 1969)

- The Kerr-Newman family - the totality of equilibrium black holes
- On condition that only allow gravity and electromagnetism - no more parameters
- True in competing theories
- In presence of extra interactions exceptions occur
  - black holes with scalar fields
  - colored black holes
  - black holes with magnetic monopole

Why black hole thermodynamics ?

- Analogy - few parameters
  - ordinary systems: E, V, N
  - black hole: M, Q, J
- Contradictions
  - ordinary systems: thermodynamical - second law
  - black hole: mechanical
  - combined system: violates second law





Formula for black hole entropy (1972)

$$A = 4\,[(m + \sqrt{m^2 - q^2 - a^2})^2 + a^2]$$

$$d(Mc^2) = \quad dA + \quad dQ + \quad dJ \quad \Longleftrightarrow \quad dE = TdS + \quad dQ + \quad dJ$$

$$\equiv c^4\,(2GA)^{-1}\sqrt{m^2 - q^2 - a^2}$$

$$\equiv 4\;Q\;(m + \sqrt{m^2 - q^2 - a^2})A^{-1}$$

$$\equiv 4\;JM^{-1}A^{-1}$$

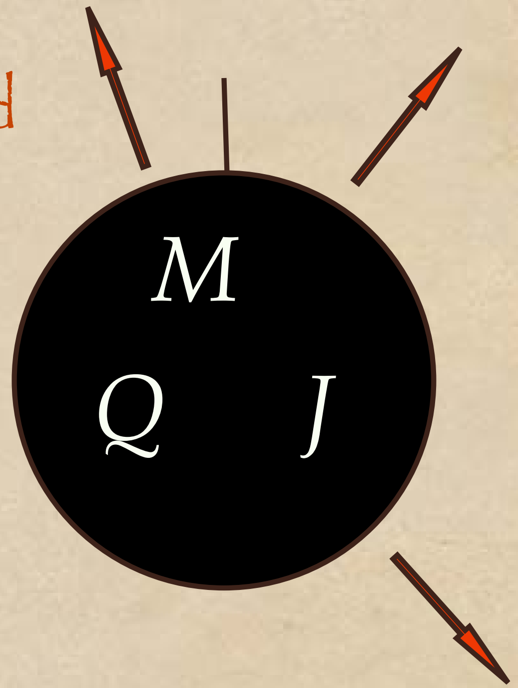
Conclusion: $S_{BH} = f(A)$

Choice $f(A) \propto \sqrt{A}$ ruled out ⟹ $S_{BH} = \; A c^3/\hbar G$

$$T_{BH} = (\; S_{BH}/\; A)^{-1} \quad = (\; c^3/\hbar G\,)$$

Hawking radiance (1974)

- Quantum fields in black hole grav. field
- Spontaneous emission
- Thermal character
  - spectrum - Planckian
  - modes - uncorrelated
  - statistics - black body radiation



$$T_{BH} = \frac{c^3\hbar}{8\ GM} \longrightarrow \quad = 1/4$$

$$S_{BH} = Ac^3/4\hbar G$$

The laws of black hole thermodynamics

0 th : In equilibrium a suitable local temperature is constant all over horizon (Carter 1970)

1 st: $d\,(Mc^2) = T_{BH} dS_{BH} + \quad dQ + \quad dJ$

2 nd: $S'_{BH} + S'_m + S'_r > S_{BH} + S_m + S_r$ (GSL, 1972)

