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Quantum Defects and Topological Quantum Numbers

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Topological Quantum Numbers

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Abstract In this lecture I discuss the nature and physical significance of topological quantum numbers and topological defects in condensed matter systems. The simplest and most familiar examples of topological quantum numbers are circulation in superfluids and magnetic flux in superconductors, and the corresponding defects are vortices and flux lines. Such things are very robust, and may be quantized with high precision. The significance of dissipative effects is emphasized, and it is pointed out that this may lead to loss of quantum coherence. The more complicated situation in quantum Hall devices will be described briefly.

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Hans Dehmelt's question:

How can precise measurements be made with poorly characterized devices such as semiconductor inversion layers or Josephson junctions?

Josephson junctions are used as secondary voltage standards, are consistent with one another to parts in 10^{17}, and have led to a revision of accepted values of fundamental constants.

The quantum Hall effect provides a secondary resistance standard that is far more reliable than its predecessors, and different devices agree to parts in 10^{10}.

The answer is related to **topological quantum numbers**, which may relate a physically observable quantity to a counting process in a way that is robust to changes of the details of a system.

It is not as simple as that, because the topological quantum number is not usually the quantity that is of direct physical interest.

Topological line defects in 3 dimensions or point defects in 2 dimensions

Topological quantum numbers, such as circulation in superfluid ^{4}He or in dilute gas of alkali atoms, magnetic flux in superconductors, Hall conductance in semiconductor inversion layers (two-dimensional electron systems), are insensitive to the symmetry of the system and to changes in the details of the structure.

In some cases they can be determined with very high precision, but not always.

Defect in simple superfluid or superconductor is characterized by winding of phase angle of condensate wave function by $2\pi n$ round the defect, which corresponds to quantized circulation nh/m or quantized magnetic flux $nh/2e$. Since these winding numbers are additive when lines are grouped together, homotopy group is just integer addition $\mathcal{Z}$.

Something similar happens for the quantum Hall effect, but it is more complicated.

Bose-Einstein condensates

Einstein (1924, 1925) showed that at sufficiently low temperatures noninteracting bosons will collapse into the lowest energy state.

Fritz London (1938) suggested that this could be an explanation for the peculiar properties of superfluid helium below 2.17 K; specific heat singularity, flow without viscosity, film flow, etc.

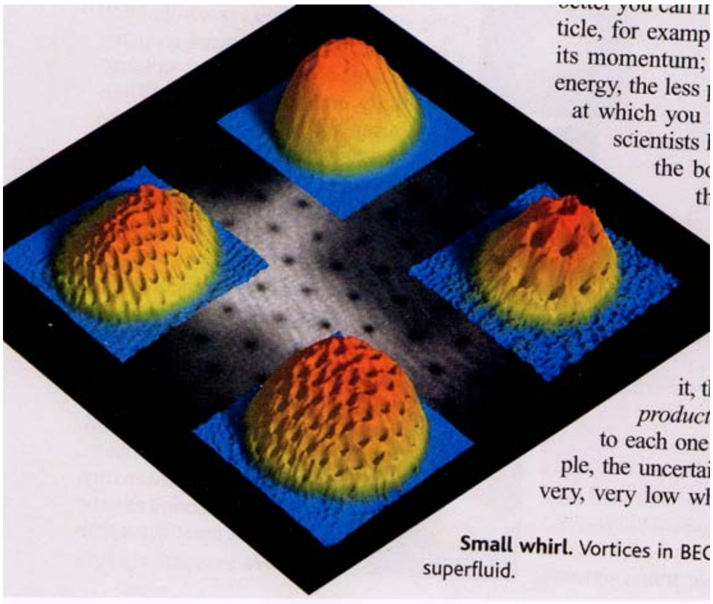

Figure 1: Quantized vortices in rotating sodium atom clusters, as shown by Ketterle et al., 2000

Bogoliubov (1947) showed that repulsion between the atoms stabilizes the condensate, the multiply occupied wave function. We now believe this repulsive interaction is essential for superfluidity, but liquid helium is very far from Bogoliubov's weakly interacting gas.

