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*SCHOOL AND WORKSHOP ON*
*QUANTUM ENTANGLEMENT, DECOHERENCE, INFORMATION, AND GEOMETRICAL PHASES IN COMPLEX SYSTEMS*
*(1 November - 12 November 2004)*

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# Stochastic Schrödinger Equations and Measurement in Quantum Optics

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# STOCHASTIC SCHRÖDINGER EQUATIONS

# AND

# MEASUREMENT IN QUANTUM OPTICS

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University of Auckland

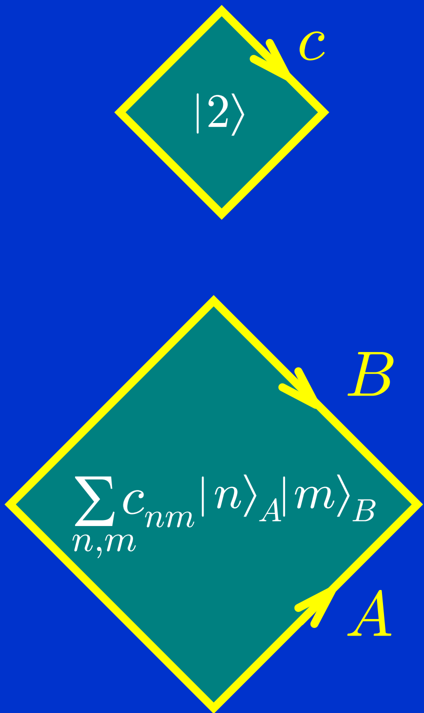
Support by the Marsden Fund of RSNZ

- Motivation
- Central idea
- Formal implementation
- Realization as a stochastic process
- What’s new?
- Development
- Return to motivation

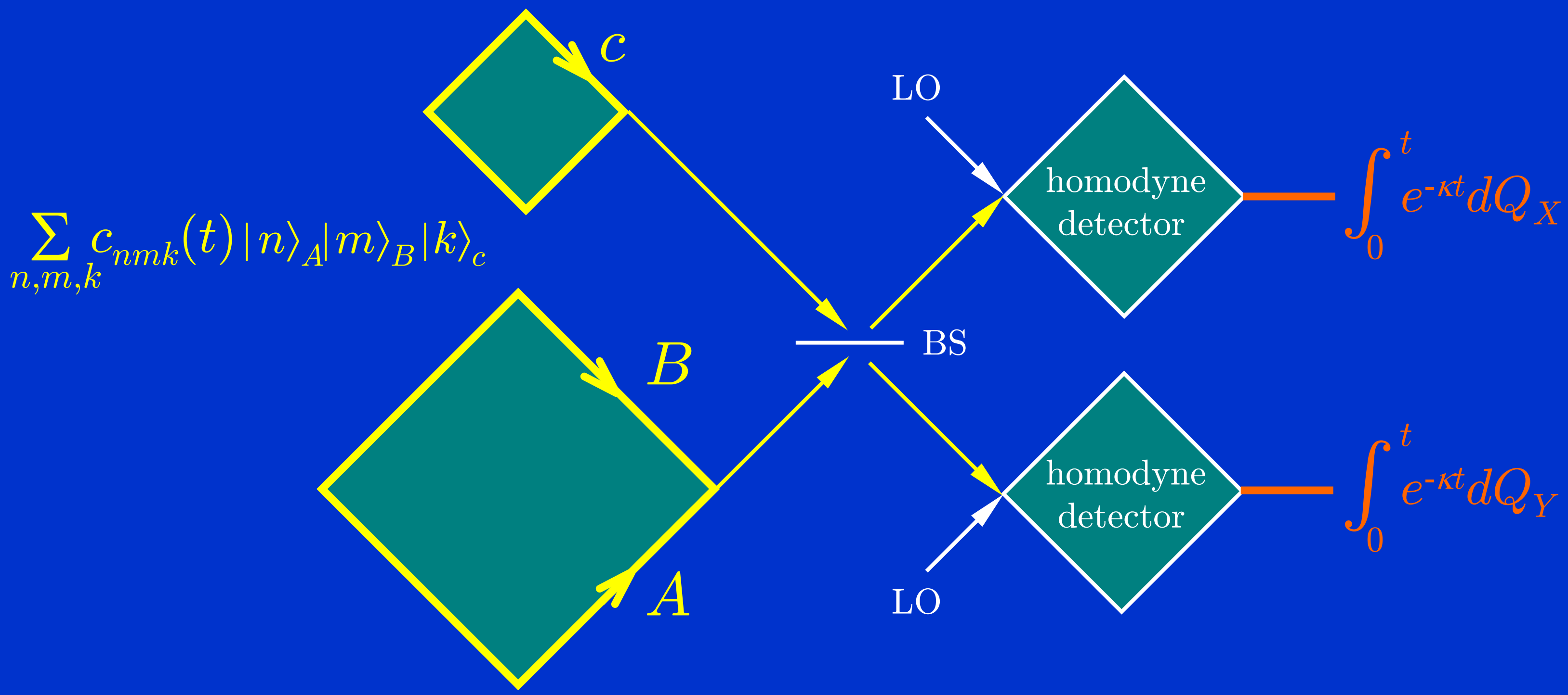
# Continuous variable quantum teleportation

elementary proposal

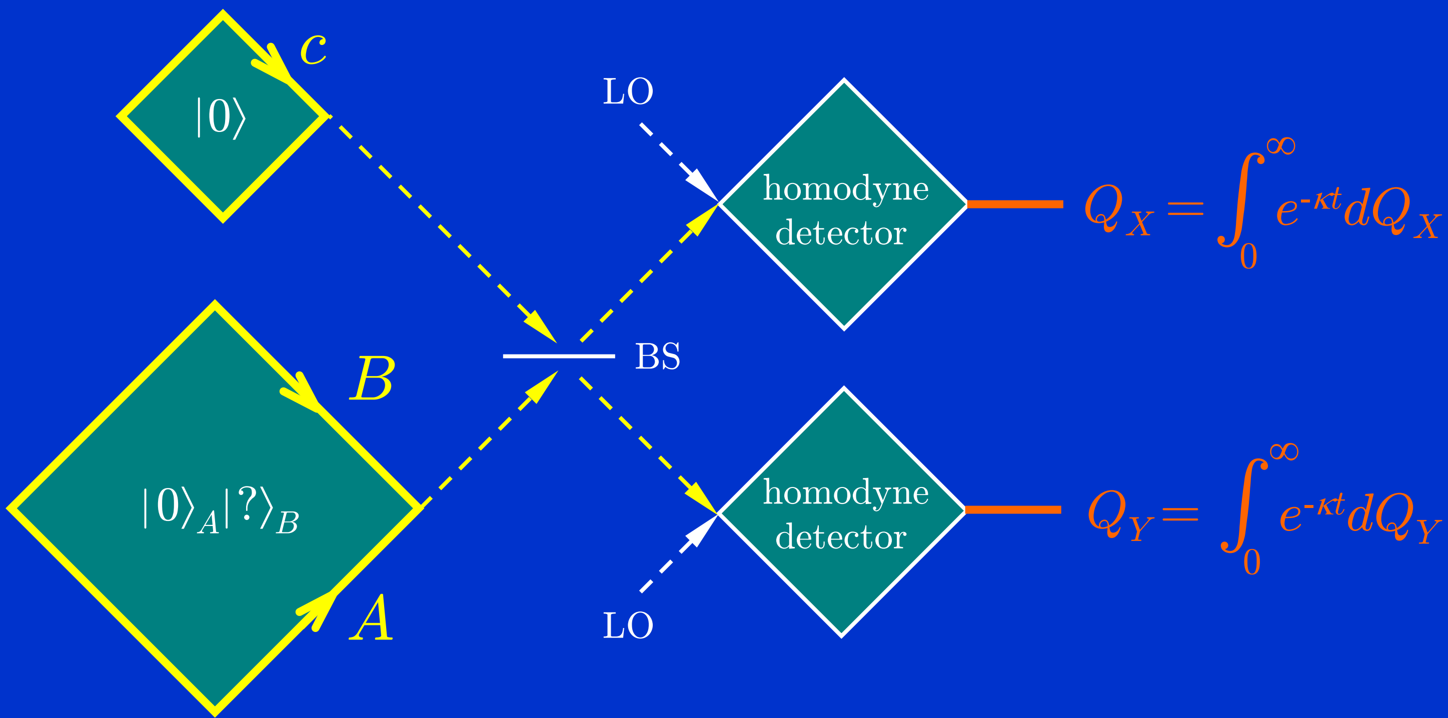
Step 1: prepare 2-mode squeezed state and input state $|2\rangle$