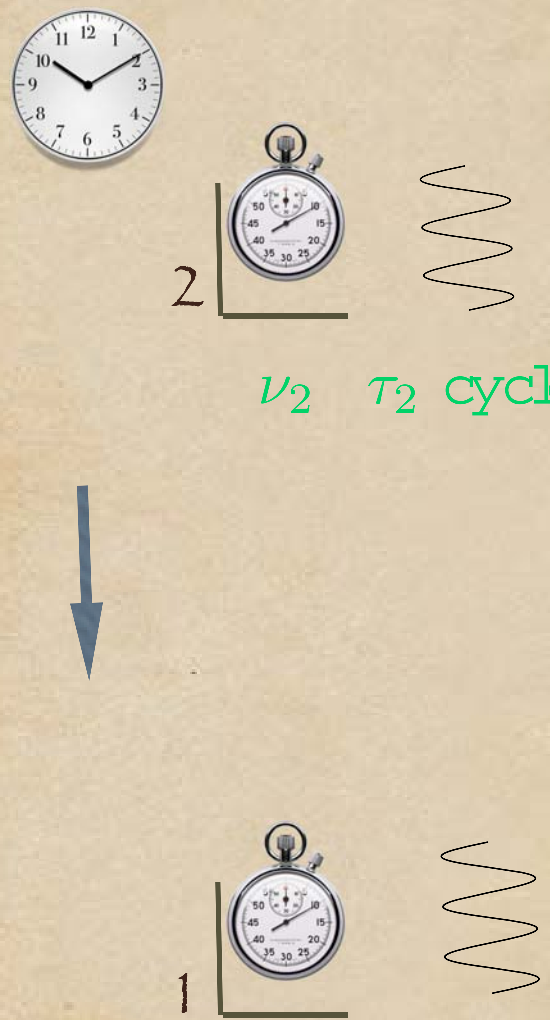
3 rd : $T_{BH} \to 0$ is hard to achieve

~~$S_{BH} \to 0$ as $T_{BH} \to 0$~~

Evidence for generalized second law (GSL)

- Black hole mergers
  - area theorem (Christodoulou, Penrose & Floyd, Hawking 1970)
  - total black hole entropy increases
  - some (gravitational) radiation entropy produced
- Infall into black hole
  - matter entropy decreases
  - black hole entropy increases
  - sum of the two increases (1972-74)
- Hawking radiation
  - radiation entropy increases
  - black hole entropy decreases
  - sum of the two increases (1975)

Gravitational redshift



$\nu_2 \quad \tau_2$ cycles

$\nu_1 \quad \tau_1$ cycles

$$c^2 d\ ^2 = c^2 dt^2 - dx^2 - dy^2 - dz^2$$

$$c^2 d\ ^2 = -g_{tt}\, dt^2 - \cdots$$

$$\frac{\ _2}{\ _1} = \frac{\sqrt{-g_{tt}(\mathbf{x}_1)}}{\sqrt{-g_{tt}(\mathbf{x}_2)}}$$

$$g_{tt} = -c^2(1 - 2GM/rc^2)$$

$$_\infty = (1 - 2GM\ /r_1 c^2)^{1/2}\ \ _1$$

$$_\infty \approx \left(\frac{r}{2GM\ /c^2}\right)^{1/2}\ \ _1$$

A *Gedanken* experiment

$$ds^2 = -(1 - 2GM/rc^2)\,dt^2 + (1 - 2GM/rc^2)^{-1}\,dr^2 + \cdots$$

$$= \int \frac{dr}{(1 - 2GM/rc^2)^{1/2}} \approx 2\,(2GM/c^2)^{1/2}(\quad r)^{1/2}$$

$$\infty \approx \left(\frac{r}{2GM/c^2}\right)^{1/2} \qquad 1$$

$$E_\infty \approx \frac{\qquad}{4GM/c^2}\,E$$

$$(\quad M)c^2 \approx \frac{R}{4GM/c^2}\,E$$

First try at an entropy bound (1981)

$$A = 4\left[\left(m + \sqrt{m^2 - \cancel{q^2} - \cancel{a^2}}\right)^2 + \cancel{a^2}\right] = 16\,(GM/c^2)^2$$

$$A \approx 32\,(G/c^2)^2 M \qquad M \approx 8\,(G/c^2) E R$$

$$S_{BH} = (c^3/4\hbar G)\,A \approx 2\,ER/\hbar c$$

$$S_{BH} + S_{\mathrm{world}} = 2\,ER/\hbar c - S \geq 0$$

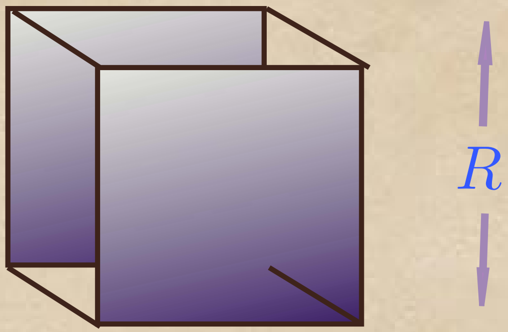
$$S \leq 2\,ER/\hbar c$$

$S \leq 2\pi ER/\hbar c$

Observations

- Bound applicable to isolated object
  - don't use for pieces of a system !
- *E* means object's total proper energy
  - don't forget rest mass energy !
- Object not strongly self-gravitating
  - nevertheless bound works for black hole itself !
- Bound independent of gravitation
  - should apply to flat spacetime physics !

