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SCHOOL AND WORKSHOP ON QUANTUM ENTANGLEMENT, DECOHERENCE, INFORMATION, AND GEOMETRICAL PHASES IN COMPLEX SYSTEMS
(1 November - 12 November 2004)

Quantum Entanglement and Quantum Information

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Quantum Entanglement & Quantum Information

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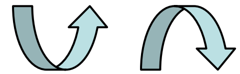
Can physics be simulated by a universal computer? .. The physical world is quantum mechanical and therefore the proper problem is the simulation of quantum physics […] the full description of quantum mechanics for a large system with R particles […] has too many variables, it cannot be simulated with a normal computer with a number of elements proportional to R […but it can be simulated with] quantum computer elements […] Can a quantum system be probabilistically simulated by a classical (probabilistic, I'd assume) universal computer? […] If you take the computer to be the classical kind I've described so far […] the answer is certainly, No!

---R. P. Feynman (1982)

Basic tools for quantum computation and information:

Qubit:

Spin: $\uparrow\downarrow$

Polarization of photon 

Atom + Resonant field $|0\rangle, |1\rangle$

A general qubit state: $\alpha|\mathbf{0}\rangle + \beta|\mathbf{1}\rangle$

Entanglement, Quantum logic gates, Quantum algorithms

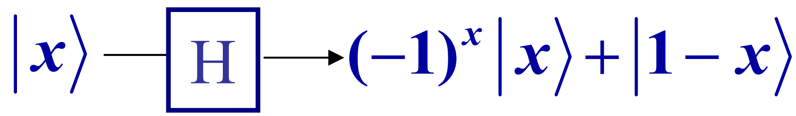
Quantum communication protocols: Quantum teleportation, quantum networking

Realization by physical systems

Single qubit gates

- Hadamard gate:

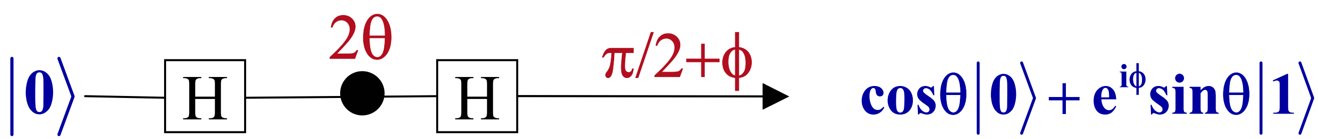
$$H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$



- Phase-shift gate:

$$\Phi = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\phi} \end{pmatrix} \qquad |x\rangle \rightarrow e^{ix\phi}|x\rangle$$

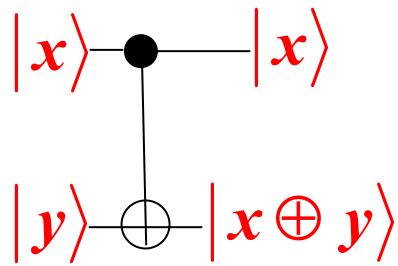
Hadamard and phase-shift gates are sufficient to construct any unitary operation on a single qubit.



Two-qubit gates

► Controlled-NOT:

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

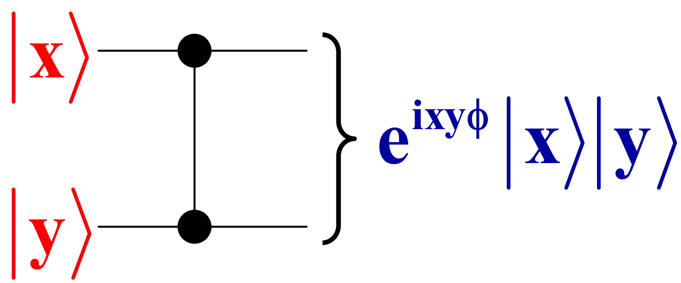


(addition modulo two)

$$|00\rangle \rightarrow |00\rangle$$
$$|01\rangle \rightarrow |01\rangle$$
$$|10\rangle \rightarrow |11\rangle$$
$$|11\rangle \rightarrow |10\rangle$$

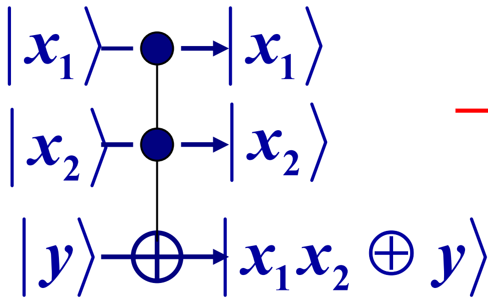
► Controlled-phase gate:

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & e^{i\phi} \end{pmatrix}$$



Basic gates ⇒ Complicated gates, Networks

► Toffoli gate
or
CCNOT gate



C-NOT → **y=0**

$$|x_1\rangle$$
$$|x_1 \oplus x_2\rangle$$
$$|x_1 x_2\rangle$$

► XOR gate

$$\mathbf{U = 1/2 + S_1^z + S_2^z - 2S_1^z S_2^z}$$

$$\rightarrow \ \mathbf{(S^+S^- - 1)}$$

(Realization in cavity-QED)

$$\mathbf{U|11\rangle = -|11\rangle;\ Rest\ +1}$$

ALGORITHMS

- Quantum Function Evaluation: $\sum_x |x,0\rangle \rightarrow \sum_x |x, f(x)\rangle$

- Deutsch-Jozsa algorithm: $f(x);\; x=(0,1);\quad f(0) \begin{cases} = f(1) \\ \neq f(1) \end{cases}$

Classically, Two queries

or

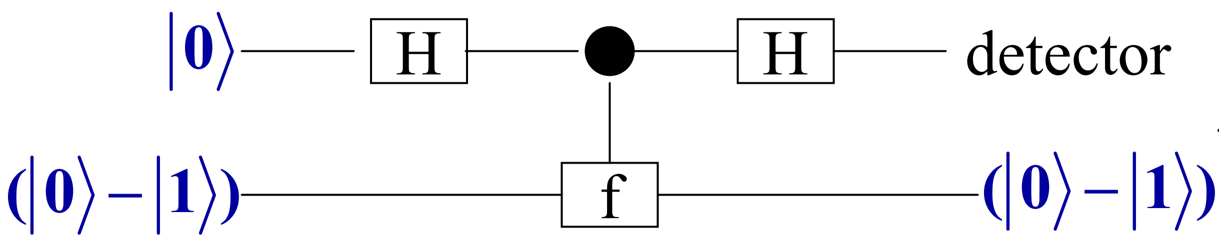
Q. mechanically, one query

$2^{n-1}+1$

vs

1

$$|x\rangle(|0\rangle-|1\rangle) \rightarrow |x\rangle(|0\oplus f(x)\rangle-|1\oplus f(x)\rangle) = (-1)^{f(x)}|x\rangle(|0\rangle-|1\rangle)$$



$|0\rangle \Rightarrow$ const

$|1\rangle \Rightarrow$ balanced

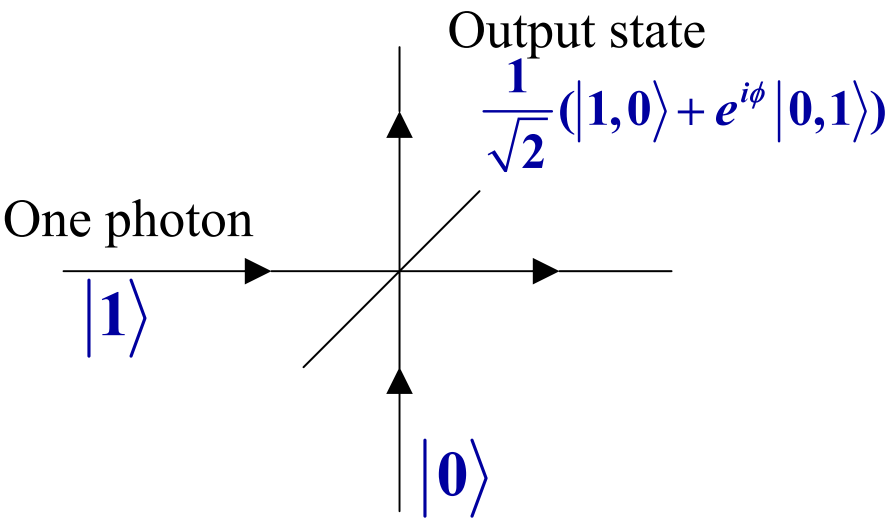
- Grover's Search algorithm: Finding a needle in haystack (unsorted database)