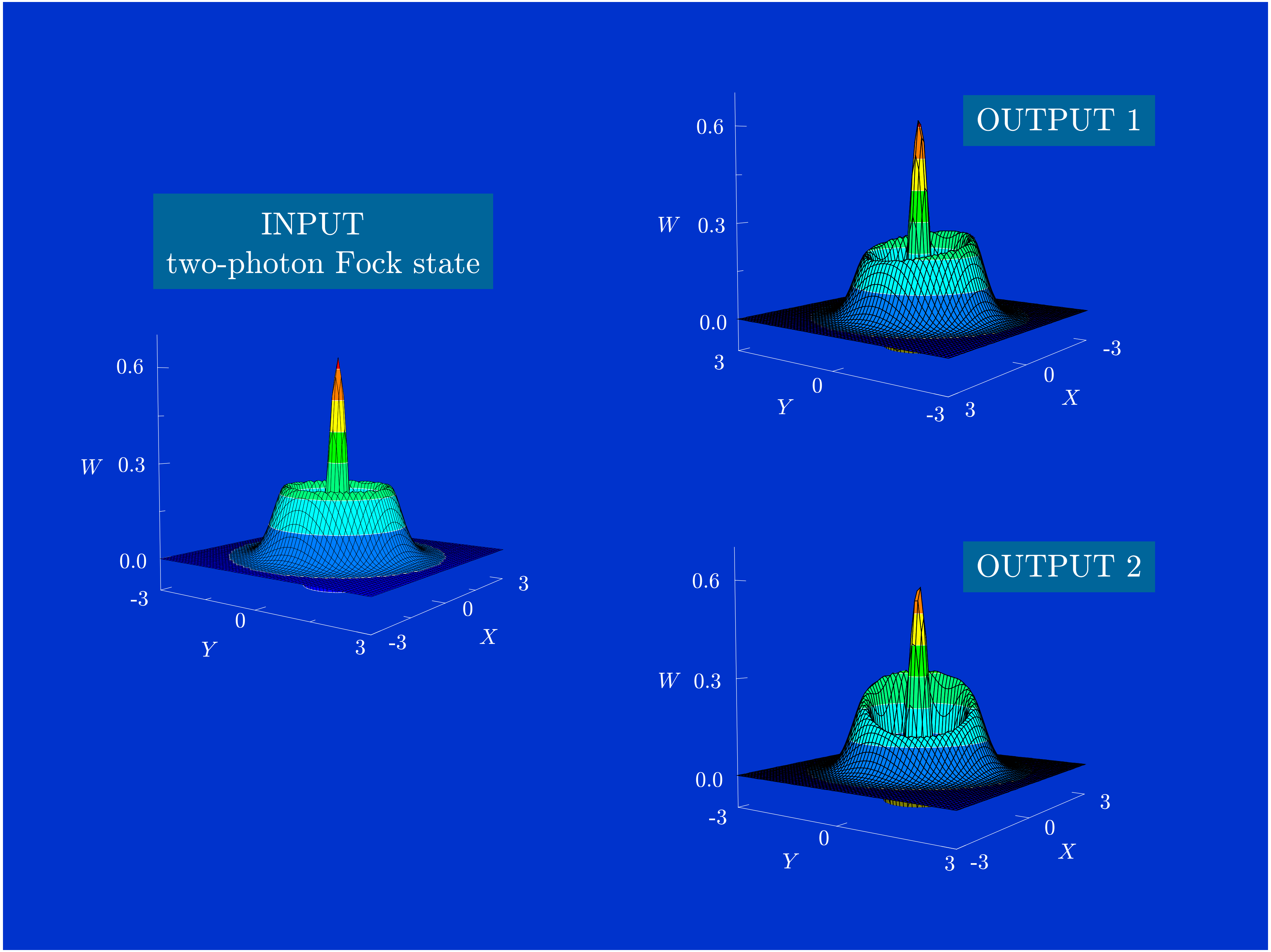


## Step 2: send mode $A$ and input mode to Alice who measures $X$ and $Y$ quadrature amplitudes

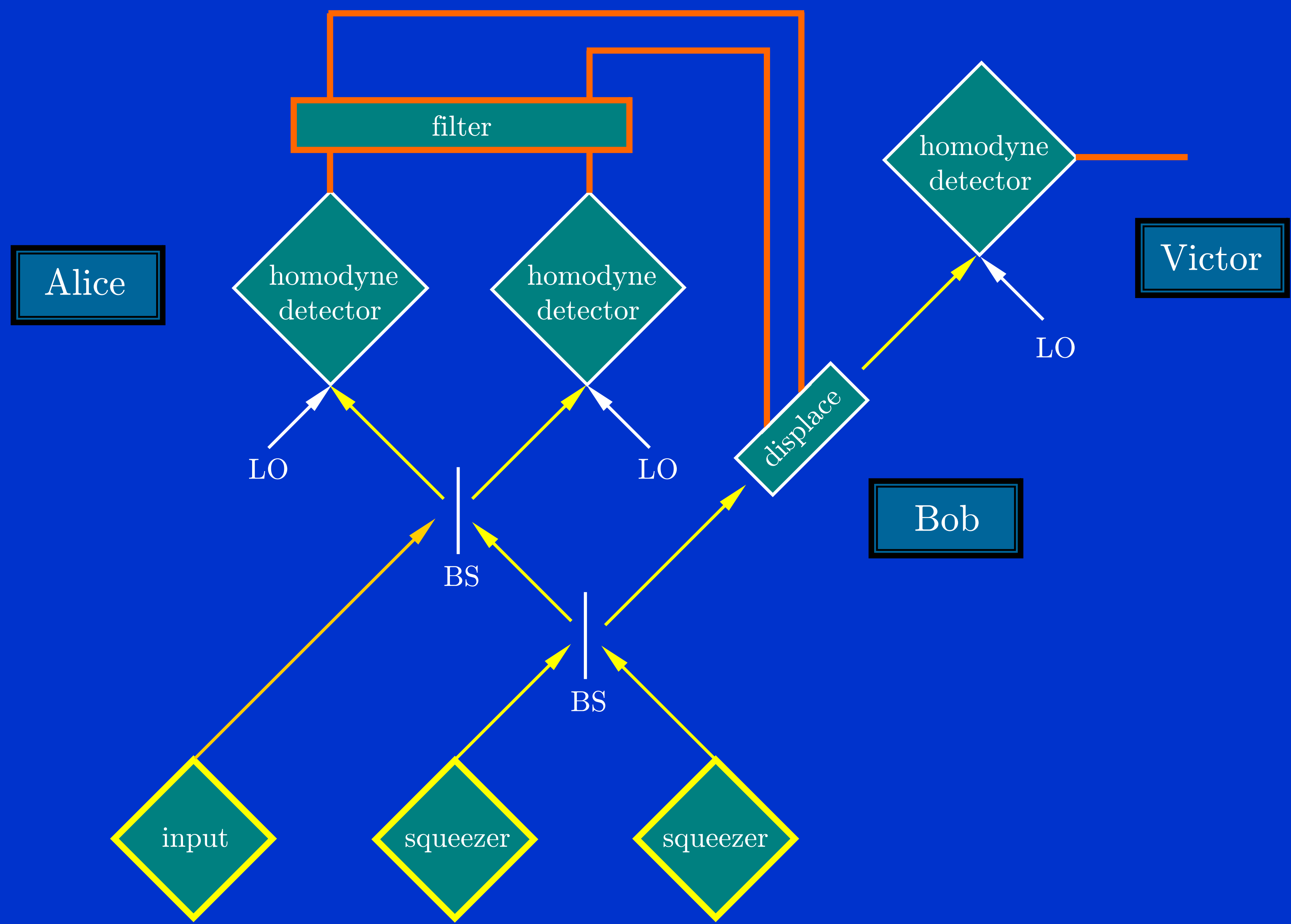


Step 3: send Alice's measurement results to Bob who displaces mode B accordingly



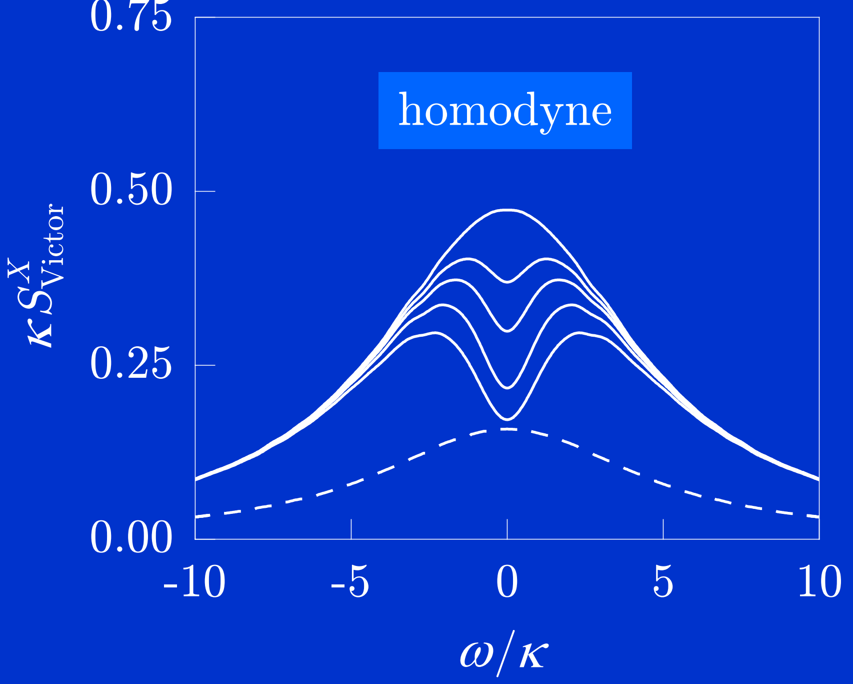

INPUT
two-photon Fock state
OUTPUT 1
OUTPUT 2
W
0.6
0.3
0.0
X
Y
-3
0
3

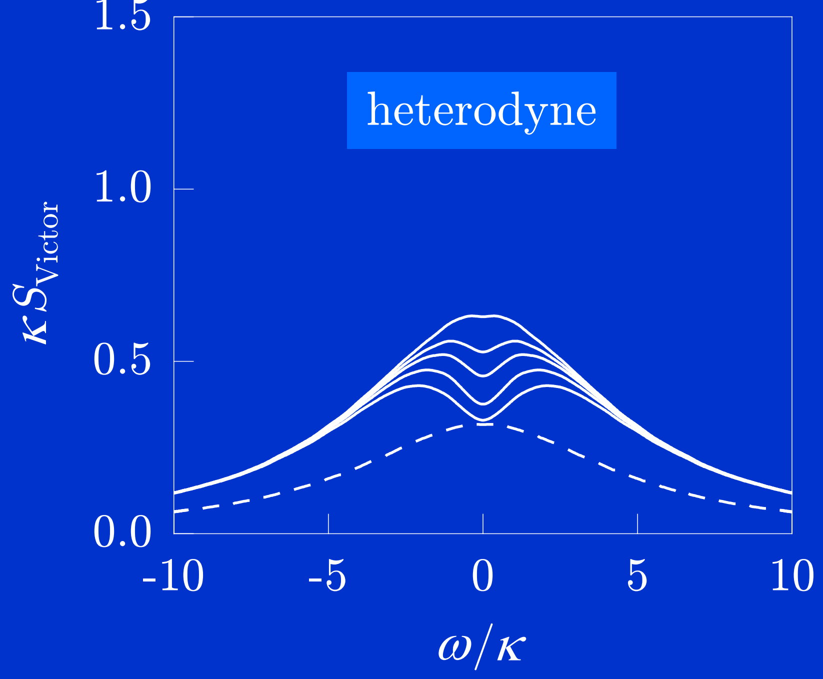
# experiments



# vacuum state input

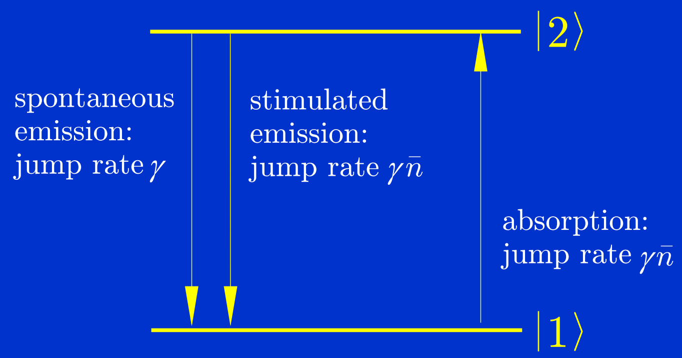
| bandwidths: | squeezing | 2 |
|---|---|---|
| | Alice's detection | 50 |
| | Victor's detection | 10 |



# Conditioning on measurement records

## Bohr-Einstein quantum jumps




density operator (unconditional state): $\rho(t) = p_2(t)|2\rangle\langle 2| + p_1(t)|1\rangle\langle 1|$

unconditional evolution:

$$\dot{p}_2 = -\gamma_d\, p_2 + \gamma_u\, p_1$$

$$\dot{p}_1 = -\gamma_u\, p_1 + \gamma_d\, p_2$$

scattering records: $\mathrm{REC} \equiv \left\{ \genfrac{}{}{0pt}{}{\cdots}{\cdots}, \not\emptyset, \genfrac{}{}{0pt}{}{\gamma_u}{T_{k\text{-}1}}, \not\emptyset, \genfrac{}{}{0pt}{}{\gamma_d}{T_k}, \not\emptyset, \genfrac{}{}{0pt}{}{\gamma_u}{T_{k+1}}, \not\emptyset, \genfrac{}{}{0pt}{}{\gamma_d}{T_{k+2}}, \not\emptyset, \genfrac{}{}{0pt}{}{\cdots}{\cdots} \right\}$

unraveling the density operator (conditional states):