Rest mass quandary



N massive nonrelativistic bosons

lowest lying states have energies of $\mathcal{O}(\hbar^2/\mu R^2)$

total energy $E = \mathcal{O}(N\hbar^2/\mu R^2)$

make $2 \;\; E R / \hbar c$ as small as you want.

Hence can violate the bound !

Wrong: forgot the rest mass part of E !

$$W = \frac{(\quad + N - 1)!}{(\quad - 1)!\,N!}$$

$$S = (\quad + N)\ln(\quad + N) - N\ln N - \quad \ln \quad + \cdots$$

$$S = \quad \ln(1 + N/\quad) + N\ln(1 + \quad/N) + \cdots$$

$$= (\ \mu c R/2\ \hbar)^3$$

$$\frac{S}{N\mu c R/\hbar} < \frac{S}{N\Omega^{1/3}} = \frac{1}{N^{1/3}} \underbrace{\frac{\ln(1+\bar{n}) + \bar{n}\ln(1+1/\bar{n})}{\bar{n}^{2/3}}}_{< 1.581}$$

$$S < \frac{1.581 N\mu c^2 R}{N^{1/3}\hbar c} < \frac{2\ N\ \mu c^2 R}{\hbar c}$$

The low temperature quandary

D. Deutsch (1982): canonical ensemble

simple case: black body radiation

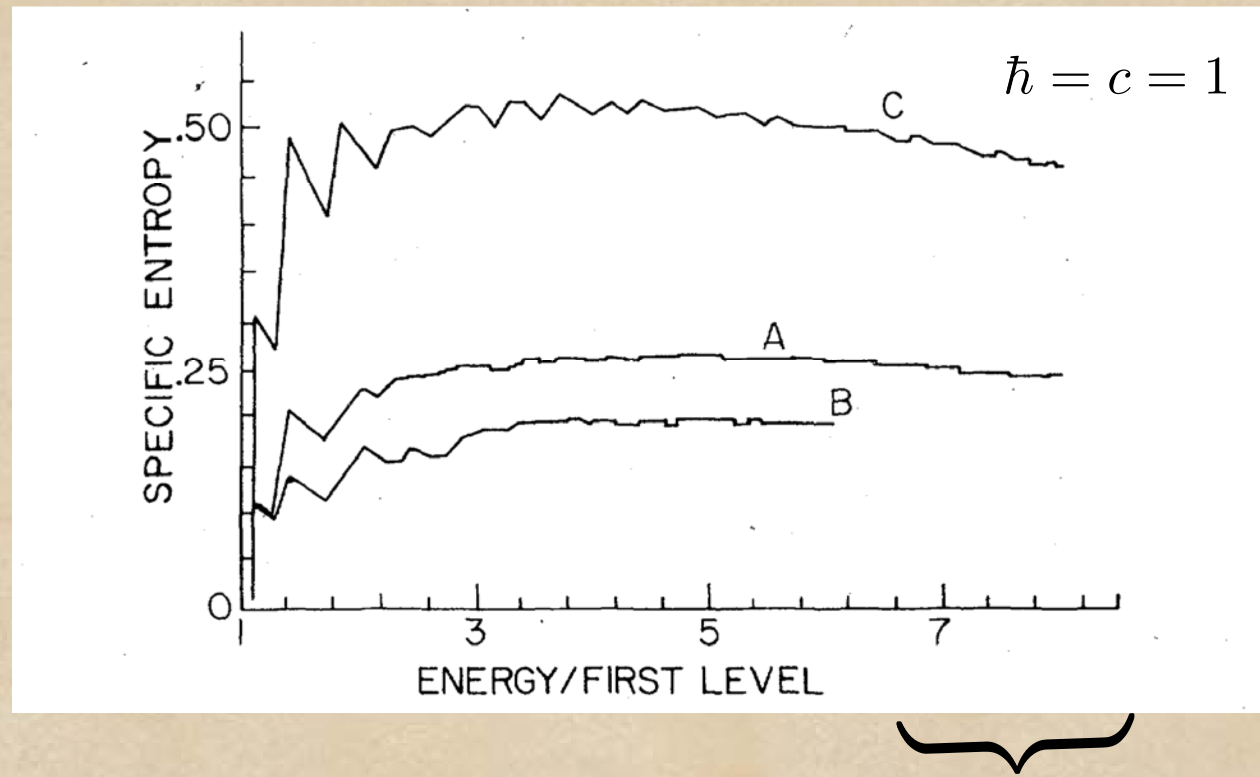