Cornell, Wieman and collaborators (1995) cooled trapped alkali metal atoms well below 1 μK and found that they formed a Bose-Einstein condensate.

Ketterle and collaborators (2000) rotated a Na atom trap and showed that the rotation creates an array of quantized vortices (**Abrikosov lattice**).

Onsager–Feynman argument for quantized circulation

Bose-Einstein condensation involves a finite proportion of bosons in system sharing a common single particle state

$$\Psi(\mathbf{r}) = |\Psi(\mathbf{r})| \exp[iS(\mathbf{r})] \, ,$$

a single-valued function of position.

Superfluid velocity is $\mathbf{v}_s = \hbar \mathbf{grad} S / m_4$, where m_4 is helium atom mass.

The phase need not be single valued, but can change by a multiple of 2π on a closed path that goes round either an obstacle, such as a wire, or when it goes round a mathematical line singularity on which $|\Psi|$ vanishes. The circulation of the superfluid velocity round a path is given by

$$\oint \mathbf{v}_s \cdot d\mathbf{r} = \frac{\hbar}{m_4} \oint \mathbf{grad} S \cdot d\mathbf{r} = n \frac{h}{m_4} .$$

The number n of quanta of circulation $\kappa_0 = h/m_4$ is given by the **winding number** of the phase of the condensate wave function.

1. The order parameter Ψ of the superfluid, the condensate wave function, is the feature that allows the topological properties of its phase to be defined and studied.

2. The phase S satisfies the Laplace equation, since $-i\hbar \mathbf{grad} S$ represents a conserved current.

3. Superfluid velocity is not directly measured;
   superfluid momentum density is measured.

Vinen **experiment** (1961)

Cylinder with stretched wire running down middle is filled with helium, which can be made to circulate round the wire.

Magnus force: $\mathbf{F}_M = \rho_s \mathbf{k}_s \times (\mathbf{v}_w - \mathbf{v}_s)$ splits vibrational modes of wire by $\Delta\nu = \rho_s \kappa / 2\pi w$, where w is mass density of wire, κ circulation round it.

Rotating apparatus was cooled through superfluid transition, then brought to a stop, leaving fluid rotating around wire.

Initially the vortex is often on only part of the wire, and the rest of it goes through the fluid. Repeated shaking of the wire usually gets rid of the free end. This leaves all the vortex on the wire, and a quantized circulation is measured.

Vinen found that measured circulations were clustered around 0 and h/m_4, with about $\pm 3\%$ precision.

Packard group (1993) has confirmed $\kappa_0 = h/2m_3$ for B phase of superfluid ^{3}He, where condensed object is a pair of ^{3}He atoms.

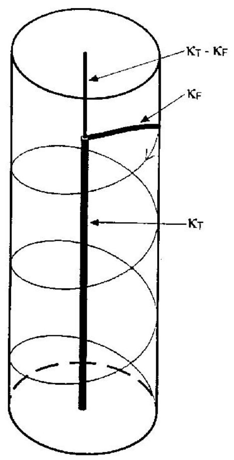

Figure 2: Vinen vibrating wire apparatus, as shown by Zieve et al., 1993

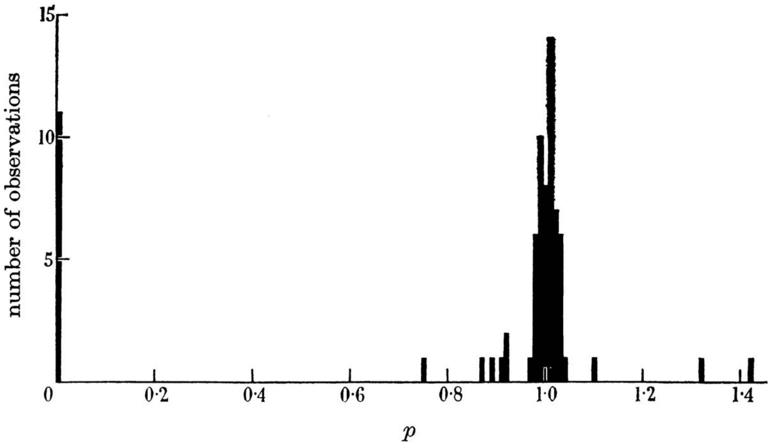

Figure 3: Histogram of measured circulation p, in units of h/m_4, from Vinen's experiment.