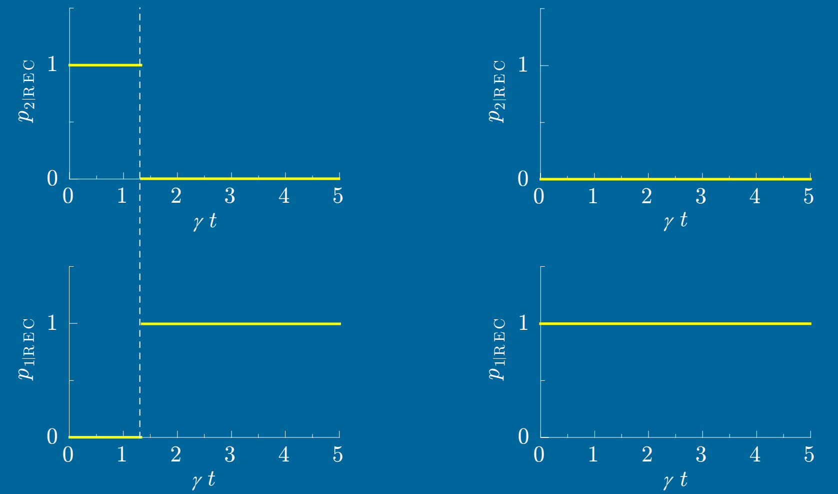
$$p_2(t) = P\left(\{A\}\right)\, p_{2|\{A\}}(t) \qquad \{A\} \equiv \left\{ \genfrac{}{}{0pt}{}{|2\rangle}{0}, \not\emptyset_t \right\}$$

$$+\, P\left(\{B\}\right)\, p_{2|\{B\}}(t) \qquad \{B\} \equiv \left\{ \genfrac{}{}{0pt}{}{|1\rangle}{0}, \not\emptyset_t \right\}$$

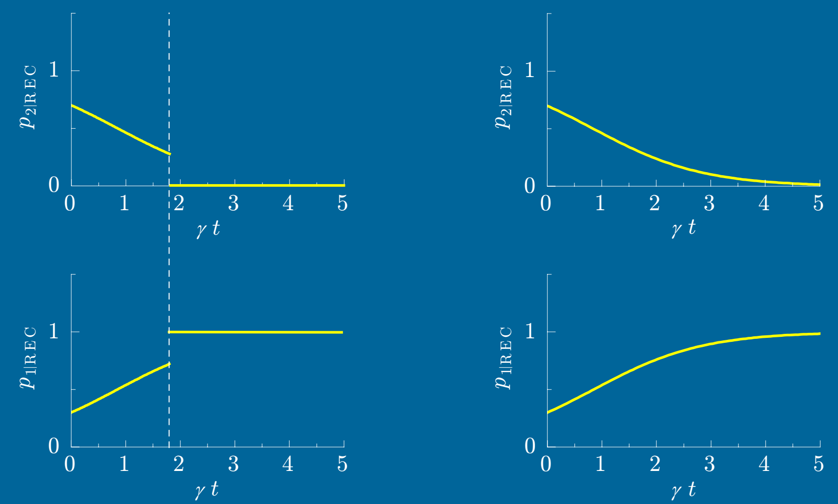
$$+ \int_0^t dt'\; P\left(\{C_{T'}\}\right)\, p_{2|\{C_{T'}\}}(t) \qquad \{C_{T'}\} \equiv \left\{ \genfrac{}{}{0pt}{}{|2\rangle}{0}, \not\emptyset, \genfrac{}{}{0pt}{}{\gamma}{T'}, \not\emptyset_t \right\}$$

record probabilities sum to unity: $P\left(\{A\}\right) + P\left(\{B\}\right) + \int_0^t dt'\, P\left(\{C_{T'}\}\right) = 1$

conditional evolution
known initial state:




conditional evolution
unknown initial state:



# Unraveling the master equation

Lindblad master equation

$$\dot{\rho} = \mathcal{L}\rho$$

$$\mathcal{L} = -i[\tfrac{1}{2}\omega_A \sigma_z, \cdot\ ] - \frac{\gamma}{2}(\sigma_+\sigma_- \cdot\ + \cdot\ \sigma_+\sigma_-) + \gamma\sigma_- \cdot \sigma_+$$

commutator

$[H_A, \cdot\ ]/ih$

anticommutator

$\mathcal{L}_0$

$\mathcal{L}_\gamma$

$$\mathcal{L} = \mathcal{L}_0 + \sum_{j=1}^{m} \mathcal{L}_{\gamma_j}$$

superoperator Dyson expansion:

$$\exp(\mathcal{L}t)$$
$$= \sum_{n=0}^{\infty} \sum_{j_n=1}^{m} \cdots \sum_{j_1=1}^{m} \int_0^t dt_n \cdots \int_0^{t_3} dt_2 \int_0^{t_2} dt_1 \, e^{\mathcal{L}_0(t-t_n)} \mathcal{L}_{\gamma_{j_n}} e^{\mathcal{L}_0(t_n-t_{n-1})} \cdots \mathcal{L}_{\gamma_{j_2}} e^{\mathcal{L}_0(t_2-t_1)} \mathcal{L}_{\gamma_{j_1}} e^{\mathcal{L}_0 t_1}$$

$$\rho(t) = \exp(\mathcal{L}t)\rho(0) = \sum_{\mathrm{REC}} \mathrm{tr}[\mathcal{K}_{\mathrm{REC}}(t)\rho(0)] \frac{\mathcal{K}_{\mathrm{REC}}(t)\rho(0)}{\mathrm{tr}[\mathcal{K}_{\mathrm{REC}}(t)\rho(0)]}$$

record probabilities

conditional states

$$\rho(t) = \sum_{\mathrm{REC}} \mathrm{tr}[\mathcal{K}_{\mathrm{REC}}(t)\rho(0)] \frac{\mathcal{K}_{\mathrm{REC}}(t)\rho(0)}{\mathrm{tr}[\mathcal{K}_{\mathrm{REC}}(t)\rho(0)]}$$

record probabilities sum to unity:

$$\mathrm{tr}[\rho(t)]=1$$

propagator factorizes:

$$\mathcal{K}_{\mathrm{REC}} = K_{\mathrm{REC}} \cdot K_{\mathrm{REC}}^{\dagger}$$

# Stochastic Schrödinger equation

pure state evolution

propagator factorizes: $\mathcal{K}_{\mathrm{REC}} = K_{\mathrm{REC}} \cdot K_{\mathrm{REC}}^{\dagger}$

$$e^{\mathcal{L}_0(t\text{-}t_n)} \mathcal{L}_{\gamma_{j_n}} e^{\mathcal{L}_0(t_n\text{-}t_{n\text{-}1})} \cdots \mathcal{L}_{\gamma_{j_2}} e^{\mathcal{L}_0(t_2\text{-}t_1)} \mathcal{L}_{\gamma_{j_1}} e^{\mathcal{L}_0 t_1}$$

$$\longrightarrow \underbrace{N_0(t\text{-}t_n) J_{\gamma_{j_n}} N_0(t_n\text{-}t_{n\text{-}1}) \cdots J_{\gamma_{j_2}} N_0(t_2\text{-}t_1) J_{\gamma_{j_1}} N_0(t_1)}_{K_{\mathrm{REC}}}$$

given $|\overline{\psi}_{\mathrm{REC}}(t)\rangle = K_{\mathrm{REC}}\, |\psi_0\rangle$:

$$|\overline{\psi}_{\mathrm{REC}}(t+dt)\rangle = \left.\begin{matrix} N_0(dt) \\ J_{\gamma_1} \\ J_{\gamma_2} \\ \vdots \\ J_{\gamma_m} \end{matrix}\right\} |\overline{\psi}_{\mathrm{REC}}(t)\rangle$$