Classical 2^{n-1}

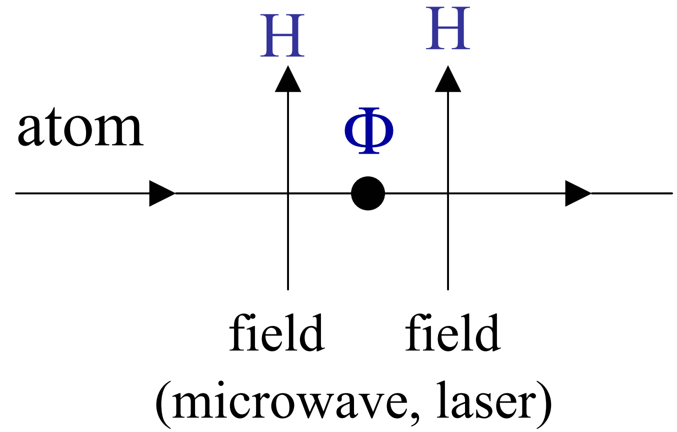
Quantum $O(2^{n/2})$

Interferometric realization of single-qubit gates

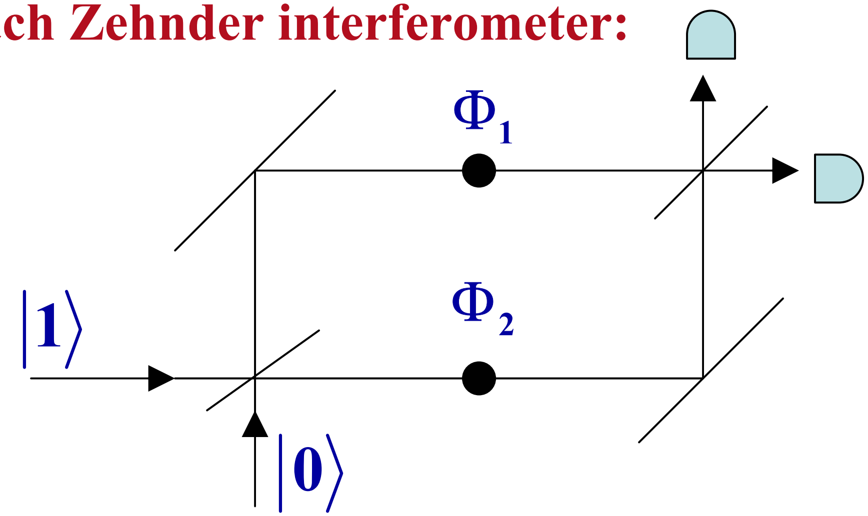
- **50-50 beam splitter:**



- **Ramsey interferometer**



- **Mach Zehnder interferometer:**



H : Beam splitter

ΦHΦ : Interferometer

Quantum entanglement in a two-body system

- Consider two coupled oscillators :

$$H = \frac{1}{2}\left[\left(\frac{p_1^2}{m} + kx_1^2\right) + \left(\frac{p_2^2}{m} + kx_2^2\right) + v(x_1 - x_2)^2\right]$$

- Diagonalization : H is a sum of two independent oscillators

$$H = \hbar\omega_0\left(n_+ + \frac{1}{2}\right) + \hbar\omega_-\left(n_- + \frac{1}{2}\right), \qquad n_\pm = a_\pm^\dagger a_\pm$$

where,

$$\omega_0 = \sqrt{k/m}, \quad \omega_- = \sqrt{(k+v)/m}$$

$a_\pm$: normal mode operators

$$= \frac{1}{\sqrt{2}}(a_1 \pm a_2)$$

- Initial state : two uncoupled oscillators

$$\psi(x_1, x_2, 0) \propto \exp\left[-\frac{1}{2}\left(x_1^2 + x_2^2\right)\right] = \exp\left[-\frac{1}{2}\left(x_+^2 + x_-^2\right)\right]$$

- Hamiltonian in terms of two normal mode operators

$$H = \frac{1}{2}k\left(x_+^2 + p_+^2\right) + \frac{1}{2}\left[\left(k + v\right)x_-^2 + p_-^2\right]$$

- Final state

$$\psi(x_+, x_-, t) = \exp\left[-A_+(t)x_+^2\right]\exp\left[-A_-(t)x_-^2\right]$$

where,

$A_+(t) = 1/2,$ $A_-(t)$: is a function of ω_-

Rewriting,

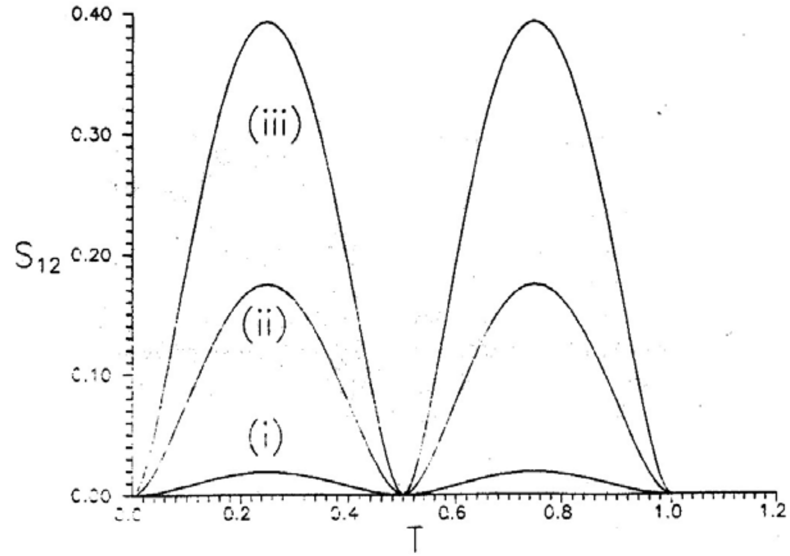
$$\psi(x_1, x_2, t) \propto \exp\left[-\left\{\frac{\left[\left(\omega_- + 1\right)^2 - \left(\omega_- - 1\right)^2 e^{2i\omega_- t}\right]\left(x_1^2 + x_2^2\right) - \left(\omega_- - 1\right)\left(1 - e^{2i\omega_- t}\right)x_1 x_2}{2\left[\omega_- + 1 - \left(\omega_- - 1\right)e^{2i\omega_- t}\right]}\right\}\right]$$

$x_1 x_2$ term : Correlation arises between two modes 1 and 2

Two particle Wigner function becomes a periodic one with frequency ω_-

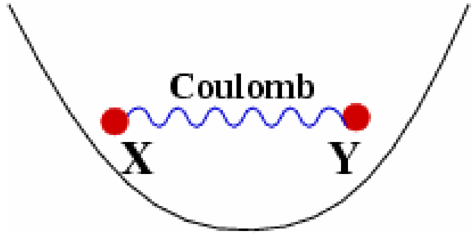
Two parts will become entangled and disentangled alternatively

$S_{12} \longrightarrow$ two-particle correlation entropy



(i) ω_-=1.1 (ii) ω_-=1.5 (iii) ω_-=2

H. Huang and G. S. Agarwal, PRA ***49****, 52 (1994)*



- Potential energy of the bound ions at **X** and **Y** $$U = \frac{1}{2}M\omega^2\left(X^2 + Y^2\right) + \frac{C}{\left|X - Y\right|}$$
- Equilibrium conditions : $\frac{\partial U}{\partial X} = 0, \quad \frac{\partial U}{\partial Y} = 0$
- Equilibrium solutions : $X = L/2, \quad Y = L/2$

  L : characteristic length scale = $\left(\frac{2c}{M\omega^2}\right)^{1/3}$

➢ Consider deviation from the equilibrium : $X = \frac{L}{2} + \delta x, \quad Y = \frac{L}{2} + \delta Y$

$$U = M\omega^2\left[\frac{3}{4}L^2 + \delta X^2 + \delta Y^2 - \delta X.\delta Y\right] + O\left[\left(\delta X\right)^3\right]$$

X and Y motions are entangled

❑ Choose :

$$\delta X = \frac{1}{\sqrt{2}}(Q + R), \quad \delta Y = \frac{1}{\sqrt{2}}(Q - R)$$