Flux quantization in superconductors

Superconductor is superfluid in which condensate consists of electron pairs.

In Hamiltonian mechanics for particles of charge q the relation between momentum $\mathbf{p}$ and velocity $\mathbf{v}$ is

$$\mathbf{v} = \frac{1}{m}(\mathbf{p} - q\mathbf{A}) \ ,$$

where $\mathbf{A}$ is the vector potential whose curl gives the magnetic field $\mathbf{B}$.

So, in quantum theory the current density is

$$\mathbf{j} = -(\frac{e\hbar}{m}\mathbf{grad}S + \frac{2e^2}{m}\mathbf{A})|\Psi|^2 \ .$$

Ψ represents condensate wave function for electron pairs, so factor $-2e$ is put in front of vector potential $\mathbf{A}$ to allow for charge $-2e$, mass $2m$.

Curl of this equation gives London equation

$$\nabla^2\mathbf{B} = \frac{e^2\mu_0}{m}n_s\mathbf{B} \ , \implies \nabla^2\mathbf{j} = \frac{e^2\mu_0}{m}n_s\mathbf{j} \ ,$$

so magnetic field and current density fall off exponentially inside superconductor or away from vortex core.

In interior region where current density vanishes,

$$\oint \mathbf{A}\cdot d\mathbf{r} = -\frac{\hbar}{2e}\oint \mathbf{grad}S\cdot d\mathbf{r} = n\frac{h}{2e} \ .$$

Since integral of the vector potential round a ring gives the flux enclosed by the ring, this shows that the flux trapped by a superconductor is equal to n times $h/2e$, where again n is the winding number of the phase of the condensate wave function.

Path enclosing quantized flux has to be in a region free of current density. It may either surround regions in which there is no superconducting material, where the flux is concentrated, or, for a type II superconductor in a magnetic field, it may surround flux lines where the superconducting order parameter has singularities.

Because London equation gives **exponential decay** of current density, corrections to flux quantization may be made exponentially small by increasing length scale of system.

Dynamics of quantized defects

Simplest description of vortex motion says quantized vortex behaves much as a vortex in classical hydrodynamics does. Vortex flows with the local superfluid velocity, and the effect of a potential gradient is to make it flow along equipotentials at a speed proportional to the potential gradient. Movement relative to the fluid requires a force perpendicular to the motion, the Magnus force.

This dynamics was put in Hamiltonian form by Kirchhoff, and, in two dimensions, the X and Y coordinates of the vortex serve as conjugate variables.

At this level, quantization of the motion of the quantized vortex (second quantization, so to speak) requires that the allowed orbits be spaced with areas

$$\frac{h}{\rho\kappa_0} = \frac{m}{\rho}$$

between them, where ρ is the density per unit area. This implies that the number of distinct states for a vortex is equal to the number of bosons in the system.

For real vortices in an extended system there are various dissipative forces on vortices, which are still a subject of controversy and confusion. These dissipative effects complicated the experimental observation of superposition of states with different topological quantum numbers, and led to the theoretical work of Caldeira and Leggett (1983).

There are many sources of energy degradation and loss of phase coherence. In a superconductor the vortex core has low-lying excitations which are thermally excited, can transfer energy from the vortex to themselves by impurity scattering, or can exchange energy with the phonon system. In a neutral superfluid the vortices have long been known (Hall and Vinen 1956) as the mechanism of transfer of energy and momentum between the superfluid component and the normal component of interacting phonons and rotons. Even at zero temperature an accelerated vortex can radiate phonons, so that the vibrational motion of a free vortex in an incompressible fluid is heavily damped by radiation (Quist, 1999, Anglin and Thouless, in progress).

Even for the ideal clean superconductor this zero temperature radiation can occur, since a moving flux line produces a dipolar elastic distortion of the underlying solid (Nozières and Vinen, 1966), so that its acceleration can also generate emission of sound waves.