Record probabilities:

$$P\left(\text{REC}\right) = \langle \psi_0 | K^{\dagger}_{\text{REC}} \; K_{\text{REC}} | \psi_0 \rangle = \; \langle \overline{\psi}_{\text{REC}} | \overline{\psi}_{\text{REC}} \rangle$$

$$P\left(\text{REC} \wedge \left\{ \begin{matrix} \gamma_j \\ T_k \end{matrix} \right\}\right) = \Delta t \; \langle \psi_0 | K^{\dagger}_{\text{REC}} \; J^{\dagger}_{\gamma_j} J_{\gamma_j} K_{\text{REC}} | \psi_0 \rangle = \Delta t \; \langle \overline{\psi}_{\text{REC}} | J^{\dagger}_{\gamma_j} J_{\gamma_j} | \overline{\psi}_{\text{REC}} \rangle$$

Bayesian inference:

$$P\left(\left\{ \begin{matrix} \gamma_j \\ T_k \end{matrix} \right\} \Big| \; \text{REC}\right) = \Delta t \; \frac{\langle \overline{\psi}_{\text{REC}} | J^{\dagger}_{\gamma_j} J_{\gamma_j} | \overline{\psi}_{\text{REC}} \rangle}{\langle \overline{\psi}_{\text{REC}} | \overline{\psi}_{\text{REC}} \rangle}$$

$$= \Delta t \; \langle \psi_{\text{REC}} | J^{\dagger}_{\gamma_j} J_{\gamma_j} | \psi_{\text{REC}} \rangle$$

jump probabilities

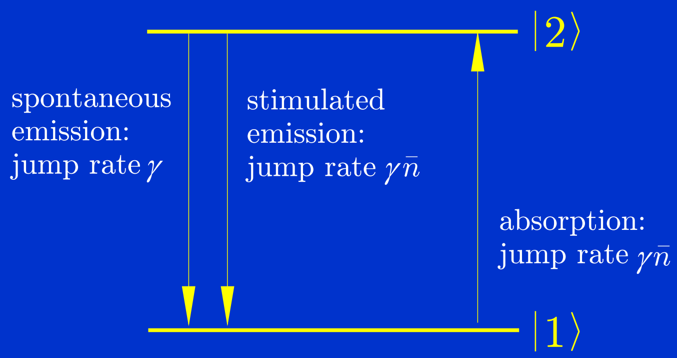
$$\Delta t \langle \psi_{\mathrm{REC}}(t) | J_{\gamma_j}^{\dagger} J_{\gamma_j} | \psi_{\mathrm{REC}}(t) \rangle$$

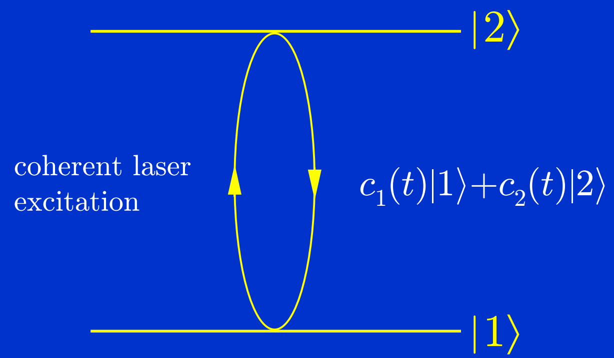
normalize

$$|\psi_{\mathrm{REC}}(t+dt)\rangle = \frac{\left.\begin{matrix} N_0(dt) \\ J_{\gamma_1} \\ \vdots \\ J_{\gamma_m} \end{matrix}\right\} |\psi_{\mathrm{REC}}(t)\rangle}{\sqrt{\langle \psi_{\mathrm{REC}}(t)| \left\{\begin{matrix} N_0(dt)^{\dagger} \, N_0(dt) \\ J_{\gamma_1}^{\dagger} \quad J_{\gamma_1} \\ \vdots \quad\quad \vdots \\ J_{\gamma_m}^{\dagger} \quad J_{\gamma_m} \end{matrix}\right\} |\psi_{\mathrm{REC}}(t)\rangle}}$$