⇓

$$U = \frac{1}{2} M\omega^2 \left(Q^2 + 3R^2 + \frac{3}{2} L^2 \right)$$

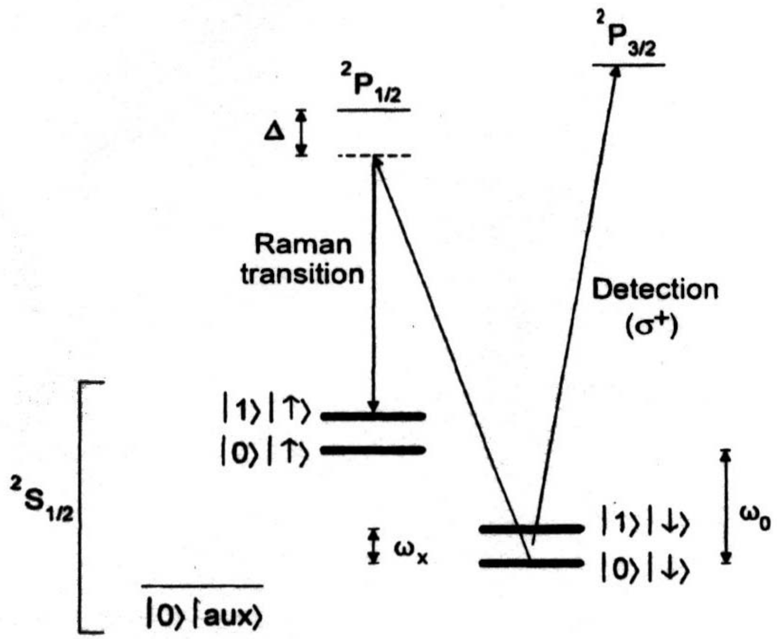
U is sum of two harmonic oscillators with frequencies :

ω for Q-motion (center of mass)

$\sqrt{3}\omega$ for R-motion (relative coordinate)

Demonstration of a Fundamental Quantum Logic Gate

C. Monroe, D. M. Meekhof, B. E. King, W. M. Itano, and D. J. Wineland
National Institute of Standards and Technology, Boulder, Colorado 80303
(Received 14 July 1995)



A single trapped $^9Be^+$ ion

- Control qubit:

  Phonon number states $|0\rangle$, $|1\rangle$ — 11 MHz

- Target qubit:

  $|\uparrow\rangle \equiv |\mathbf{F=1, m_F=1}\rangle$

  $|\downarrow\rangle \equiv |\mathbf{F=2, m_F=2}\rangle$ — 1.25 GHz

CNOT

$$|0,\downarrow\rangle \rightarrow |0,\downarrow\rangle \qquad |1,\downarrow\rangle \rightarrow |1,\uparrow\rangle$$

$$|0,\uparrow\rangle \rightarrow |0,\uparrow\rangle \qquad |1,\uparrow\rangle \rightarrow |1,\downarrow\rangle$$

Experimental Demonstration of a Controlled-NOT Wave-Packet Gate

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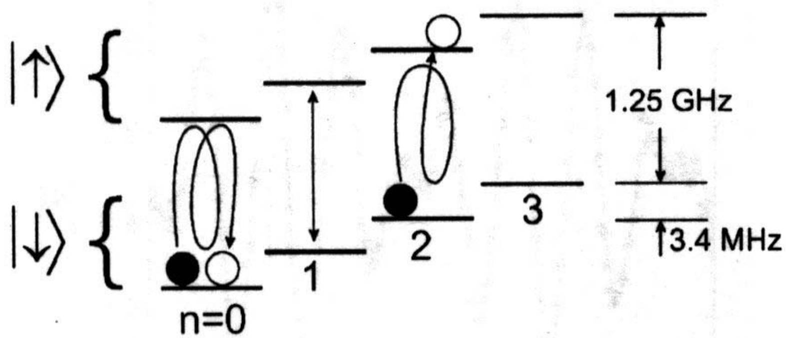


CNOT

$$|0,\downarrow\rangle \rightarrow |0,\downarrow\rangle \qquad |2,\downarrow\rangle \rightarrow |2,\uparrow\rangle$$
$$|0,\uparrow\rangle \rightarrow |0,\uparrow\rangle \qquad |2,\uparrow\rangle \rightarrow |2,\downarrow\rangle$$

By adjusting the trap strength $\Longrightarrow$ $\Omega_{0,0}/\Omega_{2,2} = 4/3$

$\Omega_{i,j} \rightarrow$ Two-photon Rabi rate for the coupling $|i,\downarrow\rangle \leftrightarrow |j,\uparrow\rangle$

Measured CNOT logic truth table

	$\downarrow$	$\uparrow$
$n=0$	0.989 ± 0.006	0.050 ± 0.007
$n=2$	0.019 ± 0.007	0.968 ± 0.007

$\Rightarrow$ **The measured probability that The ion is in $|\downarrow\rangle$ state is shown for different input eigenstates**

Collective atomic systems – N atoms

- Dicke states **(Superradiance etc.)**

$$\left|\frac{N}{2},0\right\rangle \begin{cases} \textbf{2 Atoms:} & \frac{1}{\sqrt{2}}\left(|e,g\rangle + |g,e\rangle\right) \\ \textbf{4 Atoms:} & \frac{1}{\sqrt{6}}\left(|e,e,g,g\rangle + |e,g,e,g\rangle + |e,g,g,e\rangle + |g,e,g,e\rangle + |g,e,e,g\rangle + |g,g,e,e\rangle\right) \end{cases}$$

"Entangled states"

❑ How to produce Dicke states?

Quantum entanglement in many body systems

Interaction of $\left\{ \begin{array}{c} \mathbf{N} \text{ identical two-level atoms } (\omega_0) \\ + \\ \text{Broad band squeezed radiation} \end{array} \right.$

$$\left|\{0\}\right\rangle_{sq} = \exp\left[\frac{1}{2}\int\left[a^\dagger(\omega_p+\epsilon)a^\dagger(\omega_p-\epsilon)\xi(\epsilon) - a(\omega_p+\epsilon)a(\omega_p-\epsilon)\xi^*(\epsilon)\right]\right]\left|\{0\}\right\rangle$$

$$H = \hbar\omega_0 S^z + \hbar\int d\omega a^\dagger(\omega)a(\omega) + \hbar\int d\omega[g(\omega)S^+ a(\omega) + h.c.]$$

$$\begin{aligned}\left\langle a(\omega_1)a(\omega_2)\right\rangle &= \cosh\left[\left|\xi(\omega_1-\omega_p)\right|\right]\sinh\left[\left|\xi(\omega_1-\omega_p)\right|\right] \\ &\times \xi(\omega_1-\omega_p)/\left|\xi(\omega_1-\omega_p)\right| \\ &\times \delta(\omega_1+\omega_2-2\omega_p)\end{aligned}$$

Correlation between $\omega_1 \;\&\; \omega_2 = 2\omega_0 - \omega_1$

- Steady state of atomic system : **N** even ; **pure state**

$$\left([S^{-}\cosh(|\xi|)+S^{+}\sinh(|\xi|)]/\sqrt{2\sinh(2|\xi|)}\right)|\psi\rangle=0$$

$\xi \rightarrow$ Squeezing parameter

$$|\psi\rangle=\mathbf{A_0}\,\mathbf{exp}(\theta \mathbf{S_z})\,\mathbf{exp}\left(-\mathbf{i}\frac{\pi}{2}\mathbf{S^y}\right)\left|\frac{\mathbf{N}}{\mathbf{2}},\mathbf{0}\right\rangle$$

$$\exp(2\theta)=\tanh\left(2|\xi|\right)$$

"N-Atom entangled state"

Production of Dicke State in Rotated Basis

*G. S. Agarwal and R. R. Puri, PRA **41**, 3782 (1990)*

Quantum Entanglement by Collective decay

- Two atoms in a cavity :

$$H_1 = \left[g \left(S_1^- \cos\theta + S_2^- \sin\theta \right) a + h.c. \right]$$

↓

Mode-function dependent

- Resonant cavity : $g < \kappa$

Atomic master equation:

$$\dot{\rho} \equiv -\frac{g^2}{\kappa}\left(R^+R^-\rho - 2R^-\rho R^+ + \rho R^+R^- \right), \qquad R^- = S_1^- \cos\theta + S_2^- \sin\theta$$

Steady state : $R^- \left| \psi \right\rangle = 0$

$$\left| \psi \right\rangle \equiv \left| g_1, g_2 \right\rangle$$

$$\left| \psi_E \right\rangle \equiv \sqrt{2} (\cos\theta \left| g_1, e_2 \right\rangle - \sin\theta \left| e_1, g_2 \right\rangle)$$

Entangled state

Initial state : Selective excitation

$$|g_1, e_2\rangle$$

$$|g_1, e_2\rangle \equiv \alpha[\cos\theta|g_1, e_2\rangle - \sin\theta|e_1, g_2\rangle] + \beta[\sin\theta|g_1, e_2\rangle + \cos\theta|e_1, g_2\rangle]$$

where, $\alpha = \cos\theta,\quad \beta = \sin\theta$

$$\rho = |\alpha|^2 |\psi_E\rangle\langle\psi_E| + |\beta|^2 |g_1, g_2\rangle\langle g_1, g_2|$$

The entangled state $|\psi_E\rangle$ is prepared with a probability $|\alpha|^2$

Maximal incoherent mixing : $\theta = \frac{\pi}{4}, \frac{3\pi}{4}, \ldots.$

Generation of Werner States for Atomic Qubits

Model : Coherently Driven Systems + Collective Decay

$$\rho \equiv p|\psi\rangle\langle\psi| + \frac{(1-p)}{3} \quad ; \quad |\psi\rangle \longrightarrow \text{Singlet State}$$

