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SCHOOL AND WORKSHOP ON
QUANTUM ENTANGLEMENT, DECOHERENCE, INFORMATION, AND GEOMETRICAL PHASES IN COMPLEX SYSTEMS
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Many-Particle Adiabatic Entanglement

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Many-Particle Adiabatic Entanglement

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Outline

- Problem
- Adiabatic approximation
- Entanglement measures
- Collective spin systems
  Lipkin-Meshkov-Glick (LMG) model
- Entanglement of the ground state of the LMG model
  bipartite and glogal entanglement
- Physical implementation
- Conclusion

Adiabatic Approximation

Adiabatic eigenstates

$$H\left[\,s\left(t\right)\right]\left|\Phi_n\left(s\left(t\right)\right)\right\rangle = E_n\left(s\left(t\right)\right)\left|\Phi_n\left(s\left(t\right)\right)\right\rangle$$

Caracteristic time of change of Hamiltonian $H\left(s\left(t\right)\right)$

$$T >> \frac{\mathcal{E}_{n,n_0}}{\Delta^2_{n,n_0}}$$

$\mathcal{E}_{n,n_0}$ proportional to the derivative of the $H\left(s\left(t\right)\right)$ with respect to the parameter s

Δ_{n,n_0} denotes the energy difference between the initial state

$$\left|\Psi\left(-\infty\right)\right\rangle = \left|\Phi_{n_0}\left(-\infty\right)\right\rangle$$

and an eigenstate $\left|\Phi_n\right\rangle$ of the Hamiltonian H

Evolution of the state vector

$$\left|\Psi\left(t\right)\right\rangle \approx \exp\left(-i\int_{-\infty}^{t} E_n\left(s\left(t'\right)\right)dt'\right)\exp\left(-i\gamma\left(t\right)\right)\left|\Phi_{n_0}\left(t\right)\right\rangle$$

Dynamical phase **Berry Phase**

An important question is how the time *T* for generating multipatricle entangled states scales with the number of particles.

Entanglement Measures

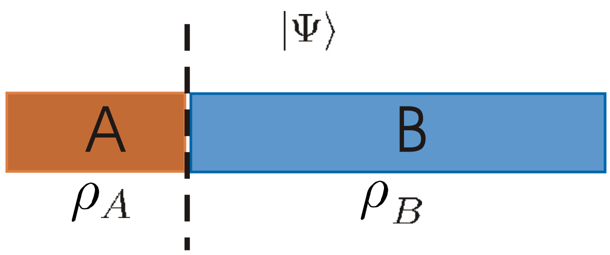
Schmidt decomposition

Given a pure state $|\Psi\rangle$, and a partition for the system, {A,B}

Schmidt's theorem asserts that any state in the Hilbert space $H_A \otimes H_B$ can be written in the form

$$|\Psi\rangle = \sum_{m-1}^{\chi} \lambda_m \left|\Phi_m^{(A)}\right\rangle \otimes \left|\Phi_m^{(B)}\right\rangle$$

Where $\chi = \min\{\chi_A, \chi_B\}$ $\sum_{m-1}^{\chi} \lambda_m^2 = 1.$



The entropy of entanglement is defined as

C.H. Bennett, *et al.*, Phys. Rev. A **53**, 2046 (1996)

$$E_{|\Psi\rangle} = -Tr(\rho_{A,B} \log \rho_{A,B})$$

$$\rho_{A,B} = Tr_{B,A}\left(|\Psi\rangle\langle\Psi|\right)$$

$$E_{|\Psi\rangle} = -\sum_{m-1}^{\chi} \lambda_m^2 \log_2 \lambda_m^2.$$

$E_{|\Psi\rangle}$ ***is the measure of the information loss due to division of the system***

