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SCHOOL AND WORKSHOP ON QUANTUM ENTANGLEMENT, DECOHERENCE, INFORMATION, AND GEOMETRICAL PHASES IN COMPLEX SYSTEMS (1 November - 12 November 2004)

Control over dynamics and decoherence of complex systems

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These are preliminary lecture notes, intended only for distribution to participants

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Control over dynamics and decoherence of complex systems

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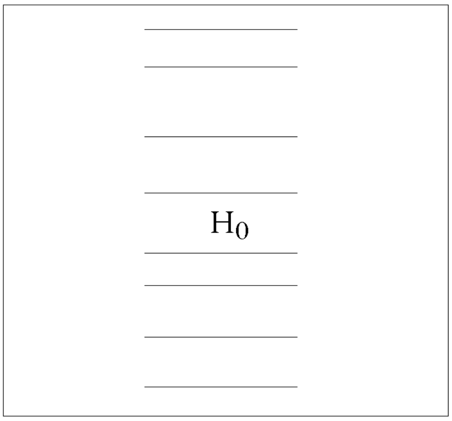
- Non-holonomic control of unitary evolution
- Coherence protection with non-holonomic control

Control of an N-level quantum system



Uncontrolled evolution: $U(t) = e^{-iH_0\,t}$

Complete control: $U(t) = e^{-iH_d\,t}$
for any desired Hermitian Hamiltonian H_d

Control by periodic guidance:

$$U(t = T) = e^{-iH_d\,\delta t}$$

$$U(t = 2T) = e^{-iH_d\,2\delta t}$$

$$\dots$$

$$U(t = nT) = e^{-iH_d\,n\delta t}$$

Control parameters for $N \times N$ Hermitian Hamiltonians

Example: $N = 2$

$$H_0 = \begin{pmatrix} E_1 & 0 \\ 0 & E_2 \end{pmatrix}$$

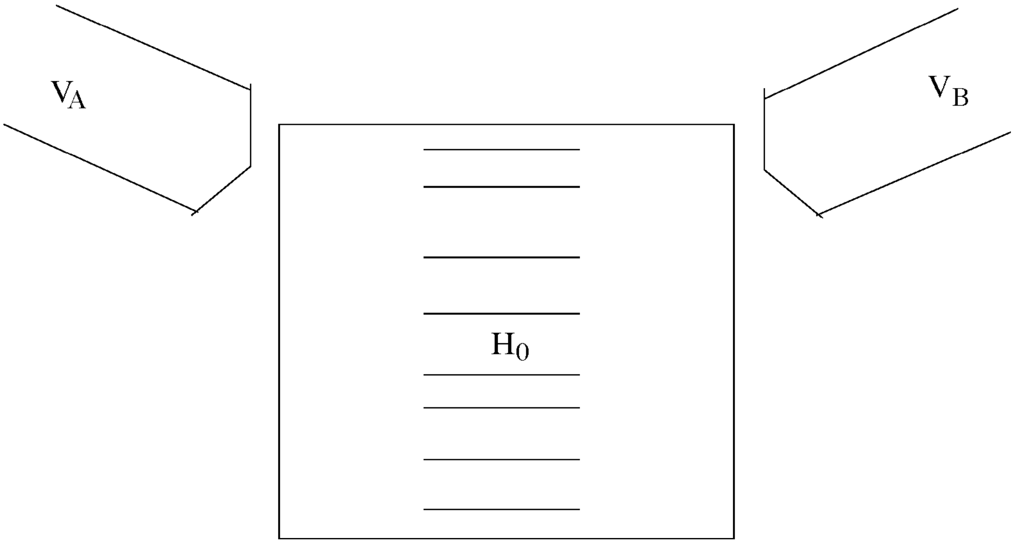
$$H_d = \begin{pmatrix} h_1 & a - ib \\ a + ib & h_2 \end{pmatrix}$$

$$H_d \longleftrightarrow (h_1, h_2, a, b)$$

- The $N \times N$ Hermitian Hamiltonians form a linear space of N^2 real dimensions