Initial State : Selective addressing $|e_1, g_2\rangle$

$$\rho = p|\psi\rangle\langle\psi| + (1-p)\rho_s \quad ; \quad \mathrm{Tr}[\rho_s] = 1$$

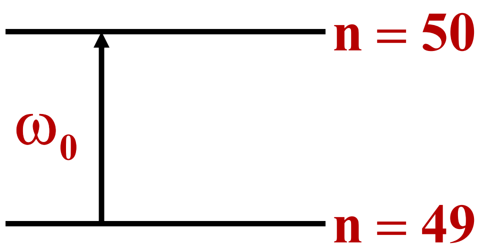
$$\rho_s \longrightarrow \text{Triplet State}$$

Master Equation for Collective Resonance Fluorescence :

$$\rho_s \; : \; \left(\varepsilon + S^-\right)^{-1}\left(\varepsilon^* + S^+\right)^{-1}$$

$$\longrightarrow \mathbf{1/3} \quad \text{in high field limit}$$

Quantum Entanglement using Dispersive interaction in a cavity



Interaction of N identical two-level atoms with a single-mode microwave cavity (ω_c)

$$H = \hbar\omega_0 S_z + \hbar\omega_c a^\dagger a + \hbar g\left(S_+ a + S_- a^\dagger\right)$$

Rotating frame frequency ω_c **:**

$$H = \hbar\Delta S_z + \hbar g\left(S_+ a + S_- a^\dagger\right) \qquad \Delta = \omega_0 - \omega_c$$

Collective basis states : $\left|n, S, m\right\rangle$

Where : $a^\dagger a\left|n\right\rangle = n\left|n\right\rangle, \qquad S_z\left|S, m\right\rangle = m\left|S, m\right\rangle$

- Consider dispersive interaction : $\Delta \gg g$

"These states experience Stark shifts"

Amount of Stark shift of the level $|i\rangle \equiv |n,S,m\rangle$:

$$\sum_j \frac{|\langle \psi_i | H | \psi_j \rangle|^2}{(E_i - E_j)}$$

$|j\rangle = |n-1,S,m+1\rangle,$
$|n+1,S,m-1\rangle$

$$\equiv \left\{ \frac{2nm}{\Delta} + \frac{S^2 - m^2 + S + m}{\Delta} \right\} \hbar g^2$$

Hence,

$$H_{eff} = \sum_{n,m} (\text{shift})_{nm} |n,S,m\rangle\langle n,S,m|$$
$$= \frac{\hbar g^2}{\Delta} \left[\frac{N}{2}\left(\frac{N}{2}+1\right) - S^{Z^2} + (2\bar{n}+1) S^Z \right]$$

Quadratic in S_z (analog of single mode field propagating through a **Kerr medium**)

$\bar{n} \rightarrow$ Mean number of photons

Consider :

$$\left|\psi(0)\right\rangle = \left|\theta, \phi\right\rangle \quad \text{: atomic coherent state}$$

$$\equiv \sum_{k=0}^{N} \sqrt{\frac{N!}{(N-k)!k!}} \exp(ik\phi) \sin^{N-k}\left(\frac{\theta}{2}\right) \cos^{k}\left(\frac{\theta}{2}\right) \left|\frac{N}{2} - k\right\rangle$$

$$\left|\psi(t)\right\rangle = e^{-iH_{eff}t/\hbar} \left|\psi(0)\right\rangle$$

$$= \sum_{k=0}^{N} \sqrt{\frac{N!}{(N-k)!k!}} \times e^{ik\phi} \sin^{N-k}\left(\frac{\theta}{2}\right) \cos^{k}\left(\frac{\theta}{2}\right)$$

$$\times \exp[-i\eta\{N + (N-1)k - k^2\}t] \left|\frac{N}{2} - k\right\rangle$$

$$\eta = \frac{g^2}{\Delta}$$

At special times : $t = \dfrac{\pi}{m\eta}$

$$|\psi(t)\rangle = \exp\left[-\frac{i\pi N}{m}\right]\sum_{q=0}^{m-1} f_q^{(o)} \times \left|\theta, \phi + \pi\frac{2q-N}{m}\right\rangle,$$ **'m' is even**

$$|\psi(t)\rangle = \exp\left[-\frac{i\pi N}{m}\right]\sum_{q=0}^{m-1} f_q^{(e)} \times \left|\theta, \phi + \pi\frac{2q-N+1}{m}\right\rangle,$$ **'m' is odd**

$|\psi(t)\rangle$: a superposition of atomic coherent states

⬇

"ATOMIC CAT STATES"

Agarwal et al., Phys. Rev. A ***56****, 2249 (1997)*

- m=2 :

$$|\psi(t)\rangle = \frac{e^{-iN\pi/2}}{\sqrt{2}}\left[e^{i\pi/4}\left|\theta, \phi - \pi\frac{N-1}{2}\right\rangle + e^{-i\pi/4}\left|\theta, \phi - \pi\frac{N-3}{2}\right\rangle\right]$$

Superposition of two coherent states with **same θ**, but **different ϕ**

- Relation of multiatom GHZ states and atomic CAT states

$$|\psi(0)\rangle = |\theta, \phi\rangle = e^{iN\phi}\prod_j\left(\cos\frac{\theta}{2}|g_j\rangle + e^{-i\phi}\sin\frac{\theta}{2}|e_j\rangle\right)$$

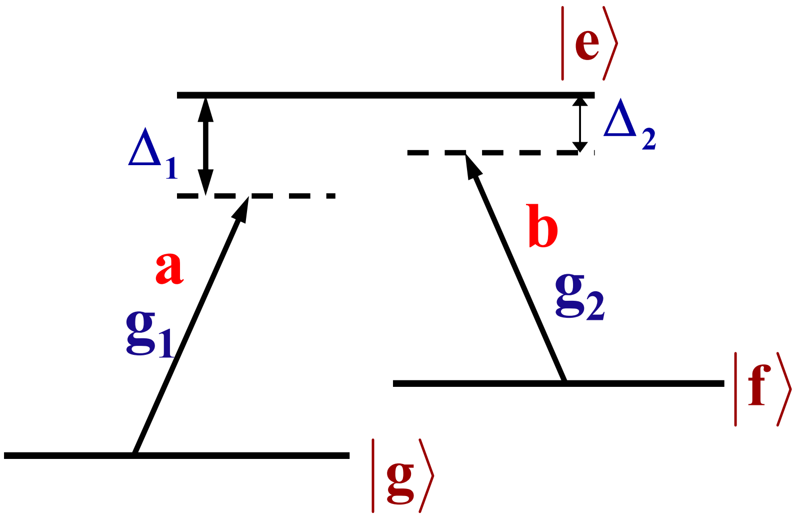
$$|\psi\rangle = \frac{e^{i\pi/4}}{\sqrt{2}}\left\{\prod_j\frac{1}{\sqrt{2}}\left(|g_j\rangle + (-i)^N|e_j\rangle\right) - i\prod_j\frac{1}{\sqrt{2}}\left(|g_j\rangle - (-i)^N|e_j\rangle\right)\right\}$$

$$\theta = \pi/2, \quad \phi = -\pi/2$$

GHZ kind of states in the basis of eigenstates of the operator,

$$\left(e^{i\chi S_+} + e^{-i\chi S_-}\right); \quad \frac{1}{\sqrt{2}}\left(|g\rangle \pm e^{i\chi}|e\rangle\right)$$

Quantum computation by dispersive interaction of a Raman-like system with a bimodal cavity



g_i 's : Atom-cavity coupling constants

Δ_i 's : Detunings

$$\Delta_{1,2} \gg g_{1,2}$$

Effective Hamiltonian

$$H_{eff} = -\frac{\hbar g^2}{\Delta_1}\left[|g\rangle\langle g|a^\dagger a + |f\rangle\langle f|b^\dagger b\right] - \frac{\hbar g^2}{\Delta_1}\left[|g\rangle\langle f|a^\dagger b + |f\rangle\langle g|ab^\dagger\right]$$
$$+\hbar(\Delta_1 - \Delta_2)|f\rangle\langle f|$$

►Hamiltonian includes interaction term
as well as the Stark shift term

Using this Hamiltonian, we perform:

- Quantum Phase gate, CNOT gate, and SWAP gate
- Quantum State Transfer (QST)
- Quantum Network
- Quantum Memory