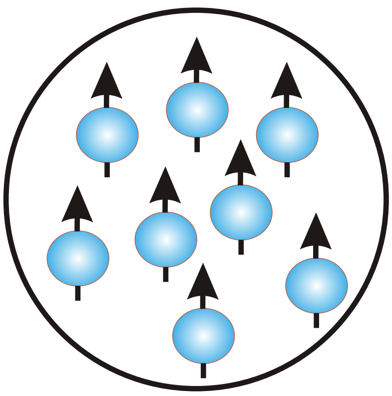
$$E_\chi(\Psi) = \log_2 \chi_{\max}$$

$$E_{|\Psi\rangle} \leq E_\chi(\Psi)$$

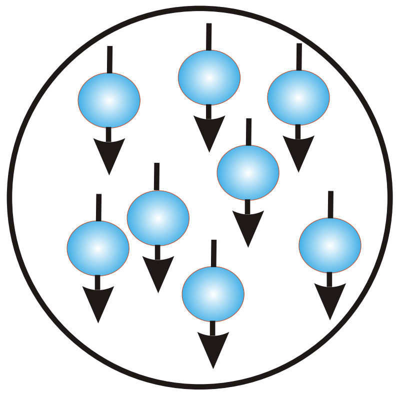
collective spin system

Consider N spins with mutual interaction

Every spin couples
to every other spin

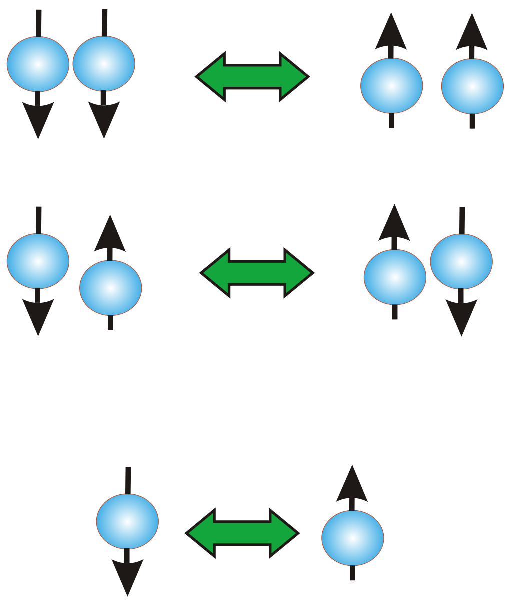


Initial state



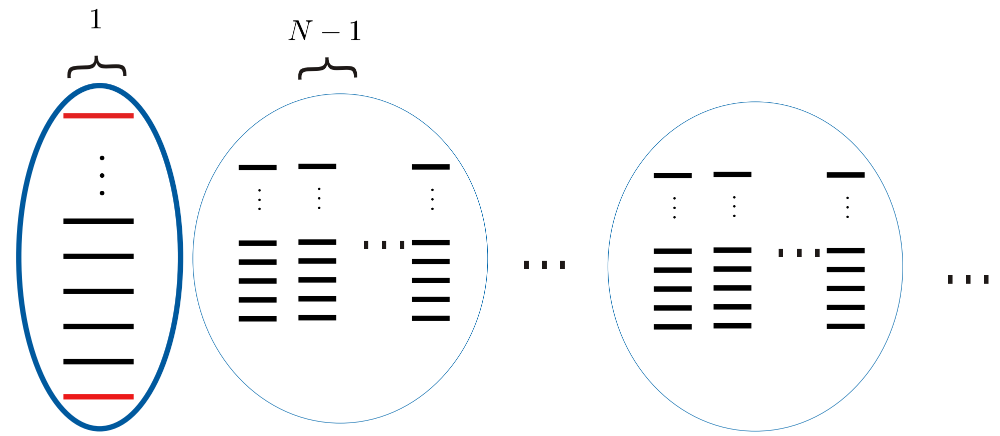
it is easily prepared in the lab (e.g. via optical pumping)

Allow transitions



N spins

Dimension of the Hilbert space 2^N



$J = \frac{N}{2}$ $\quad J = \frac{N}{2} - 1$ $\quad J = \frac{1}{2}$ or $J = 0$

Each subspace has the dimension

$$\dim(J,N) = \frac{2(2J+1)}{2J+2+N} \frac{N!}{\left(\frac{N}{2}-J\right)!\left(\frac{N}{2}+J\right)!}$$