- N^2 real *control parameters* are needed for complete control

Introducing N^2 control parameters



$$
\begin{array}{rcll}
A & = & H_0 + V_A & t_1 \\
B & = & H_0 + V_B & t_2 \\
 & \ldots & & \\
A & = & H_0 + V_A & t_{N^2-1} \\
B & = & H_0 + V_B & t_{N^2}
\end{array}
$$

$$T = t_1 + t_2 \ldots + t_{N^2-1} + t_{N^2}$$

$$U(t=T) = e^{-iBt_{N^2}} e^{-iAt_{N^2-1}} \ldots e^{-iBt_2} e^{-iAt_1}$$

S. Lloyd, Phys. Rev. Lett. **75**, 346 (1995).

Complete control of N levels

Physical means of control:

$$U(t_1, ..., t_{N^2}) \equiv e^{-iBt_{N^2}}\, e^{-iAt_{N^2-1}} \ldots e^{-iBt_2}\, e^{-iAt_1}$$

Mathematical problem:
Given Hermitian H_d and small evolution time $\delta t > 0$
find timings t_1, ..., t_{N^2} such that

$$U(t_1, ..., t_{N^2}) = e^{-iH_d\,\delta t}$$

(N^2 nonlinear equations in N^2 variables - hard)

A way to solve the problem:

1. Solve for the special case $H_d = 0$:
Find timings T_1, ..., $T_{N^2} \neq 0$ such that

$$U(T_1, ..., T_{N^2}) = 1$$

2. Compute first order corrections for $t_n \equiv T_n + \delta t_n$:

$$U(t_1, ..., t_{N^2}) \;=\; 1 - i \sum_{n=1}^{N^2} H_n\, \delta t_n \;+ o(\delta t_n)$$

3. The N^2 matrix coefficients H_n are Hermitian, and normally form a complete basis for the $N{\times}N$ Hermitian matrices

4. Given $H_d \neq 0$ and $\delta t > 0$ find timing variations $\delta t_1, ..., \delta t_{N^2}$ such that

$$\sum_{n=1}^{N^2} H_n\,\delta t_n = H_d\,\delta t$$

(N^2 linear equations in N^2 variables - easy)

5. The problem is now solved to first order in δt :
For timings $t_n = T_n + \delta t_n$ we have

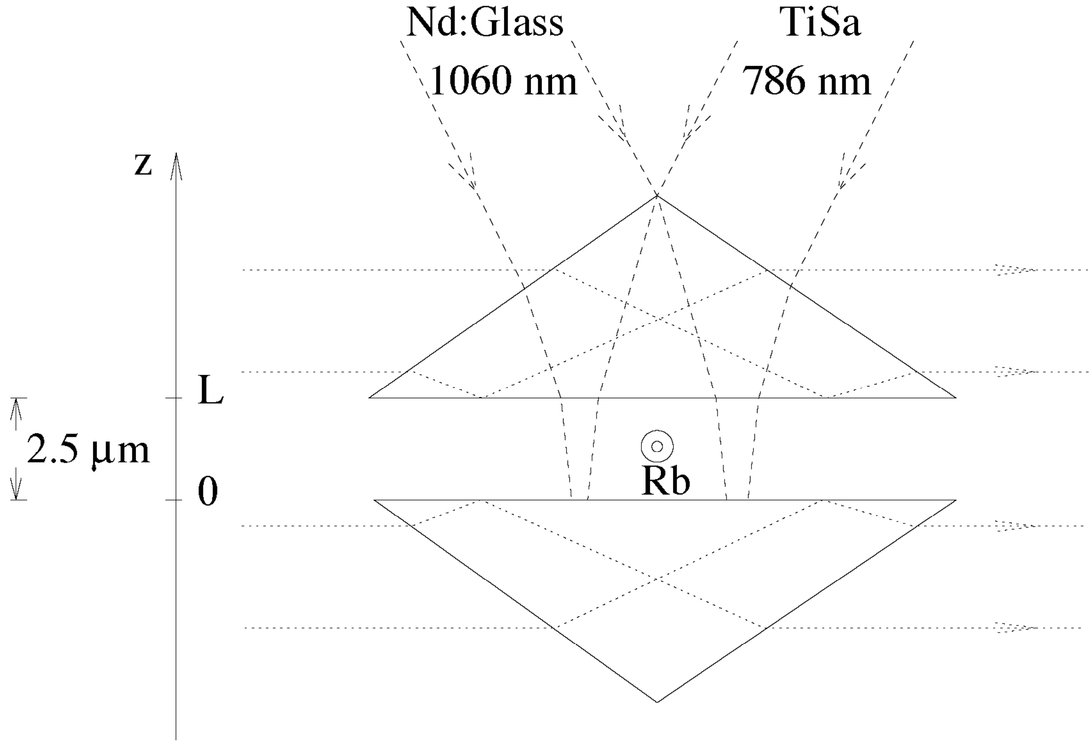
$$\begin{aligned} U(t_1, ..., t_{N^2}) &= 1 - i \sum_{n=1}^{N^2} H_n\,\delta t_n + o(\delta t_n) \\ &= 1 - i\,H_d\,\delta t + o(\delta t) \\ &= e^{-iH_d\,\delta t} + o(\delta t) \end{aligned}$$