There is some analogy between the motion of quantized vortices in a superfluid and the motion of a dislocation in a solid according to the Peierls–Nabarro theory. The 'quantization' in this case , as in the case of defects in liquid crystals, is related to the molecular nature of matter, and depends on Avogadro's number rather than on Planck's constant.

Although quantized defects are classically very robust, such dissipative processes very easily destroy quantum coherence, and must be minimized in order to preserve it.

Quantum Hall effect

Experiments done by Klitzing (1980) on two-dimensional electron systems at low temperatures in high magnetic field (MOSFETs) showed very precise Hall voltages (voltage transverse to the current) where longitudinal voltage was negligible (no Ohmic dissipation).

At these plateaus,

$$\frac{I}{V_H} = \frac{ne^2}{h}$$

with very high precision — initially better than one part in 10^5, soon shown to be more precise than any other resistance measurement.

In the fractional quantum Hall effect, found by Tsui, Störmer and Gossard (1982), integer n is replaced by simple exact fraction.

Its origin is topological, as shown by Laughlin, but the story of how a winding number appears is a little more complicated. The Chern number is the number of times the Berry phase changes by 2π around the perimeter of a cut torus or cylinder.

Laughlin's argument: generalization of Bloch's theorem that, for a loop of conductor threaded by a magnetic flux Φ, free energy is a periodic function of Φ; current $\partial F/\partial\Phi$ is periodic with period h/e and has zero average. Laughlin considers nonequilibrium situation in which current is nonzero.

Two edges of annulus are maintained at voltages V_i and V_o, and solenoid carries variable flux Φ. Fermi energy lies in mobility gap so that there is no current across annulus.

Change of the flux Φ by h/e restores the ring to its original quasiequilibrium state, except for gauge changes.

There may also be n electrons transferred from one reservoir to the other. Free energy then changes by

$$\Delta F = ne(V_o - V_i) = ne\,\delta V \ .$$

Change in flux generates voltage $d\Phi/dt$ round the annulus, and this does work

$$W = \int J\, d\Phi = \bar{J}h/e \ ,$$

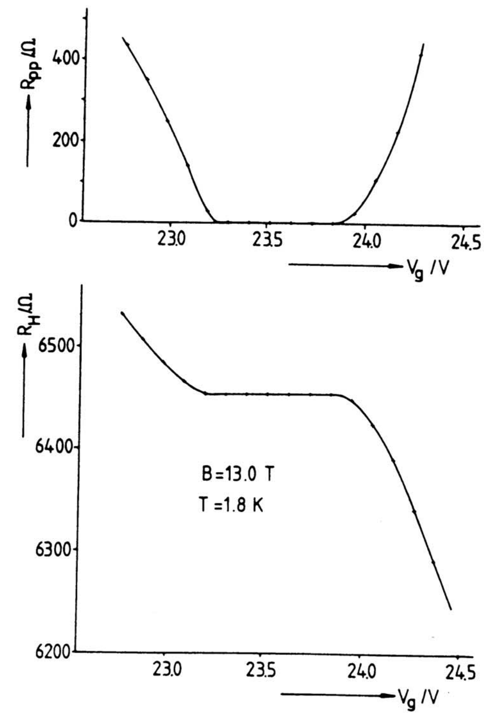

FIG. 2. Hall resistance R_H, and device resistance, R_{pp}, between the potential probes as a function of the gate voltage V_g in a region of gate voltage corresponding to a fully occupied, lowest ($n=0$) Landau level. The plateau in R_H has a value of $6453.3 \pm 0.1\ \Omega$. The geometry of the device was $L=400\ \mu m$, $W=50\ \mu m$, and $L_{pp}=130\ \mu m$; $B=13$ T.

at gate voltage close to the left side of the plateau. In Fig. 2, this minimum is relatively shallow and has a value of $6452.87\ \Omega$ at $V_g=23.30$ V.