The dimension of the space is

$$\sum_J (2J+1)\dim(J,N) = 2^N$$

The initial state of the system belongs to the first subspace

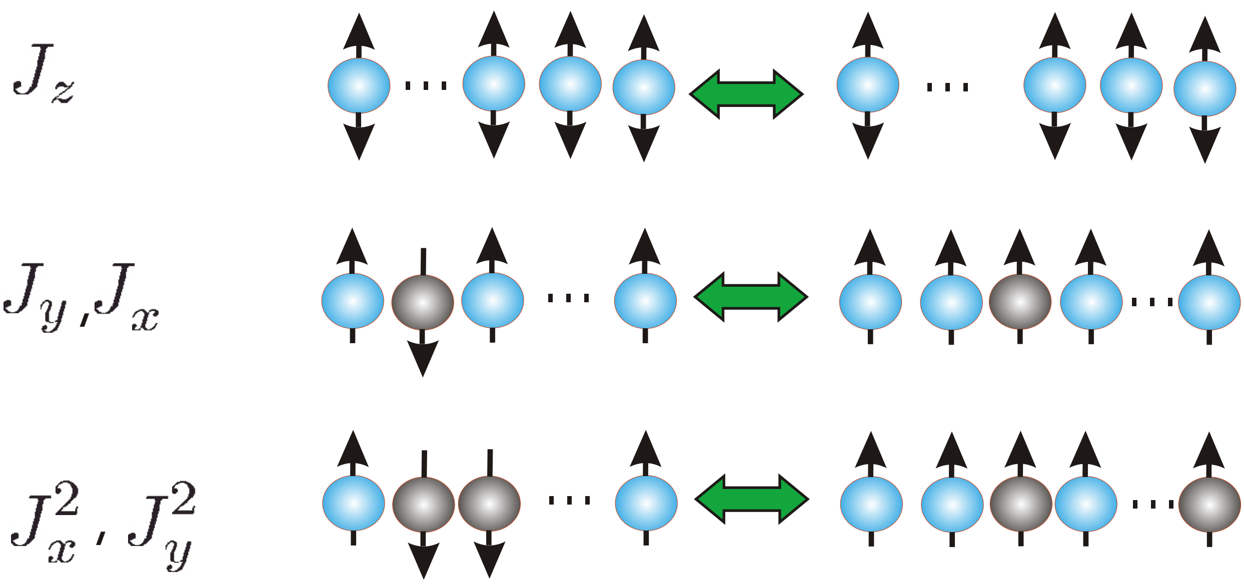
Lipkin-Meshkov-Glick Hamiltonian

$$H = \xi \left(\lambda \chi_1 \chi_2 J_z + \chi_1^2 J_x^2 + \chi_2^2 J_y^2 + 2\mu \chi_2^2 J_y \right)$$

H.Lipkin, N. Meshkov, and A. Glick, Nucl. Phys. **62**,188 (1965)

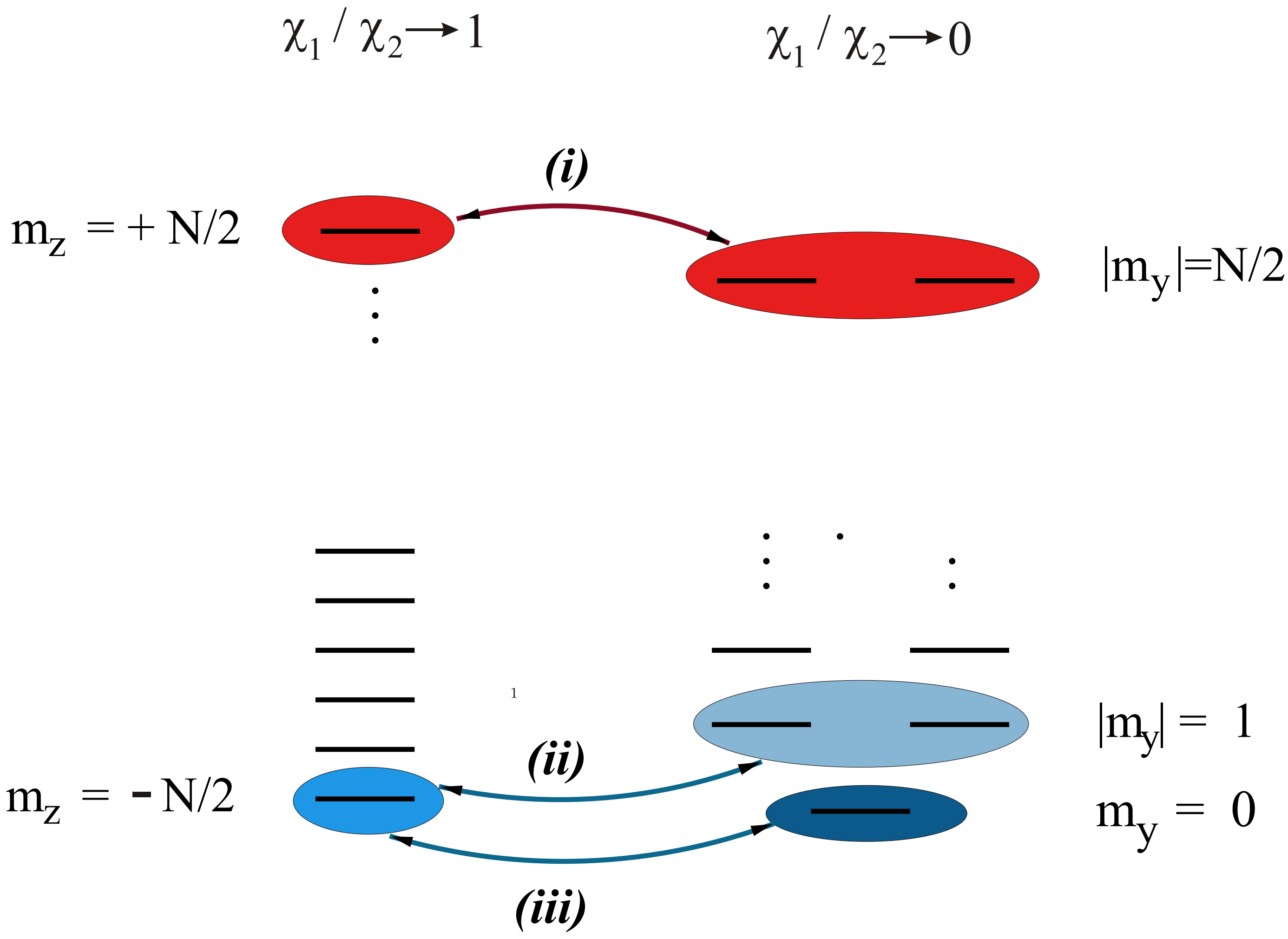
The model consideres N fermions distributed in two N-fold degenerate levels