Quantum logic gates using Stark shifts

$$\Delta_1 \neq \Delta_2 \quad \& \quad \Delta_1 - \Delta_2 \ll g_i$$

Under the condition :

$$\frac{(\Delta_1 - \Delta_2)}{g} = \frac{2}{(\Delta_1/g)}$$

and at time,

$$gT = \frac{\sqrt{2}\pi}{(\Delta_1 - \Delta_2)/g}$$

Quantum Phase gate

$$|0_a\rangle|0_b,0_A\rangle \rightarrow |0_a\rangle|0_b,0_A\rangle$$
$$|0_a\rangle|0_b,1_A\rangle \rightarrow |0_a\rangle|0_b,1_A\rangle$$
$$|0_a\rangle|1_b,0_A\rangle \rightarrow |0_a\rangle|1_b,0_A\rangle$$
$$|0_a\rangle|1_b,1_A\rangle \rightarrow -|0_a\rangle|1_b,1_A\rangle$$

CNOT gate

$$|0_a\rangle|0_b,0_A\rangle \xrightarrow{\hat{C}} |0_a\rangle|0_b,0_A\rangle$$
$$|0_a\rangle|0_b,1_A\rangle \xrightarrow{\hat{C}} |0_a\rangle|0_b,1_A\rangle$$
$$|0_a\rangle|1_b,0_A\rangle \xrightarrow{\hat{C}} |0_a\rangle|1_b,1_A\rangle$$
$$|0_a\rangle|1_b,1_A\rangle \xrightarrow{\hat{C}} |0_a\rangle|1_b,0_A\rangle$$

where

$$|0_A\rangle \equiv |g\rangle$$
$$|1_A\rangle \equiv |f\rangle$$

& $\hat{C} \equiv H_A U_Q H_A$

A. Biswas and G. S. Agarwal, Phys. Rev. A **69**, 062306 (2004)

SWAP gate under two-photon resonance

$$\Delta_1 = \Delta_2 = \Delta$$

$$\tilde{H}_{eff} = -\frac{\hbar g^2}{\Delta}\left[S^+R^- + S^-R^+ - 2S^zR^z\right]$$

$$S^+ = |f\rangle\langle g|, \quad S^- = |g\rangle\langle f|, \quad S^z = \frac{1}{2}\left(|f\rangle\langle f| - |g\rangle\langle g|\right)$$

$$R^+ = a^\dagger b, \quad R^- = ab^\dagger, \quad R^Z = \frac{1}{2}\left(a^\dagger a - b^\dagger b\right)$$

$$|0\rangle_A |0_R\rangle \xrightarrow{U_{SW}} |0\rangle_A |0_R\rangle$$

$$|0\rangle_A |1_R\rangle \xrightarrow{U_{SW}} -|1\rangle_A |0_R\rangle$$

$$|1\rangle_A |0_R\rangle \xrightarrow{U_{SW}} -|0\rangle_A |1_R\rangle$$

$$|1\rangle_A |1_R\rangle \xrightarrow{U_{SW}} |1\rangle_A |1_R\rangle$$

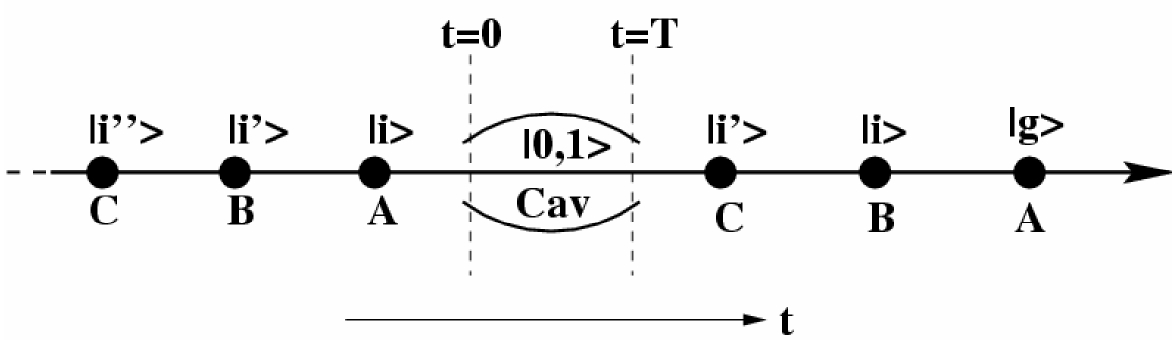
$$\left(|g\rangle \equiv |0\rangle_A, |f\rangle \equiv |1\rangle_A\right)$$

$$\left(|0_a, 1_b\rangle \equiv |0_R\rangle, |1_a, 0_b\rangle \equiv |1_R\rangle\right)$$

$$U_{SW} = \exp(-i\tilde{H}_{eff} t/\hbar) \qquad \frac{2g^2 t}{\Delta} = \pi$$

A. Biswas and G. S. Agarwal, Phys. Rev. A **69**, 062306 (2004)

Quantum State Transfer under two-photon resonance



$$\left(\alpha|\mathbf{g}\rangle+\beta|\mathbf{f}\rangle\right)_{A}|\mathbf{0,1}\rangle$$

⇓ π pulse on atom A

$$|\mathbf{g}\rangle_{A}\left(\alpha|\mathbf{0,1}\rangle-\beta|\mathbf{1,0}\rangle\right)$$

⇓ Atom B enters

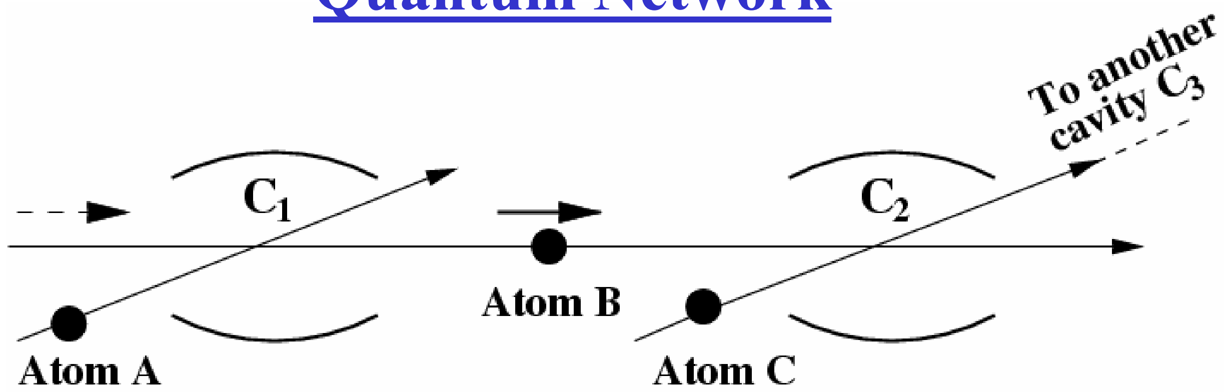
$$|\mathbf{g}\rangle_{A}\left(\alpha'|\mathbf{g}\rangle+\beta'|\mathbf{f}\rangle\right)_{B}\left(\alpha|\mathbf{0,1}\rangle-\beta|\mathbf{1,0}\rangle\right)$$

⇓ π pulse on atom B

$$|\mathbf{g}\rangle_{A}\left(\alpha|\mathbf{g}\rangle+\beta|\mathbf{f}\rangle\right)_{B}\left(\alpha'|\mathbf{0,1}\rangle-\beta'|\mathbf{1,0}\rangle\right)$$

A. Biswas and G. S. Agarwal, Phys. Rev. A **70**, 022323 (2004)

Quantum Network



Step I :

$$\left(\alpha|\mathbf{g}\rangle+\beta|\mathbf{f}\rangle\right)_A|\mathbf{0,1}\rangle_1 \rightarrow |\mathbf{g}\rangle_A\left(\alpha|\mathbf{0,1}\rangle-\beta|\mathbf{1,0}\rangle\right)_1$$

Step II:

$$\begin{cases}\left(\alpha'|\mathbf{g}\rangle+\beta'|\mathbf{f}\rangle\right)_B\left(\alpha|\mathbf{0,1}\rangle-\beta|\mathbf{1,0}\rangle\right)_1 \\ \Downarrow \\ \left(\alpha|\mathbf{g}\rangle+\beta|\mathbf{f}\rangle\right)_B\left(\alpha'|\mathbf{0,1}\rangle-\beta'|\mathbf{1,0}\rangle\right)_1\end{cases}$$

Step III:

$$\left(\alpha|\mathbf{g}\rangle+\beta|\mathbf{f}\rangle\right)_B|\mathbf{0,1}\rangle_2 \rightarrow |\mathbf{g}\rangle_B\left(\alpha|\mathbf{0,1}\rangle-\beta|\mathbf{1,0}\rangle\right)_2$$