$E \sim T^4 V$ and $S \sim T^3 V$

specific entropy $S/E \sim T^{-1} =$

to violate bound take $> 2 \; R/\hbar c$

Wrong: forgot effect of boundary

Microcanonical ensemble calculations



thermal

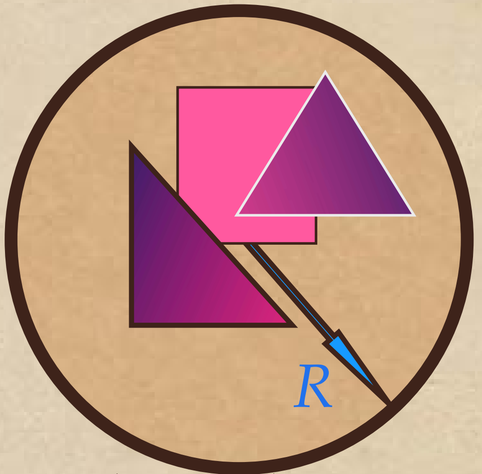
A - scalar radiation in a unit cube

B - photons in box 1 x 2/3 x 1/5

C - neutrinos in unit ball

Universal entropy bound - summary

$$S \leq 2\pi ER/\hbar c$$



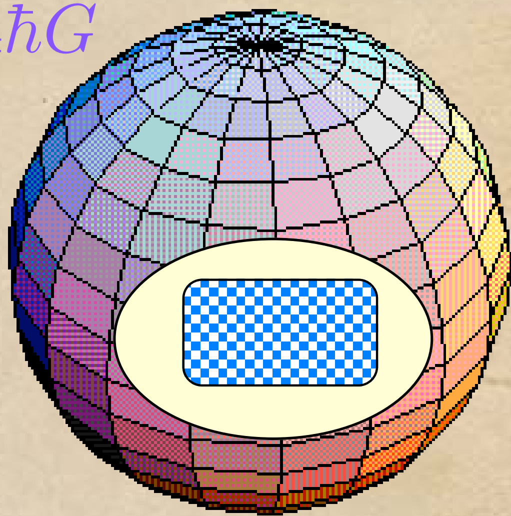
- originated from black hole physics
- *E* is gravitating energy
- system must be complete, isolated
- is example tested
- may fail for strong ly gravitating or rapidly evolving systems
- tightest of known bounds

Holographic entropy bound

('t Hooft 1993; Gonzales-Diaz 1983)

- A area of any closed circumscribing surface
- is example tested
- includes G
- valid for weak and strong self-gravity
- fails for rapidly evolving systems
- is overly generous