$J_{x,y,z}$ are total spin operators of the ensemble



Spectrum

$$H = \xi\left(\lambda\chi_1\chi_2 J_z + \chi_1^2 J_x^2 + \chi_2^2 J_y^2 + 2\mu\chi_2^2 J_y\right)$$



Interesting scenarios

case (i)

$\xi < 0$ and $\mu = 0$ $\lambda \geq N$

$\chi_1 = \chi_2$ $H \rightarrow \xi\chi_2^2 \left(\lambda J_z - J_z^2 + J(J+1)\right)$

Ground state $|\uparrow\uparrow\uparrow \ldots\ldots\ldots \uparrow\rangle$

$\chi_1 = 0$ $H \rightarrow \xi\chi_2^2 J_y^2$

Two degenerate ground states $| \, m_y = \pm\frac{N}{2} \, \rangle$

Due to symmetry of LMG Hamiltonian only one state coupled

$$|\uparrow\uparrow\uparrow \ldots \uparrow\rangle \Longrightarrow \frac{1}{\sqrt{2}} \left[|+ + + \ldots +\rangle + \exp\left(i\pi J\right) |- - - \ldots -\rangle \right]$$

$|\pm\rangle$ Eigenstates of J_y

Separable (ground) state $\Rightarrow$ Greenberger-Horne-Zeilinger (GHZ), entangled (ground) state

Interesting scenarios

case (ii)

$\xi > 0$ and $\mu = 0$

Odd number of particles

$\chi_1 = \chi_2$ Ground state $|\downarrow\downarrow\downarrow \cdots\cdots \downarrow\rangle$

$\chi_1 = 0$ $|\, m_y = \pm\frac{1}{2}\,\rangle$

Due to symmetry of LMG Hamiltonian only one state coupled

$$|\downarrow\downarrow\downarrow \ldots \downarrow\rangle \Longrightarrow \frac{1}{\sqrt{2}}\left[|(+)^n(-)^{n-1}\rangle + i|(-)^n(+)^{n-1}\rangle\right]$$

$|\pm\rangle$ Eigenstates of J_y

$n=[N/2]$

<u>**Separable (ground) state $\Rightarrow$ entangled (ground) state**</u>

Problem:

in case (i) & case (ii) merging of energy levels susceptible to environment coupling

Interesting scenarios

case (ii)

$\xi > 0$ and $\mu = 0$

Odd number of particles

$\chi_1 = \chi_2$ Ground state $|\downarrow\downarrow\downarrow \cdots\cdots \downarrow\rangle$