# Quantum jumps plus coherence

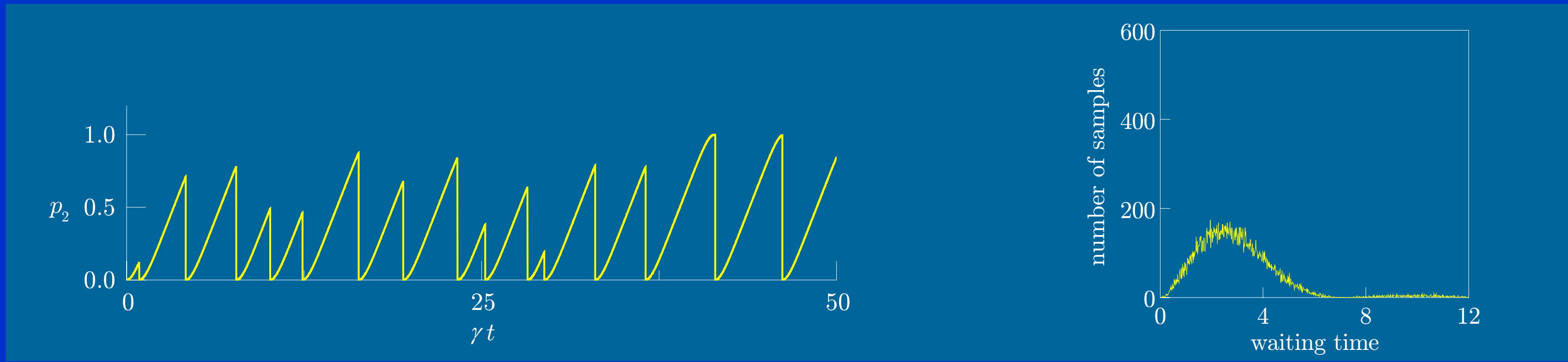
## resonance fluorescence

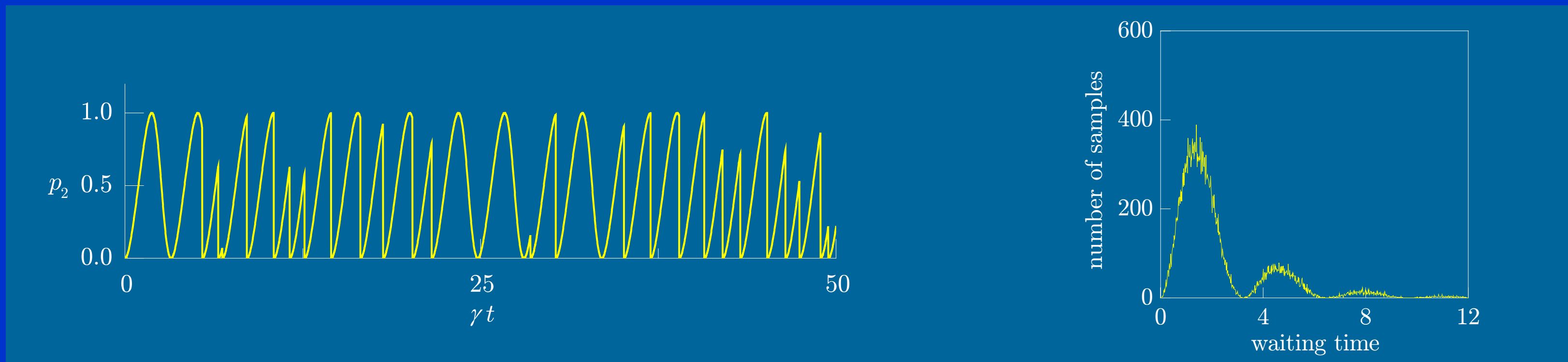


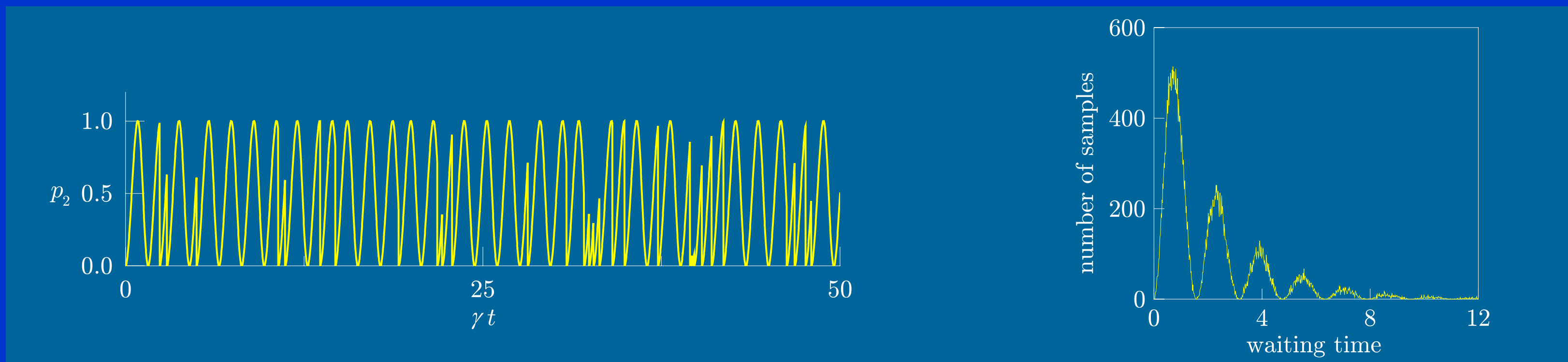




non-Hermitian Hamiltonian: $H_0 = i\hbar E(\sigma_+ - \sigma_-) - i\hbar \frac{\gamma}{2}\sigma_+\sigma_-$

jump operator: $J_\gamma = \sqrt{\gamma}\,\sigma_-$

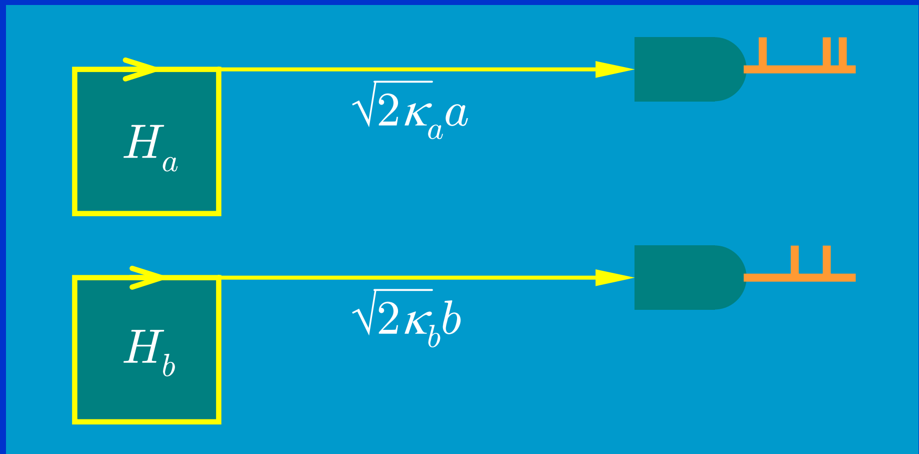

$p_2$
1.0
0.5
0.0
0
25
50
$\gamma t$
600
400
200
0
number of samples
0
4
8
12
waiting time


$p_2$
1.0
0.5
0.0
0
25
50
$\gamma t$
600
400
200
0
number of samples
0
4
8
12
waiting time


$p_2$
1.0
0.5
0.0
0
25
50
$\gamma t$
600
400
200
0
number of samples
0
4
8
12
waiting time

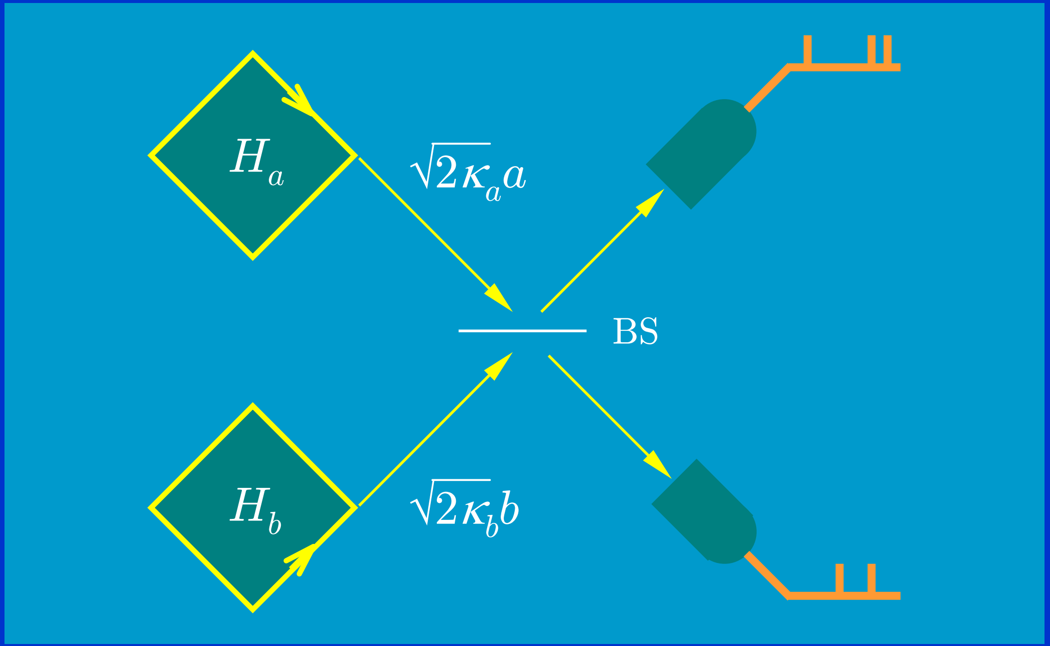
# More master equation unravelings

photoelectron counting records




non-Hermitian Hamiltonian: $H_0 = H_a + H_b - i\hbar(\sqrt{\kappa_a}a^\dagger a + \sqrt{\kappa_b}b^\dagger b)$