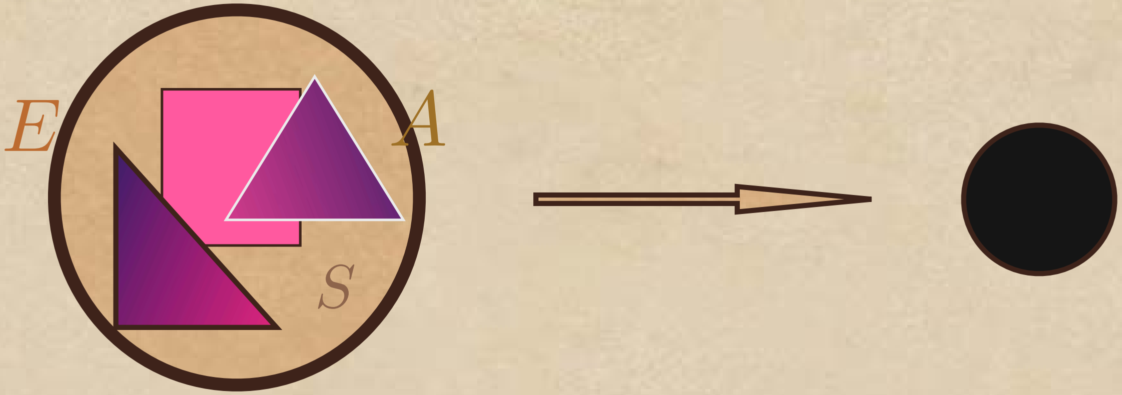
$$S \leq c^3 A/4\hbar G$$



$R = 1$ cm; $\mu = 1$ g

$$S \leq \begin{cases} 2 \times 10^{38}; \text{universal} \\ 1 \times 10^{66}; \text{holographic} \end{cases}$$

How do we know ? (Susskind 1995)



$$S < S_{BH} = c^3 A_{BH}/4\hbar G < c^3 A/4\hbar G$$

$$S_{BH} = (4\ \ GM^2/c\hbar) \approx 8\ \ GM(E/c^2)/c\hbar = c^3 A/4\hbar G$$

$$2GM/c^2 > \sqrt{A/4\ \ } \equiv R$$

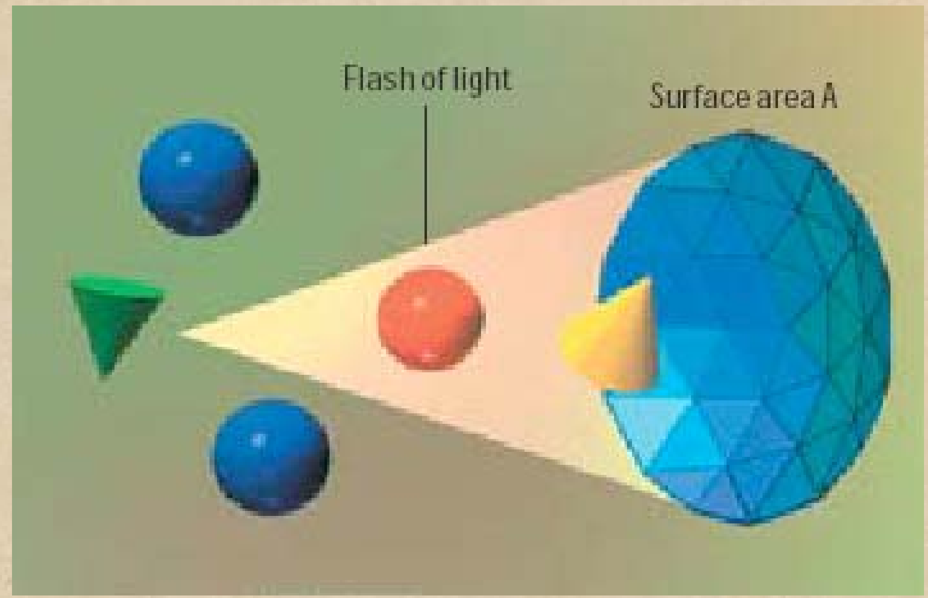
$$M = \frac{c^6 A}{32\ \ G^2 E}$$

Bousso covariant entropy bound

$S \leq c^3 A / 4\hbar G$

A - area of surface with fixed sign of Gaussian curvature

S - entropy that is illuminated by "light rays" up to caustics

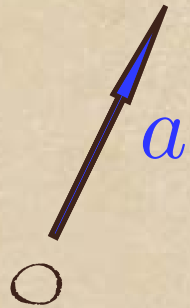


but still overly generous

- covariant
- example tested
- valid for strong gravity
- valid for rapidly evolving systems