$\chi_1 = 0$ $| \, m_y = \pm\frac{1}{2} \, \rangle$

Due to symmetry of LMG Hamiltonian only one state coupled

$$|\downarrow\downarrow\downarrow \dots \downarrow\rangle \Longrightarrow \frac{1}{\sqrt{2}} \left[|(+)^{n} (-)^{n-1} \rangle + i |(-)^{n} (+)^{n-1} \rangle \right]$$

$|\pm\rangle$ Eigenstates of J_y

$n = [N/2]$

<u>**Separable (ground) state $\Rightarrow$ entangled (ground) state**</u>

<u>**Problem:**</u>

in case (i) & case (ii) merging of energy levels susceptible to environment coupling

Interesting scenarios

case (iii)

$\xi > 0$ and $\mu = 0$

Even number of particlles

$\chi_1 = \chi_2$ Ground state $|\downarrow\downarrow\downarrow \ldots\ldots\ldots \downarrow\rangle$

$\chi_1 = 0$ $|m_y = 0\rangle$

$$|\downarrow\downarrow\downarrow \ldots \downarrow\rangle \Longrightarrow |(+)^{n/2}(-)^{n/2}\rangle$$

$|\pm\rangle$ Eigenstates of J_y

Separable (ground) state $\Rightarrow$ entangled (ground) state

case (iv)

Similar to case (iii) but $\mu = M$, $M=-J,-J+1,\ldots J$

$$|\downarrow\downarrow\downarrow \ldots \downarrow\rangle \Longrightarrow |m_y = M\rangle$$

There is always a finite energy gap between ground state and excited state

Condition of adiabaticity

- cases (i) and (ii)

  associated with a merging of pairs of energies
  (two-fold degeneracy)

- Only in cases (iii) and (iv) there is a finite energy gap to all other states

  Thus the entanglement generation will be robust against

  collective and individual decoherence processes.

$\chi_1^{max} T^{1/2}$, $\chi_2^{max} T^{1/2} >> 1$ **with** $\lambda > N$ **for case (i)**

$\lambda > 1$ **for cases (ii)-(iv)**

Supersymmetry (SUSY)

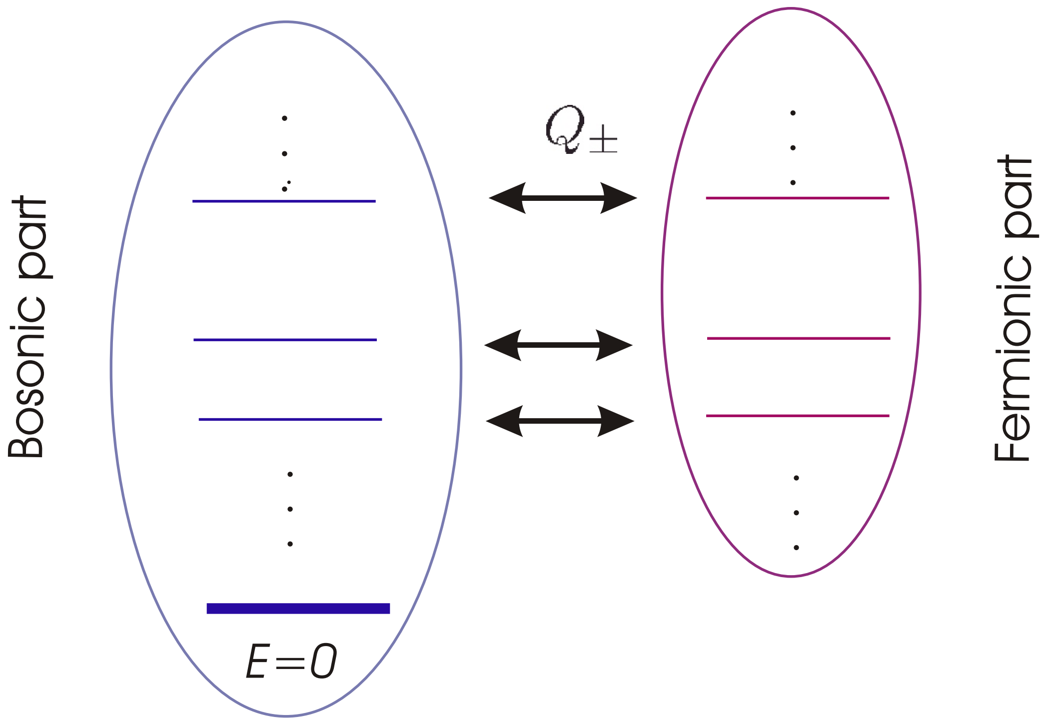
Property of SUSY:

$$H = Q_1^2 = Q_2^2 = \{Q_+, Q_-\}\ ,\{Q_1\ , Q_2\} = 0$$

$$Q_1 = Q_+ + Q \quad , \qquad Q_2 = i\,(Q_- - Q_+)$$

the operators $Q_\pm$ convert a boson into fermion and vice versa. They are nilpotent i.e.

$$Q_+^2 = Q^2 = 0$$



Ground State

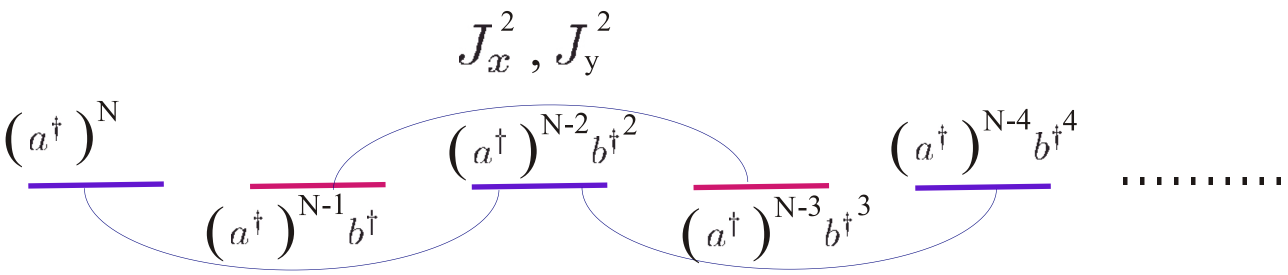
SUSY of LMG Hamiltonian

$$Q_1 = \sigma_x \chi_1 J_x + \sigma_y \chi_2 J_y, \qquad Q_2 = \sigma_y \chi_1 J_x - \sigma_x \chi_2 J_y$$

$$J_x = \frac{1}{2}(ab^\dagger + ba^\dagger), \quad J_y = \frac{1}{2i}(ba^\dagger - ab^\dagger),$$

$$J_z = \frac{1}{2}(a^\dagger a - b^\dagger b)$$

N: even number and λ=1



$$H = \begin{bmatrix} \mathbf{H}_{even} & \mathbf{0} \\ \mathbf{0} & \mathbf{H}_{odd} \end{bmatrix}$$

Ground State of the SUSY Hamiltonian

$$Q_1 |\Psi\rangle_{SUSY} = 0$$

$$(\chi_1 J_x - i\chi_2 J_y) |\Psi\rangle_{SUSY} = 0$$

$$|\Psi\rangle_{SUSY} = \mathcal{N} \exp(-\gamma J_z) |M_x = 0\rangle \qquad tanh(\gamma) = \chi_2 / \chi_1$$

Entanglement of the ground state

Bipartite decomposition of the ground state

Symmetric state $|\Psi\rangle_{SUSY} = \mathcal{N} \exp\left(-\gamma J_z\right) |M_x = 0\rangle$

Schmidt decomposition ⇔ Clebsch-Gordan decomposition

$\gamma = 0$ ⇨

$$|J, M = 0\rangle = \sum_{m=-\min\{J_A, J_B\}}^{\min\{J_A, J_B\}} C_{m,-m,0}^{J_A, J_B, J} |J_A, m\rangle \otimes |J_B, -m\rangle$$

$$E_{Sym} = -\sum_{\alpha=1}^{d} \lambda_\alpha^2 \log_2 \lambda_\alpha^2 \qquad \sum_{\alpha=1}^{d} \lambda_\alpha^2 = 1$$

$$\lambda_m^2 = \left(C_{m,-m,0}^{J_A, J_B, J}\right)^2$$

$$\left(C_{m,-m,0}^{J_A, J_B, J}\right)^2 = \frac{\exp\left[-\frac{m^2}{J_B}\right]}{\sqrt{J_B \pi}}, \quad J = J_A + J_B, \quad J >> 1, \quad J_A >> J_B$$

$$E_{Sym} \approx \frac{1}{2} \log_2 N_B + \frac{1}{2} \left(\log_2 e\pi - 1\right)$$