6. The δt_n are improved iteratively (Newton method); hence, for $T \equiv \sum_{n=1}^{N^2} T_n + \delta t_n$ we get

$$U(t = T) \equiv U(t_1, ..., t_{N^2}) = e^{-iH_d\,\delta t}$$

G. Harel, V.M. Akulin, Phys. Rev. Lett. **82**, 1 (1999).

Control of a trapped cold atom



unperturbed Hamiltonian $H_0 = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial z^2} + mgz$

boundary conditions $\psi(0) = \psi(L) = 0$

perturbation A $V_A = v_A \sin^2(2\pi z/\lambda_A)$

perturbation B $V_B = v_B \sin^2(2\pi z/\lambda_B)$

Timings for complete control of $N = 10$ translational levels of a trapped cold atom

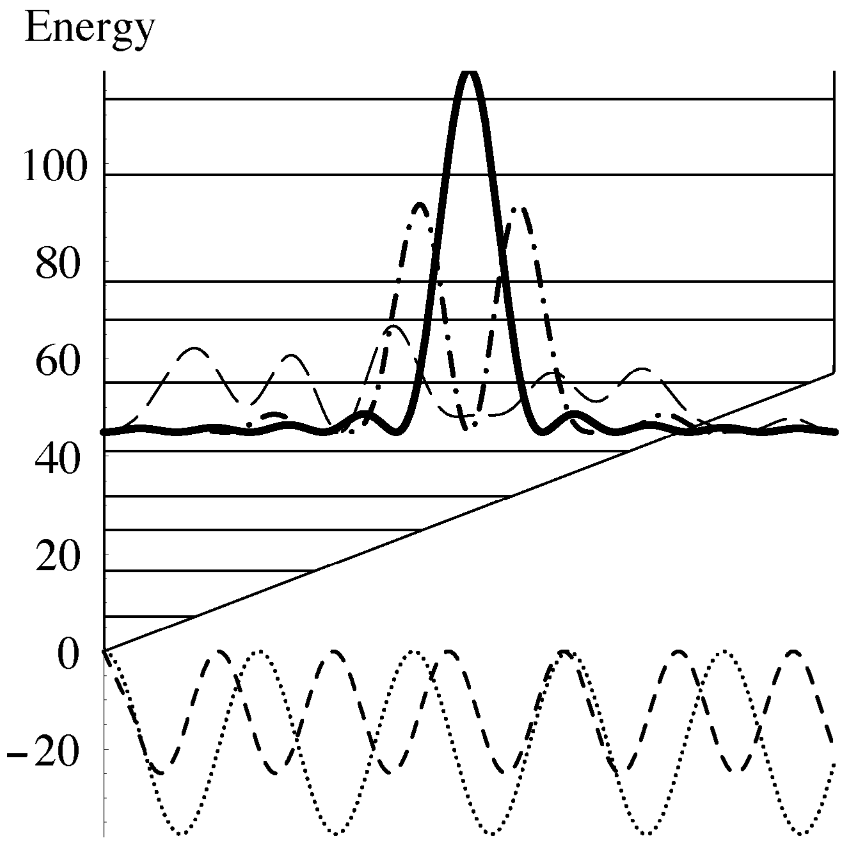
$$H_d = 0$$

n	T_n (μ sec)	n	T_n (μ sec)
1	230.667	6	343.634
2	333.121	7	277.201
3	259.938	8	235.567
4	248.654	9	331.637
5	320.099	10	326.410

$$T = 10\,(T_1 + T_2 + ... + T_{10}) = 29.069280 \text{ m sec}$$

$$U(t = T) = 1 \pm 10^{-4}$$

Complete control of $N = 10$ levels



n	T_n (μ sec)	t_n (μ sec)	δt_n (μ sec)
1	230.667	230.335	-0.332
2	333.121	330.169	-2.952
3	259.938	257.599	-2.339
4	248.654	247.880	-0.774
5	320.099	321.819	1.720
6	343.634	344.247	0.613
7	277.201	275.666	-1.535
8	235.567	239.881	4.314
9	331.637	330.457	-1.180
10	326.410	329.022	2.612