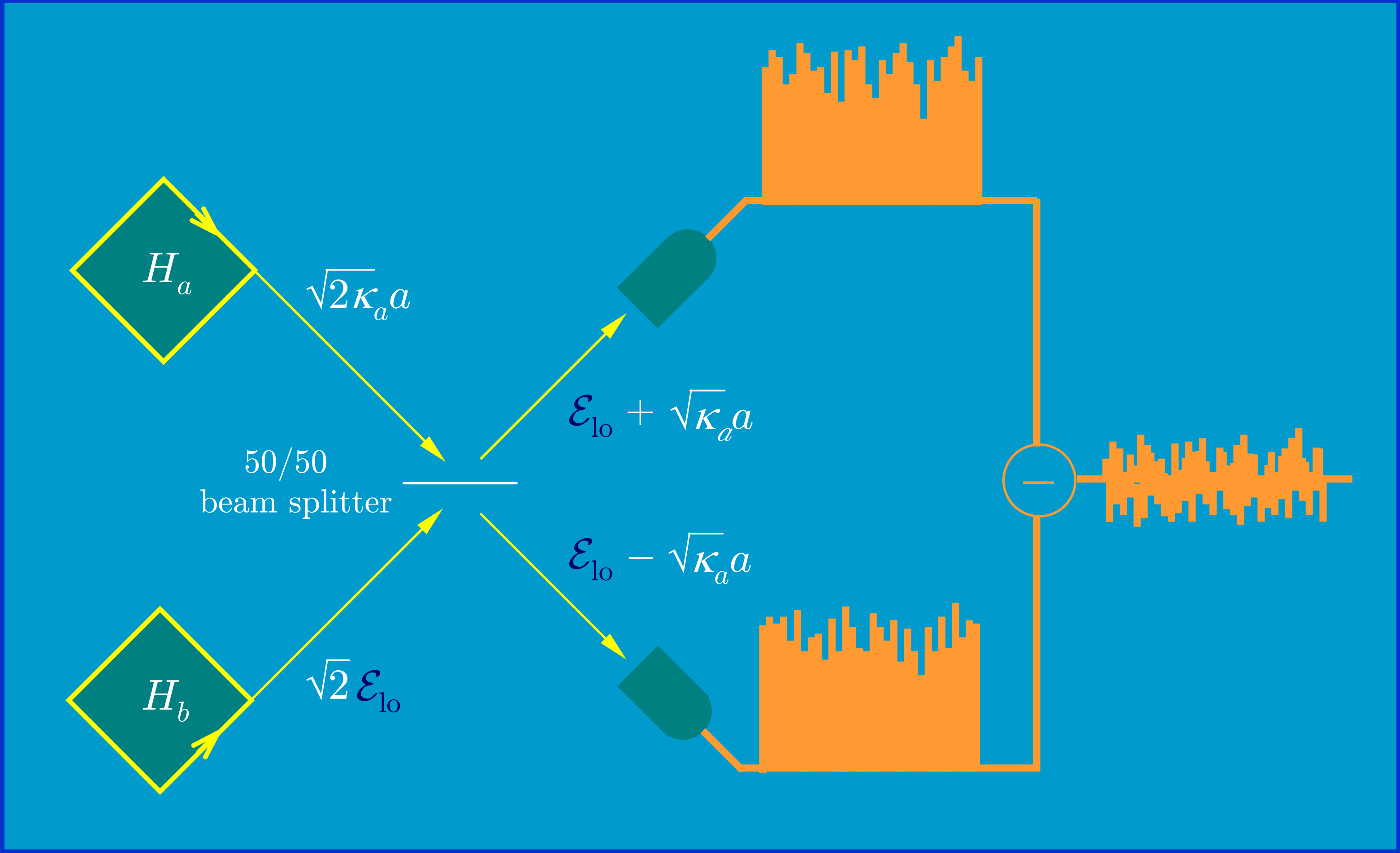
jump operators: $\sqrt{2\kappa_a}a$ and $\sqrt{2\kappa_b}b$




non-Hermitian Hamiltonian: $H_0 = H_a + H_b - i\hbar(\kappa_a a^\dagger a + \kappa_b b^\dagger b)$

jump operators: $\sqrt{r}\sqrt{2\kappa_a}a + \sqrt{t}\sqrt{2\kappa_b}b$ and $\sqrt{r}\sqrt{2\kappa_a}a - \sqrt{t}\sqrt{2\kappa_b}b$

# homodyne detection records



homodyne current record:

$$dI_\theta = -\Gamma(I_\theta dt - dQ_\theta) \qquad \text{(filtered)}$$

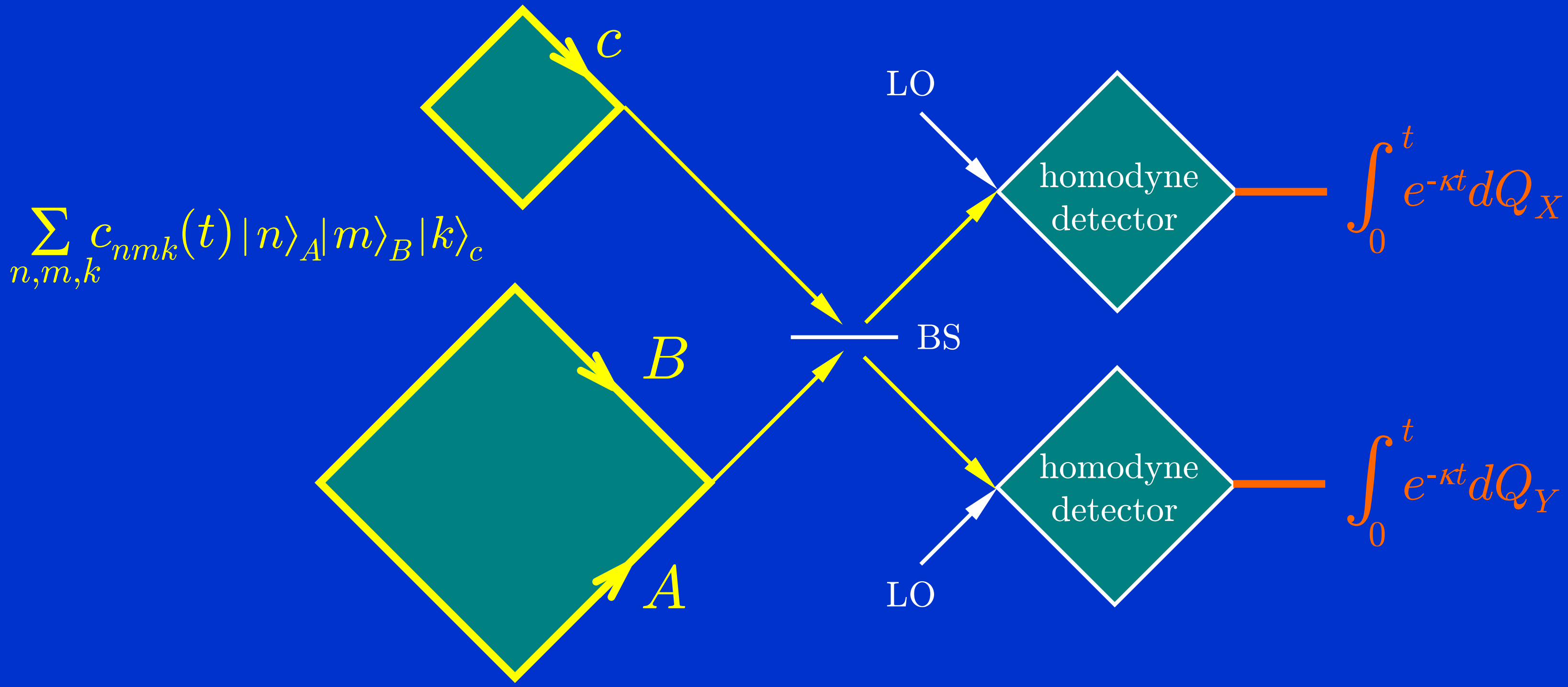
$$dQ_\theta = \sqrt{2\kappa_a}\langle a^\dagger e^{i\theta} + a e^{-i\theta} \rangle_{\mathrm{REC}}\, dt + dW \qquad \text{(unfiltered)}$$

stochastic
Schrödinger equation:

$$d|\bar{\bar{\psi}}_{\mathrm{REC}}\rangle = \left[ \left( \frac{1}{i\hbar} H_a - \kappa_a a^\dagger a \right) dt + \sqrt{2\kappa_a}\, a e^{-i\theta} dQ_\theta \right] |\bar{\bar{\psi}}_{\mathrm{REC}}\rangle$$