Quantum buoyancy

Unruh effect (1976) $T_U = \frac{\hbar a}{2 \; c}$



$$a = c^2 \frac{d}{dr}(1 - 2GM/rc^2)^{1/2}$$

$+ p = T s$ (Gibbs-Duhem relation)

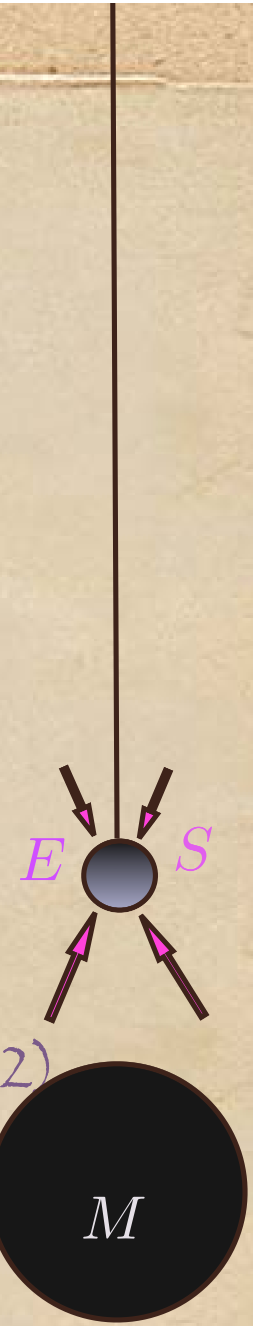
W reaches its maximum at $E = E_{\rm rad}$

$$E - W = T_{BH} S_{\rm rad}$$

$$S_{BH} = (E - W)/T_{BH}$$

$$S_{BH} + \quad S_{\rm world} = S_{\rm rad} - S$$

Unruh and Wald (1982)



Not everything is so simple

$$3p = \quad = \frac{T^4}{15\hbar^3 c^3} \qquad R \gg \frac{\hbar c}{E}$$

For $r - 2GM/c^2 \ll 2GM/c^2$,

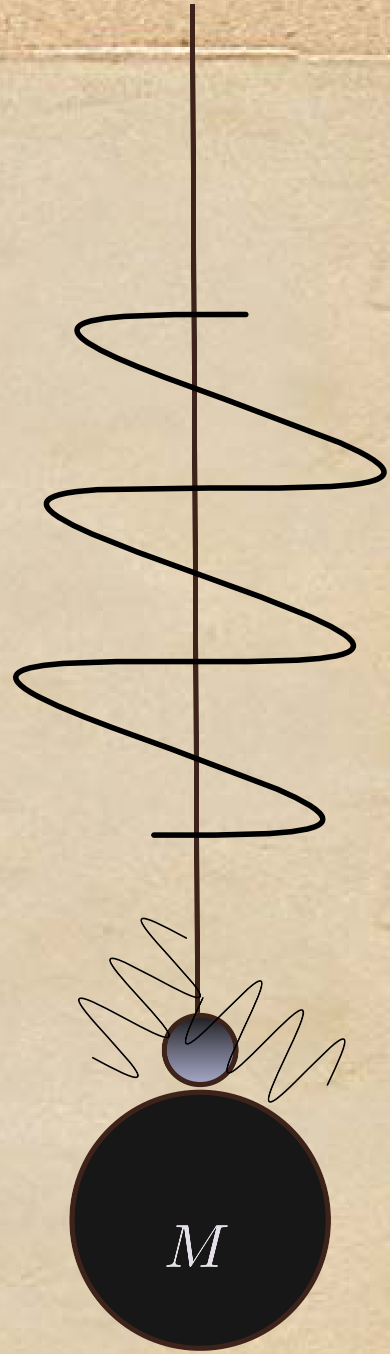
$$T \approx \frac{\hbar}{2} \qquad \Longrightarrow \lambda_{\text{typical}} \approx \ell$$

Buoyant force drops rapidly with ℓ

1. Drop sphere from a few radii out

- buoyancy makes little difference

Weak version of universal bound



More precisely

b) Account for buoyant force in detail

- fluid model of radiation a poor one (1999)
- wave scattering problem
- the important parameter $\sqrt{\frac{N \hbar}{180 \; E R}}$