Solving for the special case $H_d = 0$

Physical means of control:

$$U(t_1, ..., t_{N^2}) \equiv e^{-iBt_{N^2}} e^{-iAt_{N^2-1}} ... e^{-iBt_2} e^{-iAt_1}$$

Mathematical problem:

Find timings T_1, ..., $T_{N^2} \neq 0$ such that

$$U(T_1, ..., T_{N^2}) = 1$$

(N^2 nonlinear equations in N^2 variables - hard)

A way to reduce the problem:

1. Define

$$U_r(t_1, ..., t_N) \equiv e^{-iBt_N} e^{-iAt_{N-1}} ... e^{-iBt_2} e^{-iAt_1}$$

2. Find timings T_1, ..., $T_N \neq 0$ such that

$$U_r(T_1, ..., T_N)^N = 1$$

3. Construct the desired timings T_1, ..., T_{N^2} by repeating N times the sequence T_1, ..., T_N, thus obtaining

$$U(T_1, ..., T_{N^2}) = U_r(T_1, ..., T_N)^N = 1$$

Solving the reduced problem

Physical means of control:

$$U_r(t_1, ..., t_N) \equiv e^{-iBt_N}\, e^{-iAt_{N-1}} \dots e^{-iBt_2}\, e^{-iAt_1}$$

Mathematical problem:
Find timings $T_1, ..., T_N \neq 0$ such that

$$U_r(T_1, ..., T_N)^N = 1$$

A way to solve the problem:
Ensure that the eigenvalues λ_q of U_r will be

$$\lambda_q = e^{2\pi iq/N} \qquad q = 1, 2, ..., N,$$

that is, find timings $T_1, ..., T_N \neq 0$ such that

$$\det[\lambda - U_r(T_1, ..., T_N)] = \lambda^N - 1$$

1. Define coefficient functions a_j by

$$\sum_{j=0}^{N} a_j(T_1, ..., T_N)\lambda^j \equiv \det\left[\lambda - U_r(T_1, ..., T_N)\right]$$

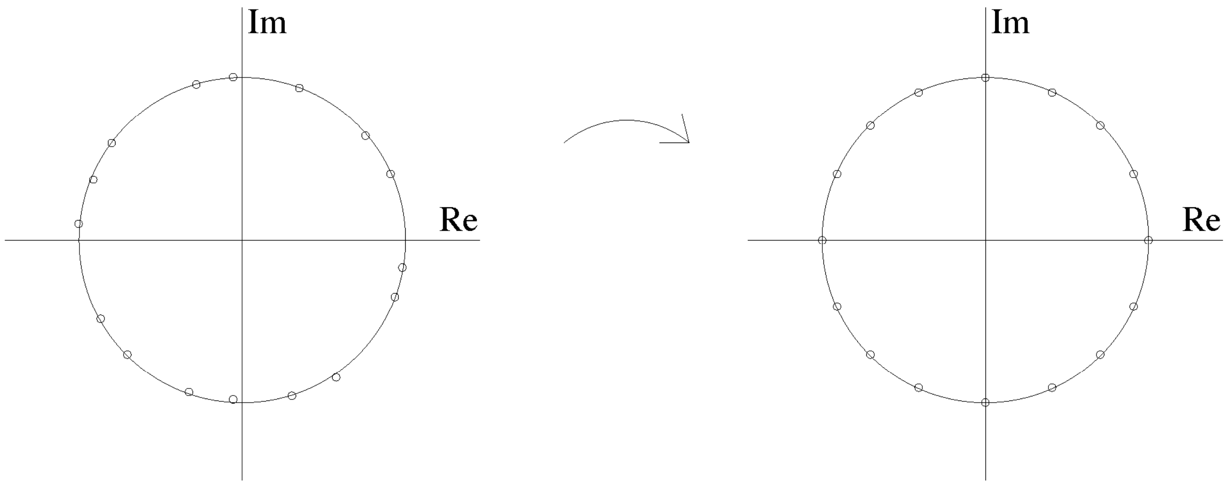
2. Minimize $\sum_{j=0}^{N} |a_j(T_1, ..., T_N)|^2$ with respect to $T_1, ..., T_N$.

Spectral rigidity

The spectral rigidity property of ensembles of random unitary matrices allows for easy convergence to a solution for

$$U_r(T_1, ..., T_N)^N = 1$$

that has a non-degenerate spectrum.