A. Biswas and G. S. Agarwal, Phys. Rev. A **70**, 022323 (2004)

Quantum Memory

$$\left(\alpha'|\mathbf{g}\rangle+\beta'|\mathbf{f}\rangle\right)_A\left(\alpha|\mathbf{0,1}\rangle-\beta|\mathbf{1,0}\rangle\right)_c$$

$$\Downarrow$$

$$\left(\alpha|\mathbf{g}\rangle+\beta|\mathbf{f}\rangle\right)_A\left(\alpha'|\mathbf{0,1}\rangle-\beta'|\mathbf{1,0}\rangle\right)_C$$

Storage of information of the cavity into long-lived atomic states

$$\left(\alpha|\mathbf{g}\rangle+\beta|\mathbf{f}\rangle\right)_A|\mathbf{0,1}\rangle_C\rightarrow|\mathbf{g}\rangle_A\left(\alpha|\mathbf{0,1}\rangle-\beta|\mathbf{1,0}\rangle\right)_C$$

or

$$\left(\alpha|\mathbf{g}\rangle+\beta|\mathbf{f}\rangle\right)_A|\mathbf{1,0}\rangle_C\rightarrow-|\mathbf{f}\rangle_A\left(\alpha|\mathbf{0,1}\rangle-\beta|\mathbf{1,0}\rangle\right)_C$$

Retrieval of information using another cavity

A. Biswas and G. S. Agarwal, Phys. Rev. A **70**, 022323 (2004)

Fidelity of state transfer with available parameters

Bimodal **microwave** cavity, $|g\rangle$ and $|f\rangle$ are **Rydberg** states

$$\mathbf{g = 2\pi \times 50 \;\; KHz}, \quad \kappa_a = \kappa_b = \kappa = \mathbf{2\pi \times 100 \;\; Hz},$$

$$\kappa/\mathbf{g} = \mathbf{0.002}, \qquad \Delta = \mathbf{10g}$$

$\mathbf{T = 50 \;\; \mu s}$, For a '$\pi$' pulse

Atom $\longrightarrow$ Cavity

$\mathbf{F(T)}$ remains more than 90% for the above parameters

Atom $\longrightarrow$ Atom

$\mathbf{F(2T+\tau)}$ remains above 80% for $\tau = 63\ \mu s$

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Coherent Operation of a Tunable Quantum Phase Gate in Cavity QED

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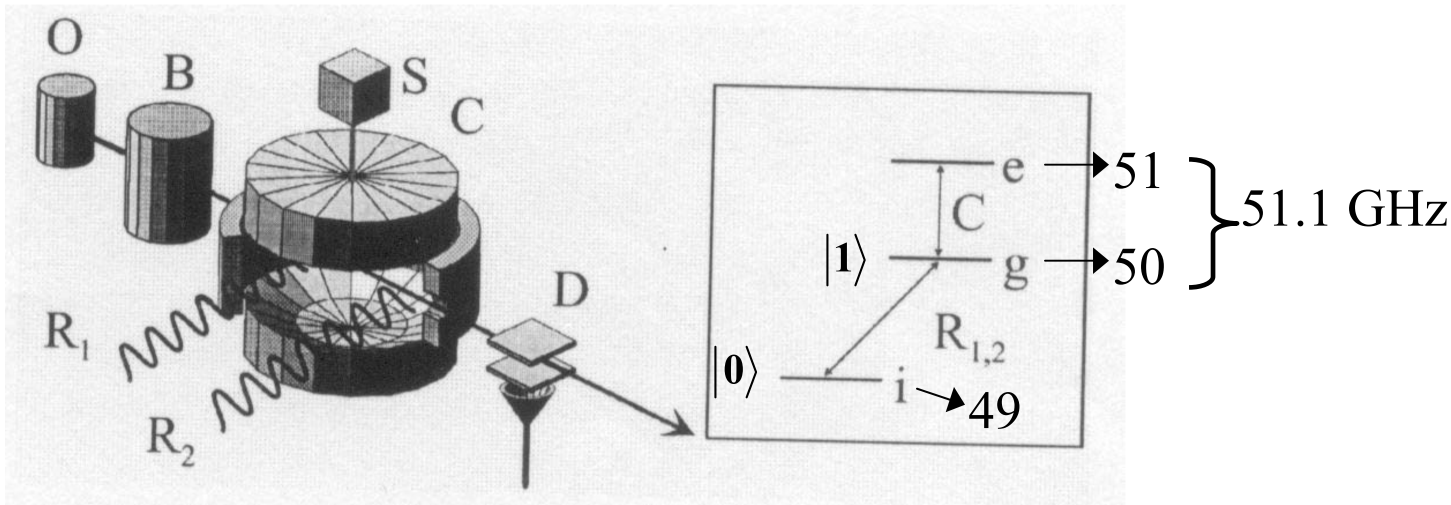
(Received 29 July 1999)



Interaction of Rydberg atom with a bimodal cavity

Atom relaxation time : **30 ms** Cavity relaxation time : **1 ms**

Atom-cavity interaction time t ~ **20 μs**

► The level $|i\rangle$ is decoupled from the cavity

QPG:

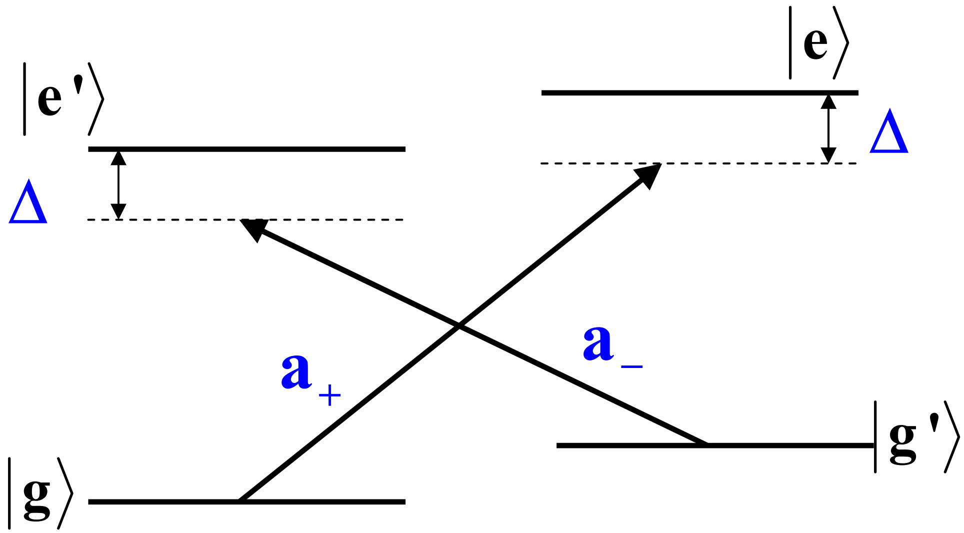
$$|0,i\rangle \rightarrow |0,i\rangle,$$
$$|0,g\rangle \rightarrow |0,g\rangle \qquad \phi = \pi$$
$$|1,i\rangle \rightarrow |1,i\rangle$$
$$|1,g\rangle \rightarrow e^{i\phi}|1,g\rangle$$

Atomic velocity v = 503 m/s, Cavity waist w = 6 mm

$\mathbf{t} = \sqrt{\pi}\mathbf{w}/\mathbf{v}$ such that $\Omega t = 2\pi$ $\quad \Omega \rightarrow$ 47 KHz

Arbitrary value of ϕ can be obtained by changing the cavity detuning

Implementation of Deutsch-Jozsa algorithm using ac Stark shift



Dispersive interaction of atom with photon

$$\Delta \gg g$$

g = atom-photon coupling constant

Ground levels |g> and |g'> get Stark-shifted by an amount

$$\lambda = -g^2/\Delta$$

- Effective Hamiltonian:

$$\mathbf{H}_{\mathrm{eff}} = \hbar\lambda\left[\sum_{i=1}^{N}|\mathbf{g}\rangle_{i\ i}\langle\mathbf{g}|\,|1,0\rangle\langle 1,0| + \sum_{i=1}^{N}|\mathbf{g}'\rangle_{i\ i}\langle\mathbf{g}'|\,|0,1\rangle\langle 0,1|\right]$$

$$= -2\hbar\lambda\left[\hat{\mathbf{S}}^{Z}\hat{\mathbf{R}}^{Z} - \frac{1}{4}\hat{\mathbf{N}}\hat{\mathbf{1}}\right]$$

N : number of atoms in the ensemble

where

$$\hat{\mathbf{S}}^{Z} = \frac{1}{2}\sum_{i=1}^{N}\left(|\mathbf{g}'\rangle_{i\ i}\langle\mathbf{g}'| - |\mathbf{g}\rangle_{i\ i}\langle\mathbf{g}|\right)$$

$$\hat{\mathbf{R}}^{Z} = \frac{1}{2}\left(|1,0\rangle\langle 1,0| - |0,1\rangle\langle 0,1|\right)$$