# Continuous variable quantum teleportation

elementary proposal



Alice's homodyne current records:

$$Q_X = \int_0^t e^{-\kappa t}\, dQ_X$$

$$dQ_X = 2(\sqrt{\kappa}\,\langle c_X\rangle_{\mathrm{REC}} + \sqrt{\kappa}\,\langle A_X\rangle_{\mathrm{REC}})dt + dW_X$$

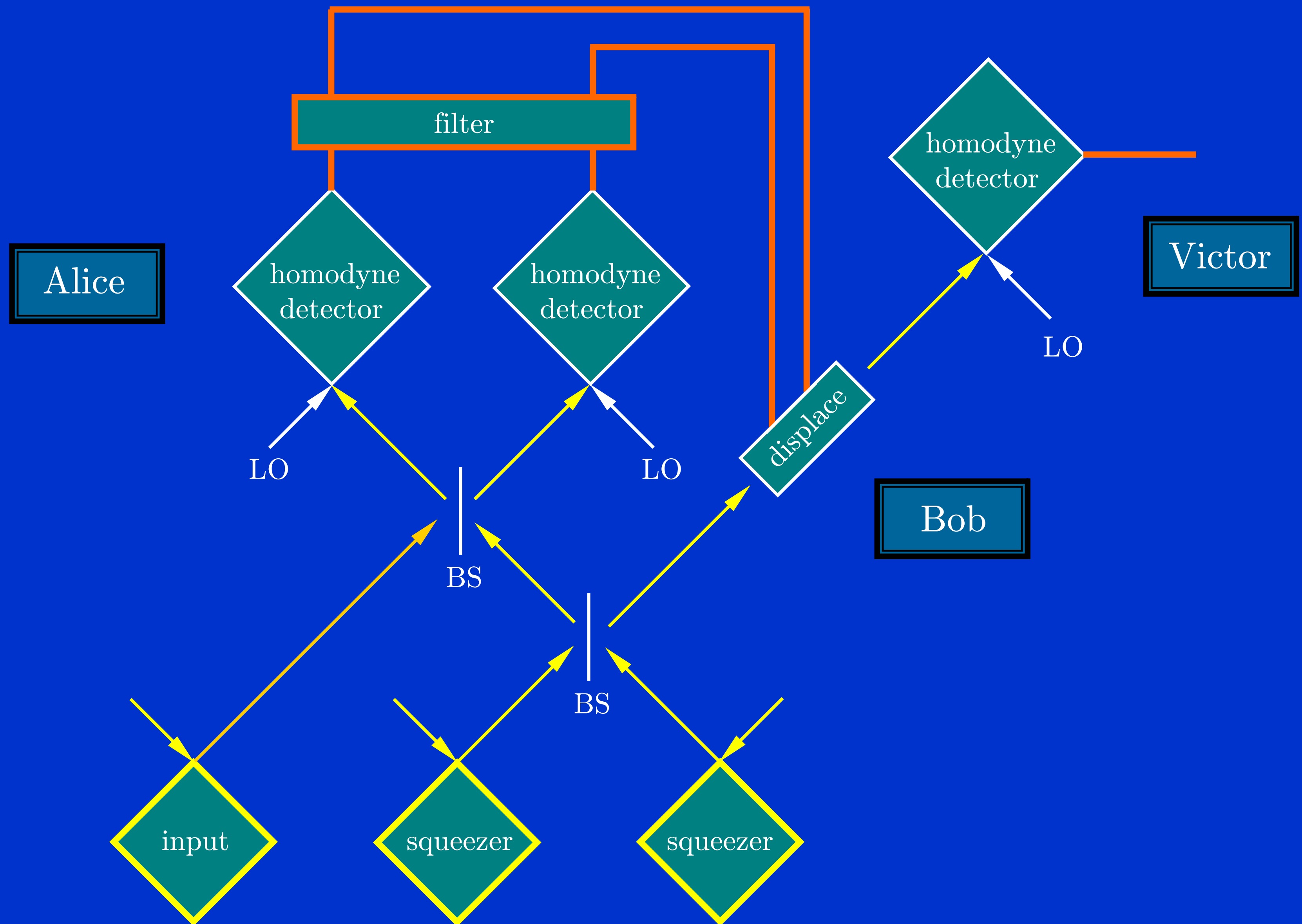
$$Q_Y = \int_0^t e^{-\kappa t}\, dQ_Y$$

$$dQ_Y = 2(\sqrt{\kappa}\,\langle c_Y\rangle_{\mathrm{REC}} + \sqrt{\kappa}\,\langle A_Y\rangle_{\mathrm{REC}})dt + dW_Y$$

Stochastic Schrödinger equation:

$$d\,|\overline{\overline{\psi}}\rangle_{\mathrm{REC}} = \Big\{ -\kappa\,(c^\dagger c + A^\dagger A)dt + (dQ_X - i dQ_Y)\sqrt{\kappa}\,c + (dQ_X + i dQ_Y)\sqrt{\kappa}A \Big\}\, |\overline{\overline{\psi}}\rangle_{\mathrm{REC}}$$

# experiments



Alice's homodyne current records:

$$dI_X = -\Gamma\,(I_X\,dt - dQ_X)$$

$$dQ_X = 2(\sqrt{\gamma}\,\langle c_X\rangle_{\mathrm{REC}} + \sqrt{\kappa}\,\langle A_X\rangle_{\mathrm{REC}})dt + dW_X$$

$$dI_Y = -\Gamma\,(I_Y\,dt - dQ_Y)$$

$$dQ_Y = 2(\sqrt{\gamma}\,\langle c_Y\rangle_{\mathrm{REC}} + \sqrt{\kappa}\,\langle A_Y\rangle_{\mathrm{REC}})dt + dW_Y$$

Victor's homodyne current record:

$$dI_{\mathrm{Victor}} = -\Gamma'(I_{\mathrm{Victor}}\,dt - dQ)$$

$$dQ = 2(\sqrt{2\kappa}\,\langle B_X\rangle_{\mathrm{REC}} + I_X/\sqrt{2}\,)dt + dW$$

Stochastic Schrödinger equation:

$$
\begin{aligned}
d\,|\bar{\bar{\psi}}\rangle_{\mathrm{REC}} = \Big\{ & \big(\frac{1}{i\hbar} H_{\mathrm{in}} - \gamma c^{\dagger} c\big) dt + (dQ_X - i dQ_Y)\sqrt{\gamma}\, c \\
& - \kappa \big[ \lambda (A^{\dagger} B^{\dagger} - AB) + A^{\dagger} A + B^{\dagger} B \big] dt \\
& + (dQ_X + i dQ_Y)\sqrt{\kappa} A + (I_X - i I_Y)\sqrt{\kappa} B dt \\
& + dQ \Big[ \sqrt{2\kappa} B + \frac{1}{\sqrt{2}} (I_X + i I_Y) \Big] \Big\} |\bar{\bar{\psi}}\rangle_{\mathrm{REC}}
\end{aligned}
$$