This is because for a generic choice of the perturbations V_A and V_B and an initial guess for the timings T_1, ..., T_N the initial spectrum will already be well spread around the unit circle in the complex plane.

Related formulas

Generators of $su(2)$ (Pauli matrices):

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

General element of $SU(2)$:

$$e^{-i\alpha\,\sigma_z}\, e^{-i\beta\sigma_y}\, e^{-i\gamma\sigma_z} = \begin{pmatrix} e^{-i\alpha}\cos\beta\, e^{-i\gamma} & -e^{-i\alpha}\sin\beta\, e^{i\gamma} \\ e^{i\alpha}\sin\beta\, e^{-i\gamma} & e^{i\alpha}\cos\beta\, e^{i\gamma} \end{pmatrix}$$

$$[\sigma_y, \sigma_z] = 2i\sigma_x$$

$$e^A B e^{-A} = e^{[A,}B] = 1 + [A, B] + \tfrac{1}{2}[A, [A, B]] + \cdots$$

$$e^{A\delta} e^{B\delta} e^{-A\delta} e^{-B\delta} = e^{[A,\, B]\delta^2} + O(\delta^3)$$

$$e^A e^B = e^{A + B + \frac{1}{2}[A, B] + \cdots}$$

(3) $\qquad + \frac{1}{12}[[A, B], A] + \frac{1}{12}[[A, B], B]$

(4) $\qquad - \frac{1}{24}[[[A, B], B], A]$

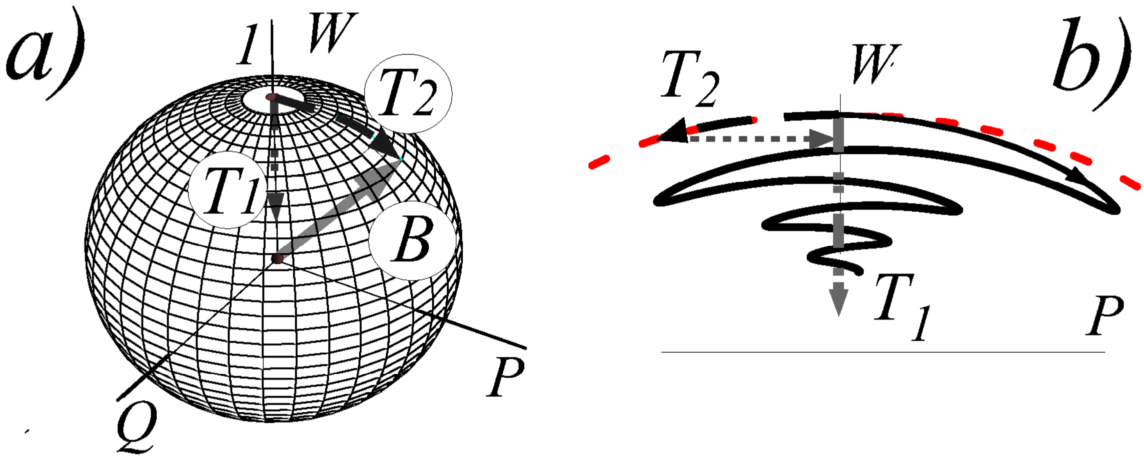
(5) $\qquad - \frac{1}{720}[[[[A, B], B], B], B] - \frac{1}{720}[[[[B, A], A], A], A]$

(5) $\qquad + \frac{1}{180}[[[[A, B], A], A], B] - \frac{1}{180}[[[[B, A], A], B], B]$

(5) $\qquad - \frac{1}{120}[[[A, B], A], [A, B]] - \frac{1}{360}[[[B, A], B], [A, B]]$

Coherence protection by the Zeno effect

Decoherence in a two-level system:



B - Bloch vector representation of the density operator

$$\rho = \frac{1}{2}(I + P\sigma_x + Q\sigma_y + W\sigma_z)$$

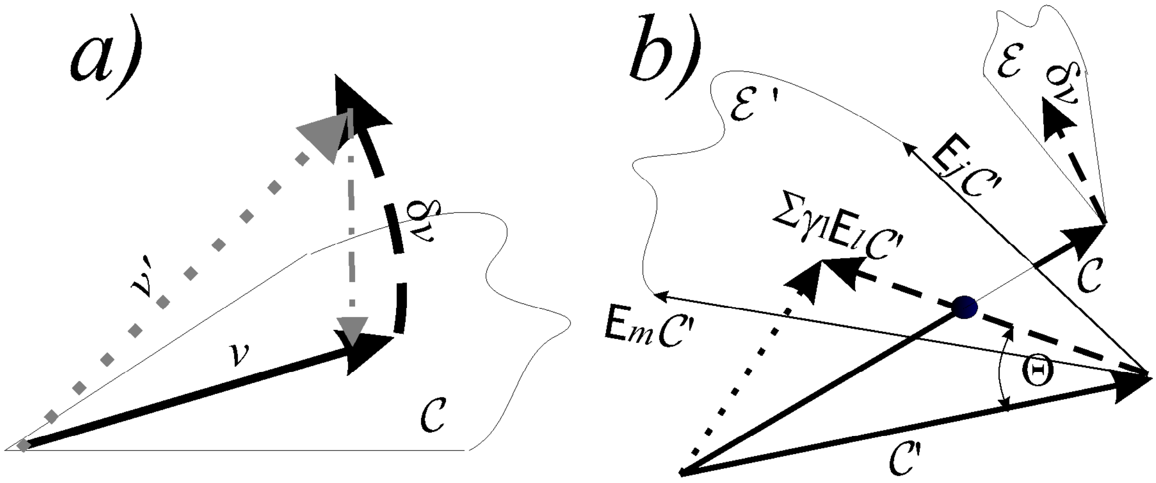
T_2 - coherence loss (uncontrolled Hamiltonian)

T_1 - decay of a pure state into a statistical mixture

Zeno effect in a two-level system:

For an initial state $W = 1$ evolving with a σ_y Hamiltonian for a short time $t \ll T_2$, measurement of σ_z projects the state at t onto the initial state, restoring the initial state with probability 1 up to second order in t/T_2.

Multidimensional Zeno effect:
Restore any state ν in a subspace of possible initial states $\mathcal{C}$ by applying a projective measurement.



To protect quantum data in a K-dim space $\mathcal{K}$ against decoherence due to error Hamiltonians E_m, $m = 1..M$:

1. Add an A-dim ancilla space $\mathcal{A}$, with $A > M$, to obtain a KA-dim space $\mathcal{N} = \mathcal{K} \otimes \mathcal{A}$; prepare the ancilla in a fixed state $|\alpha\rangle \in \mathcal{A}$.

2. Move the data to an "error-orthogonal" subspace $\mathcal{C} \subset \mathcal{N}$ by a unitary transformation C implemented with non-holonomic control (MK^2 control parameters).

3. Transform back with C^{-1} after a short time $t \ll T_2$.

4. Measure the ancilla in the state $|\alpha\rangle$, restoring the initial state of the data space with probability 1 up to second order in t/T_2.

$$i\frac{d\rho_{\mathcal{K}}}{dt} = [h_e, \rho_{\mathcal{K}}]; \quad h_e = \sum_{m=1}^{M} f_m \langle\alpha|C^{-1} E_m C|\alpha\rangle = 0$$

E. Brion et al., Europhys. Lett. **66**, 157 (2004).

Coherence protection by random coding

When the data and ancilla systems are built with a large number of particles, e.g. with k and $n-k$ qubits, it is impractical to satisfy exactly the orthogonality conditions

$$\langle\kappa|\langle\alpha|C^{-1}E_m C|\alpha\rangle|\kappa'\rangle = 0; \qquad m = 1..M$$

for all $|\kappa\rangle, |\kappa'\rangle \in \mathcal{K}$, because of the exponentially-high dimensions: $K = 2^k$.

In this case however, typical error Hamiltonians E_m are sparse matrices because they arise from single-particle or binary interactions.

Therefore, a generic unitary transformation C will spread the matrix elements of E_m over all the 2^n states of the system, and the subsequent measurement of the ancilla will project the matrix to 2^k dimensions, reducing its effect by the exponential factor 2^{n-k}.

In contrast, the number and strength of relevant error Hamiltonians E_m are only polynomial in n, say

$$M \sim \binom{n}{s} \sim n^s \qquad ||E_m|| \sim n^2$$

assuming that at most s qubits are affected. Hence, decoherence will be strongly suppressed: $h_e \sim 2^{-(n-k)}$.

Non-holonomic control can afford such a generic C (random coding) when Hamiltonians H_A and H_B that allow complete control are switched $\sim n$ times, with control timings that ensure big acquired actions. If $-H_A$ and $-H_B$ are also feasible, C^{-1} can be performed at the same level of complexity.

E. Brion et al., quant-ph/0211003.