Geometric measure of entanglement

$\gamma \neq 0$ ⇨ $|\Psi\rangle_{SUSY} = \mathcal{N} \exp(-\gamma J_z) |M_x = 0\rangle$

$$\Lambda_{\max}(J,\gamma) = \frac{\sqrt{(2J)!}}{2^J J!} \frac{\exp(\gamma J)}{\sqrt{P_J^{(0,0)}(\cosh 2\gamma)}}$$

⇩

$$\Lambda_{\max}(J,\gamma=0) = \frac{\sqrt{(2J)!}}{2^J J!} \approx \frac{1}{\sqrt[4]{2J}}$$

Tzu-Chieh Wei and Paul M. Goldbart, PRA, **68**, 042307 (2003)

$$\Lambda_{\max}(J,\gamma) \to (1-\exp(-4\gamma))^{1/4}, \; J >> 1, \; \gamma \neq 0$$

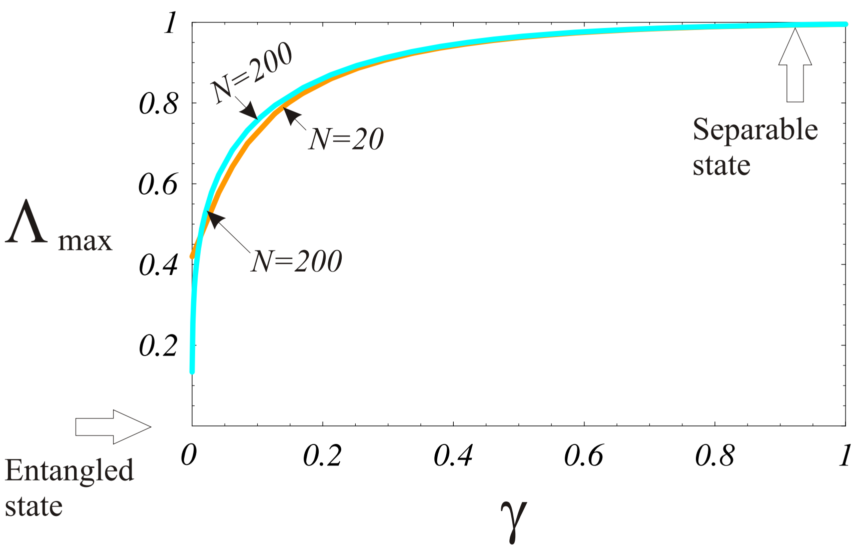
$\Lambda_{\max}(J,\gamma)$ does not depend on number of qubits if $\gamma \neq 0$