QUBITS

Photon: $\begin{cases} |0\rangle_F \equiv |0,1\rangle \\ |1\rangle_F \equiv |1,0\rangle \end{cases}$

Atoms : $\begin{cases} |0\rangle_A \equiv \prod_{i=1}^{N} |g\rangle_i \\ |1\rangle_A \equiv \prod_{i=1}^{N} |g'\rangle_i \end{cases}$

Two-qubit operation :

$$|0\rangle_F |0\rangle_A \to |0\rangle_F |0\rangle_A$$
$$|0\rangle_F |1\rangle_A \to -i|0\rangle_F |1\rangle_A$$
$$|1\rangle_F |0\rangle_A \to -i|1\rangle_F |0\rangle_A$$
$$|1\rangle_F |1\rangle_A \to |1\rangle_F |1\rangle_A$$

$$\boxed{\lambda NT = \frac{\pi}{2}}$$

Sequence of above operation and single-qubit operations helps to implement DJA

One bit operations : for atom

Apply a resonant microwave field between the ground levels

$$\mathbf{H}_{\mathbf{micro}} = -\hbar\Omega\left[\mathbf{e}^{\mathbf{i}\phi}\left|\mathbf{g}'\right\rangle_{\mathbf{i}}\ {}_{\mathbf{i}}\left\langle\mathbf{g}\right| + \mathbf{h.c.}\right]$$

One bit operations : for photon

- 50/50 beam splitter

  creates equal superposition

- Phase shifter

  introduces relative phase in
  the superposition

Implementation of U_{f_n}

- ✓ U_{f_1} (identity) ⟹ Trivial

$$H_1^A (H_1^A)^{-1}$$

- ✓ U_{f_2} (NOT) ⟹ Requires microwave interaction

$$\exp[-iH_{micro}t] \quad \text{for} \quad \Omega t = \pi/2, \quad \phi = 0$$

- ✓ U_{f_3} (CNOT) ⟹ Requires atom-photon interaction

$$H_1^A Q_1 H_1^A, \quad Q_1 = (H_1 H_4 H_3)_A (H_1 H_4 H_3)_F U_{eff}$$

- ✓ U_{f_4} (Z-CNOT) ⟹ Requires atom-photon interaction

$$H_1^A Q_2 H_1^A, \quad Q_2 = (H_1 H_2 H_3)_A (H_1 H_4 H_3)_F U_{eff}$$

$$\boxed{U_{DJ} = H_1^F U_{f_n} H_1^F H_1^A}$$

Final state

$$\left\{\begin{matrix} |1\rangle_F \\ |0\rangle_F \end{matrix}\right. \qquad \prod_{i=1}^{N} \frac{1}{\sqrt{2}}\left[|g\rangle_i + |g'\rangle_i\right] \equiv \left\{\begin{matrix} \text{constant} \\ \text{balanced} \end{matrix}\right.$$

Experimental Feasibility

Clock transition in ^{133}Cs atomic cloud

Transition frequency: $2\pi \times 3.517 \times 10^{14} s^{-1}$

Dipole moment: 3.797×10^{-29} coulomb – meter

Cloud length: 5 mm

Cloud cross-section: $0.1 \ mm^2$

$$g = 1.84 \times 10^6 s^{-1}, \quad N = 10^8, \quad T = 1.666 \times 10^{-11} s$$

$$\Delta = 3.59 \ Hz = 1951 g \gg g$$

Implementation of the Deutsch–Jozsa algorithm on an ion-trap quantum computer

Stephan Gulde*, Mark Riebe*, Gavin P. T. Lancaster*, Christoph Becher*, Jürgen Eschner*, Hartmut Häffner*, Ferdinand Schmidt-Kaler*, Isaac L. Chuang*† & Rainer Blatt*

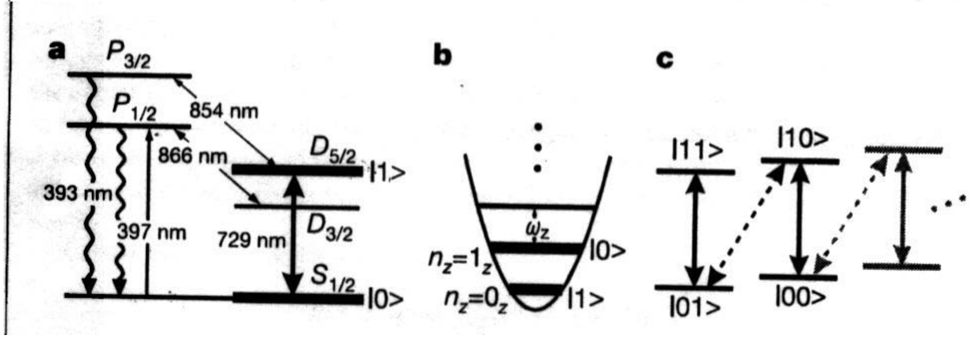


Table 2 Expected and measured results of the complete Deutsch–Jozsa algorithm

	Constant		Balanced	
	Case 1	Case 2	Case 3	Case 4
Expected $\|\langle 1 \| a\rangle\|^2$	0	0	1	1
Measured $\|\langle 1 \| a\rangle\|^2$	0.019(6)	0.087(6)	0.975(4)	0.975(2)
Expected $\|\langle 1 \| w\rangle\|^2$	1	1	1	1
Measured $\|\langle 1 \| w\rangle\|^2$	–	0.90(1)	0.931(9)	0.986(4)

The numbers in brackets are statistical 1σ uncertainties.

$^{43}Ca^{+}$ in a linear Paul trap

Qubit 1:

$$S_{1/2}(m=-1/2)\equiv|0\rangle,$$

$$D_{5/2}(m=-1/2)\equiv|1\rangle$$

Qubit 2: (phonon number of the axial vibration mode of the single ion)

$$n_z=0_z\equiv|1\rangle, n_z=1_z\equiv|0\rangle$$

- Carrier rotation:

$$R(\theta,\phi)=\exp\left[i\theta(e^{i\phi}\sigma^{+}+e^{-i\phi}\sigma^{-})/2\right]$$

- Transition of the blue sideband

$$R^{+}(\theta,\phi)=\exp\left[i\theta(e^{i\phi}\sigma^{+}b^{\dagger}+e^{-i\phi}\sigma^{-}b)/2\right]$$

Entanglement between Macroscopic Systems

B. Deb and G.S. Agarwal, Phys. Rev. A ***67****, 023603 (2003); ibid.* ***65****, 063618 (2002)*

Bose - Einstein Condensate

Condensate of Na atoms: Parameters used in Ketterle's Experiments

Size: Length ~ 200 μm, Diameter ~ 20 μm Temp ~ 100 nK

Atom number ~ 10^7 , Density ~ 10^{14} cm^{-3}

Imaging Technique ⇒ Density

$$H = \sum_{\vec{k}} \hbar\omega \hat{a}^{\dagger}_{\vec{k}} \hat{a}_{\vec{k}} + \frac{4\pi\hbar^2 a_s}{2mV} \sum_{\vec{k}} \hat{a}^{\dagger}_{\vec{k}_3} \hat{a}^{\dagger}_{\vec{k}_4} \hat{a}_{\vec{k}_5} \hat{a}_{\vec{k}_6} \delta_{\vec{k}_3+\vec{k}_4,\vec{k}_5+\vec{k}_6}$$

$$a_s = \underset{k\to 0}{\mathrm{Lt}} \frac{-\delta_0(k)}{k}$$ ⟶ Scattering Length

$\delta_0(k)$ is s-wave scattering phase shift

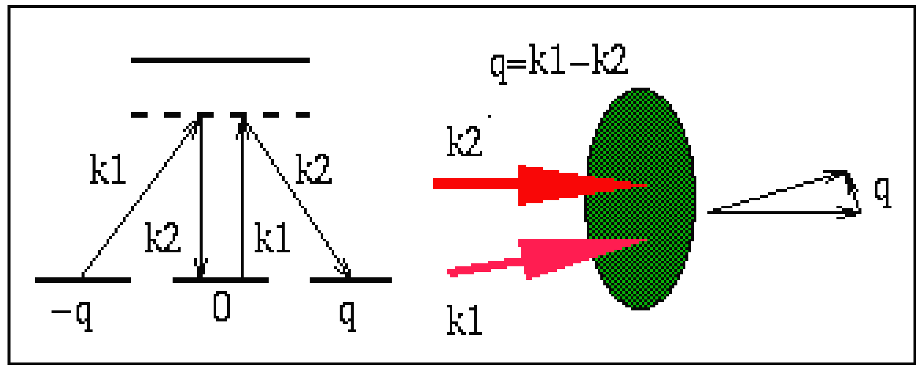
For Na atoms, a_s=2.8 nm

• Probe : Bragg Scattering

$$H_I = \hbar\Omega \hat{c}^\dagger_{\vec{k}_2}\hat{c}_{\vec{k}_1}\sum_{\vec{k}}\left(\hat{a}^\dagger_{\vec{q}+\vec{k}}\hat{a}_{\vec{k}} + \hat{a}_{-\vec{q}+\vec{k}}\hat{a}^\dagger_{\vec{k}}\right) + \text{h.c.} \simeq \left[\hbar\eta\hat{c}^\dagger_{\vec{k}_2}\left(\hat{\alpha}^\dagger_{\vec{q}} + \hat{\alpha}^\dagger_{-\vec{q}}\right) + \text{h.c.}\right]$$