In order to demonstrate the insensitivity of the Hall resistance on the geometry of the device, measurements on two samples with a length-to-width ratio of $L/W=0.65$ and $L/W=25$, respectively, are plotted in Fig. 3. The gate-voltage scale

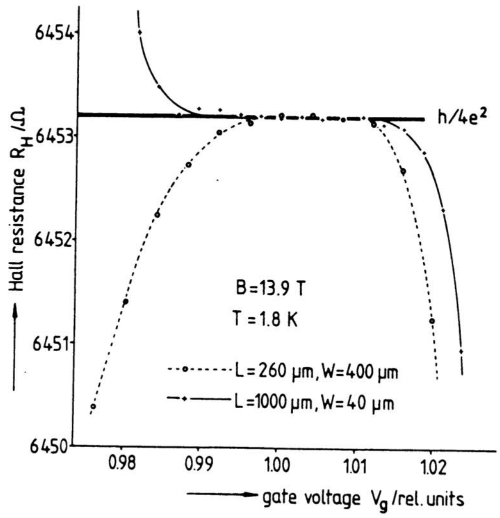

FIG. 3. Hall resistance R_H for two samples with different geometry in a gate-voltage region V_g where the $n=0$ Landau level is fully occupied. The recommended value $h/4e^2$ is given as $6453.204\ \Omega$.

Figure 4: Original measurement of quantum Hall effect by Klitzing Dorda and Pepper (1980)

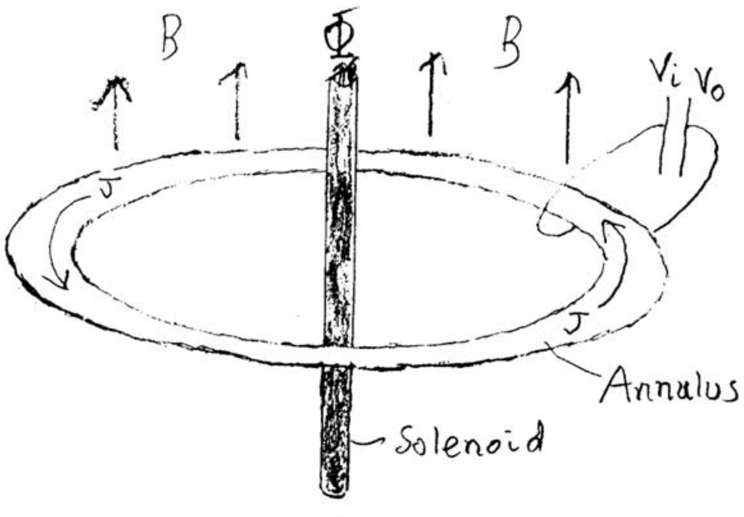

Figure 5: Annulus in uniform field B has Hall current J induced by voltage $V_o - V_i$ between the edges. Change of the flux Φ through the solenoid by h/e induces an emf round the annulus, and this causes n electrons to pass between the edges.

where $\bar{J}$ is the average current around the annulus.

Equality of work done and change in free energy gives

$$ne\,\delta V = \bar{J}h/e\ .$$

Laughlin's profound 1981 explanation of the integer quantum Hall effect has its topological aspects somewhat hidden, and I understood the relation between Hall conductance and Chern numbers (Thouless, Kohmoto, Nightingale, den Nijs, Avron, Seiler, etc) long before I grasped how Laughlin's earlier argument was related to ours.

In this quasiequilibrium argument the flux quantum crosses the annulus in company with n electrons, maintaining the system in its equilibrium state with n electrons for each flux quantum. A flux line accompanied by $n \pm 1$ electrons is a negatively or positively charged topological defect for the integer quantum Hall state.

In the fractional quantum Hall effect (FQHE) the situation is a little different, since the state with quantized conductance q/p has a p-fold degeneracy, so that p flux lines have to be moved across the annulus to transport q electrons. I will restrict attention to the case $q/p = 1/3$, the simplest of the fractional states described in Laughlin's 1983 work.

In Jain's description this can be made up from composite fermions, each made up from one electron and two flux lines, filling the lowest Landau level, so that there is a total of one electron for every three flux lines, two associated with the electron, and one from the background field in which the composite electrons condense.

Each uncharged flux line forms a defect with charge $-e/3$, while each composite electron in excess of those that fill the Landau level has charge $+e/3$.

It is better to regard the external field B as given, but each composite electron carries a pseudomagnetic gauge field whose effects cancel the gauge field of two real magnetic quantized flux lines, to leave an effective total gauge field $B/3$.