$$|\Psi\rangle_{SUSY} = \mathcal{N} \exp(-\gamma J_z) |M_x = 0\rangle$$

Local, nonunitary ⇦ $\exp(-\gamma J_z)$

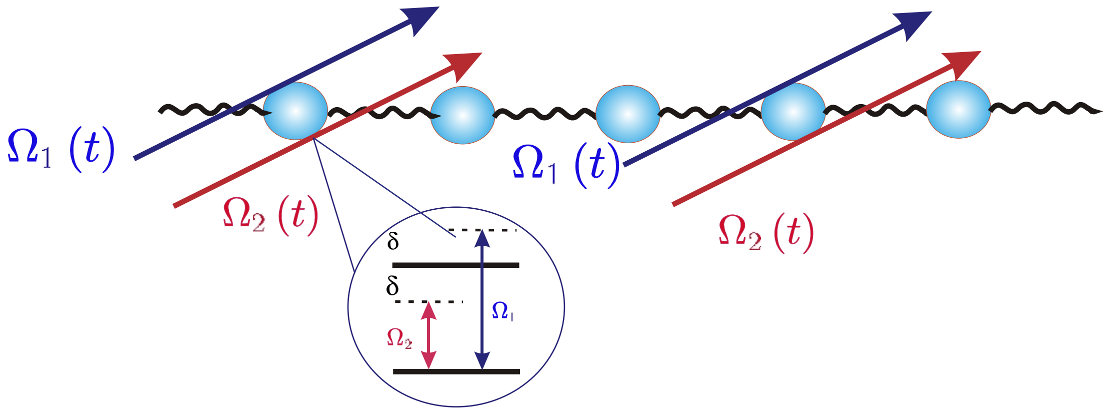
$|M_x = 0\rangle$ ⇨ Entangled state

the entanglement should be decreased



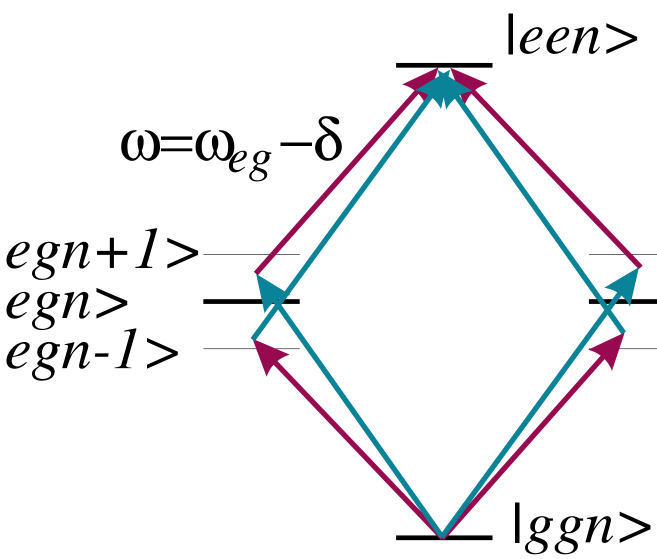
Physical implementation in an ion trap

ion trap, low T: only one oscillation mode ν



Sørensen and Mølmer, Phys. Rev. Lett. **82**,1971 (1999)

fields detuned off resonance, but two-photon resonance in two-ion system



Hamiltonian

$$H = J_+ \exp(i\eta(c + c^\dagger)) \left[\Omega_1 \exp(-i\omega_1 t) + \Omega_2 \exp(-i\omega_2 t)\right] + H.c.$$

elimination of phonons $\delta, \quad |\delta \pm \nu| >> \alpha$

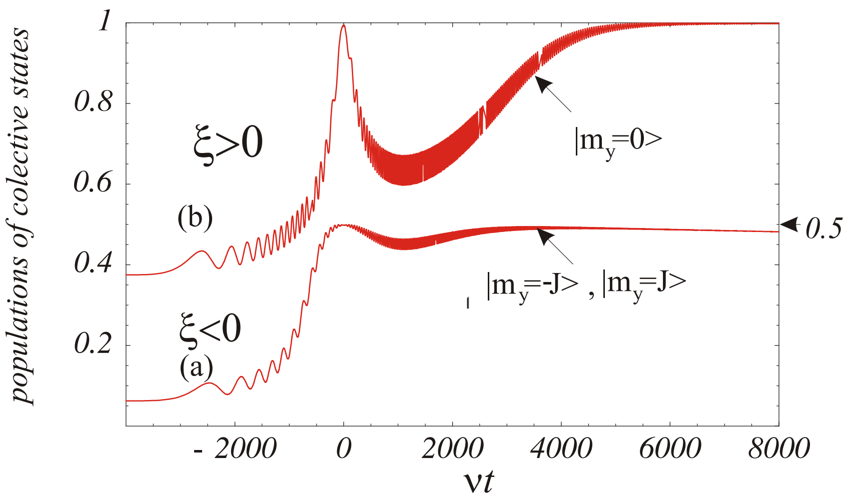
Lamb-Dicke limit $(n+1)\,\eta^2 << 1$

Lamb-Dicke parameter $\eta = \sqrt{\frac{\hbar k^2}{2M\nu}}$

$$H \longrightarrow H_{LMG}$$

$$\xi = \frac{2\nu\eta^2}{\delta^2 - \nu^2} \qquad \lambda = \frac{2}{\xi\delta} \qquad \chi_{1,2} = \Omega_1 \mp \Omega_2$$

numerical example *N=4*



$$\Omega_1(t) = \frac{\alpha}{2}\left(1 + \tanh\frac{t}{T_1}\right) \qquad \Omega_2(t) = \frac{\alpha}{2}\left(1 + \tanh\frac{t}{T_2}\right)$$

Conclusion

- Uncertainties of external parameters can be eliminated by adiabatic change of parameters

  decoherence due to reservoir couplings can be suppresed in the presence of a finite energy gap

- LMG Hamiltonian has SUSY, ground state can be explicitly constructed

- bipartite entanglement scales logarithmically with size of subsystem

- global entanglement saturates and decreases monotonically with respect to anistropy parameter