Ω : Two-photon Rabi frequency, $\eta = \sqrt{N} f_q \Omega$; $f_q = u_q - v_q$

The centre-of-mass momentum states $-q$, 0 and q are in same electronic ground state



$$\vec{k}_f = \vec{k}_i + \vec{q} \quad ; \quad \omega_1 - \omega_2 = \frac{\hbar\left(k_f^2 - k_i^2\right)}{2m}$$

For a condensate in ground state, $\hat{k}_i \simeq 0$

$\omega_1 - \omega_2 = \omega_q$
drop $\hat{\alpha}_{-q}$
$\hat{c} \longrightarrow$ classical

$$H_I \sim \left(\beta\alpha^\dagger_{\vec{q}} + \text{h.c.}\right)$$

Displacement

All $\hat{\alpha}_{\vec{k}}$ modes in vacuum state $|0\rangle_{ph}$

$$\hat{\alpha}_{\vec{k}}|0\rangle_{ph} = 0$$

$|0\rangle_{ph} \rightarrow$ two-mode entangled state in terms of the atomic operators $\hat{a}_{\vec{q}}, \hat{a}_{-\vec{q}}$

$$\left(\hat{a}^{\dagger}_{\vec{q}}\hat{a}^{\dagger}_{-\vec{q}} + \mathrm{h.c.}\right)$$

$$\left\langle \hat{a}^{\dagger}_{\vec{q}}\hat{a}^{\dagger}_{-\vec{q}} \right\rangle \neq 0$$

Q: How to generate coherent states of phonon

Displacing field needed

Bogoliubov Theory : Weakly Interacting Gas

Macroscopic occupation of $\vec{k} = 0$ (ground state)

$$\hat{a}_0, \hat{a}_0^\dagger \rightarrow \sqrt{N_0} \quad ; \quad H \simeq \sum \hbar\omega_k^B \left(\hat{\alpha}_{\vec{k}}^\dagger \hat{\alpha}_{\vec{k}} + \hat{\alpha}_{-\vec{k}}^\dagger \hat{\alpha}_{-\vec{k}} \right)$$

Bogoliubov's Transformation: $\hat{a}_{\vec{k}} = u_{\vec{k}} \hat{\alpha}_{\vec{k}} - v_{\vec{k}} \hat{\alpha}_{-\vec{k}}^\dagger$

Healing Length : $\xi = 1 / \left(8\pi n_0 a_s\right)^{1/2} \sim 0.1 \quad \mu m$

Chemical Potential : $\mu = \dfrac{\hbar^2 \xi^{-2}}{2m}$

$$\omega_k^B = \left[\left(\frac{\hbar k^2}{2m} + \frac{\mu}{\hbar} \right)^2 - \left(\frac{\mu}{\hbar} \right)^2 \right]^{1/2}$$

Properties :

Phonon regime : $k < \xi^{-1}, \quad \omega_k^B \propto k, \quad \omega_k^B \sim 10 \quad KHz$

Atom : Single Particle regime : $k \gg \xi^{-1}, \quad \omega_k^B \propto k^2$

- Probe is quantized :

$$\hat{C}_k^\dagger \hat{a}_k^\dagger + h.c.$$

Parametric interaction – simultaneous production of two Bosonic models

$\hat{\alpha}_{-\mathbf{q}}$ starts growing as the interaction strength increases

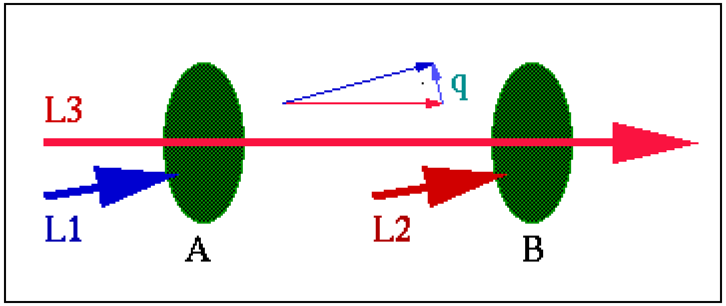
- Two mode entanglement parameter :

$$\xi_{\alpha\beta} = \frac{\left\langle \left[\Delta\left(\hat{\mathbf{n}}_\alpha - \hat{\mathbf{n}}_\beta\right)\right]^2 \right\rangle}{\left\langle \hat{\mathbf{n}}_\alpha + \hat{\mathbf{n}}_\beta \right\rangle} \quad ; \qquad \alpha, \beta = \mathbf{q}, \ -\mathbf{q}, \ \mathbf{k}_2$$

Particle operators

$\xi_{\alpha\beta} < 1$ Signature of entanglement

Entanglement of Two condensates



'A' and 'B' are two condensates, L1 and L2 are pump lasers, L3 is a common entangling probe laser. Both the pumps have same wave vectors k_1, probe's wave vector k_2. The probe is red detuned from the pumps. The laser are in Bragg resonance with a particular momentum mode q of both the condensates.

Coupling through the stimulating field

Dynamics critical

Entanglement parameters

Different observables : number of collective excitations

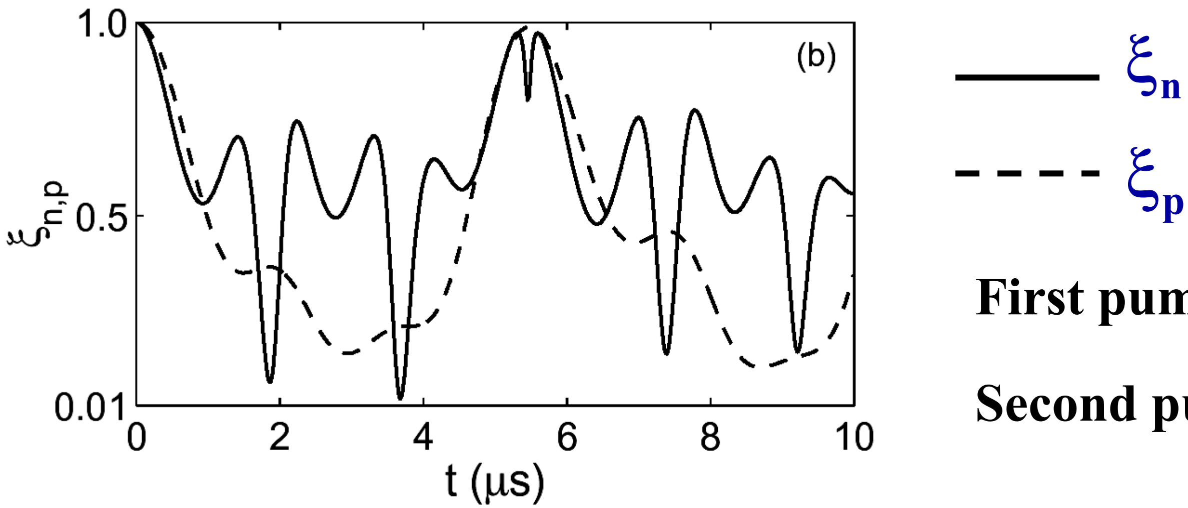
$$\xi_n(1,2) = \frac{\left\langle [\Delta(\hat{n}_1 - \hat{n}_2)]^2 \right\rangle}{\left(\langle \hat{n}_1 \rangle + \langle \hat{n}_2 \rangle \right)} < 1$$

Phase of collective excitations :

$$\xi_n(1,2) = \frac{1}{2}\left[\left\langle [\Delta(X_1 + X_2)]^2 \right\rangle + \left\langle [\Delta(p_1 + p_2)]^2 \right\rangle \right] < 1$$

Coupling $\longrightarrow \eta = \sqrt{N} f_q \Omega$

Control by changing density (temp), light intensity



First pump blue–detuned and

Second pump Red-detuned

$$\delta_1 = \delta_2 = 2.92 \quad \text{MHz}, \qquad q = 8.33\xi^{-1}$$

$$\omega_q^B = 2.96 \quad \text{MHz}, \quad \eta_A = \frac{4}{5}\eta_B = 2.22 \quad \text{MHz}$$

Light scattering events in two condensates correlated