For the quasiparticles electric charge is topological, but, as with superfluid singularities, external effects can give problems with phase coherence.

Fractional quantum Hall effect does give examples of the kind of topologically protected entity that has been explored by Kitaev and by Freedman, Shtengel and Nayak.

Work by Haldane and Rezayi (1985) showed that for the $1/m$ FQHE on a torus there is an m-fold degeneracy asscoiated with m independent states of the center of mass coordinate.

A torus in a uniform field may be hard to realize, but may still be useful for a theorist to think about.

Different states of the center of mass coordinate differ only very slightly in electron density, so are only very slightly perturbed by external fields. Correspondingly they will be very hard to measure.

A balance may have to be struck by keeping the size of the system and the number of electrons small, in order to keep the coordinate robust but measurable.

An explicit representation of the Haldane-Rezayi wave function on a torus was worked out by Lan Yin. For a torus with two equal periods the wave function can be written as a product of three factors. The center of mass

$$\bar{Z} = N^{-1} \sum_{j=1}^{N} \bar{z}_j = N^{-1} \sum (x_j - iy_j)$$

has wave function

$$\psi_{\rm cm} = \sigma^m (N\bar{Z} - N\bar{D}) \exp\left(-\frac{BN}{4}|Z|^2 + NBD\bar{Z}\right) ,$$

where B is the magnetic field, N the number of electrons, D is a constant, and σ the Weierstrass σ function, an entire, quasiperiodic function with a simple zero at the origin. The wave function for the relative coordinates is

$$\psi_{\rm rel} = \prod_{i>i'} \left[\sigma^m(\bar{z}_i - \bar{z}_{i'}) \exp(-\frac{B}{4N}|z_i - z_{i'}|^2)\right] ,$$

which is of the Laughlin form with $(\bar{z}_i - \bar{z}_{i'})^m$ replaced by σ^m. Finally there is a phase factor

$$\psi_{\rm gauge} = \prod_{i=1}^{N} \prod_{j=1}^{mN} \sqrt{\frac{\sigma(z_i - c_j)}{\sigma(\bar{z}_i - \bar{c}_j)}} ,$$

where the c_j are the positions of incoming flux lines.

For given values of the positions c_j of the entering flux lines there are m allowed values of the constant D determined by the periodicity of the wave function on the torus. This gives the m-fold degeneracy of the state.

Movement of one of the flux lines around a nontrivial closed loop round the torus gives a continuous change of the constant D, and maps the center of mass wave function into a different linear superposition of its m degenerate components. One representation of these mappings for the case $m = 3$ represents translation by one period in the x direction or in the y direction by the matrices

$$T_x = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} , \quad T_y = \begin{pmatrix} e^{2\pi i/3} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{-2\pi i/3} \end{pmatrix} .$$

These transformations generate nine different, but linearly dependent, ground states that differ in the positions of the maxima and minima of the very weak modulation of the charge density of the electrons.

Conclusions.

1. Topological quantum numbers are robust, insensitive to perturbations and details of fabrication.

2. The simplest form of topological quantum number is characterized by a winding number of a condensate wave function.

3. Robustness of topological quantum number is semiclassical result. High degeneracy of defect states and significant coupling to environment can lead to loss of phase coherence.

4. Hall conductance in quantum Hall systems gives another high precision quantum number. Charged quasiparticles are topological defects.

5. The usual abelian states should also have phases sensitive to perturbations. Do nonabelian states have less sensitive phases?

6. Can measurements be made readily on such protected states?

7. Dirac (1931) gave a topological argument for the high precision of the quantization of electric charge.

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Detailed references

Further details and references can be found in

- D.J.Thouless. Vortices in Superfluids and Superconductors, and Topological Defects in Other Materials. In *Topology of Strongly Correlated Systems: Proceedings of the XVIII Lisbon Autumn School*, ed. P.Bicudo, J.E.Ribiero, P.Sacramento, J.Seixas and V.Viera (World Scientific, Singapore, 2001). A version of this article can be seen on http://www.phys.washington.edu/users/thouless/lisbon.ps.gz.

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