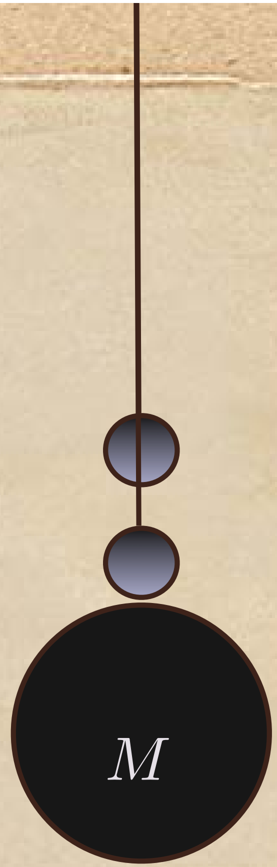
take $\sigma \gg 1$

at floating $E \neq E_{\text{rad}}$ and $\Delta S_{BH} < S_{\text{rad}}$

$$S_{BH} > 2 \; E R / \hbar c$$

now consider $\sigma \ll 1$

$$S_{BH} = 2 \; E R / \hbar c \, [(1 + \quad /2) \; \overline{1 + \quad} \; - \; \cdots \;]$$



Alternative way

$$F(r) = \frac{\mathcal{N}\hbar c^2}{61{,}440(\ m\, r)^2}$$

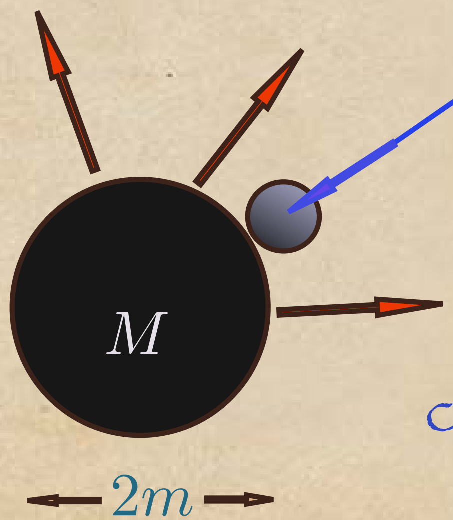
S E $R = m\ /$

$$t \quad 5 \times 10^4\, \frac{E(m/c)^2}{\hbar N}$$

$$d \approx 2(c^2 t^2 m\ /\ \ ^2)^{1/3}$$

$$d \quad 1\,2 \times 10^3 \left(\frac{\ \ }{N}\,\frac{R}{\hbar c/E}\right)^{2/3} m$$

$$\frac{f_{rad}(r)}{f_{grav}(r)} = \frac{\mathcal{N}_e}{61{,}440\ \ ^2}\,\frac{(\hbar c/E\ )R^{\ 2}}{m^{\ 3}}$$



choose parameters so that M is unchanged overall

Entropy accounting

$S \qquad \mathrm{R} \;= \mathrm{m}\;/$

E

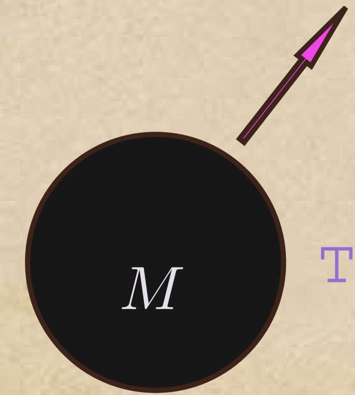
$\mathrm{S}_{\rm rad} = 8 \qquad \mathrm{E}\,\mathrm{R}\;/\hbar\mathrm{c}$

$S_{\rm rad} - S > 0$

$\mathrm{S}_{\rm rad} = \dfrac{\mathrm{E}}{\mathrm{T}_{BH}}$

$\mathrm{S} \;<\; 8 \qquad \mathrm{E}\,\mathrm{R}\;/\hbar\mathrm{c}$

$\mathrm{S} \leq 2 \;\; \mathrm{E}\,\mathrm{R}\;/\hbar\mathrm{c}$



$\mathrm{T}_{BH} = \dfrac{\mathrm{c}\hbar}{8 \quad \mathrm{m}}$