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SCHOOL AND WORKSHOP ON
QUANTUM ENTANGLEMENT, DECOHERENCE, INFORMATION, AND GEOMETRICAL PHASES IN COMPLEX SYSTEMS
(1 November - 12 November 2004)

Geometric phases in the presence of noise

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These are preliminary lecture notes, intended only for distribution to participants

GEOMETRIC PHASES

IN THE PRESENCE OF NOISE

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OUTLINE

- *A brief introduction to Berry phase*
- *Berry phase and fault tolerant Quantum Computation*
- *Classical noise in the evolution of spin ½*
- *Dynamic vs. geometric decoherence*
- *Quantum Noise in the evolution of a spin ½*
- *Berry phase and squeezed quantum noise*

THE BERRY PHASE

- a Hamiltonian H depending on a set of control parameters $\boldsymbol{R}$

$$H(\boldsymbol{R}(t))\,|\Psi\rangle = E(\boldsymbol{R}(t))\,|\Psi\rangle \qquad \boldsymbol{R}(T) = \boldsymbol{R}(0)$$

- the control parameters are changed adiabatically and cyclically

A non degenerate energy eigenstate at the end of the cyclic evolution differs from the initial one by a phase factor

$$|\Psi(T)\rangle = \exp\{i\delta\}\,\exp\{i\gamma\}\,|\Psi(0)\rangle$$

- Dynamic phase $\delta = \int_0^T E(t)dt$
- *Geometric (Berry) phase* γ

BERRY CONNECTION

$$\gamma = \int_C \boldsymbol{A}\cdot d\boldsymbol{R} \qquad \boldsymbol{A}(\boldsymbol{R})_n \equiv i <n(\boldsymbol{R}(t))|\, \nabla_{\boldsymbol{R}}\, |\, n(\boldsymbol{R}(t))>$$

BERRY PHASE

- The Berry connection is the analogue of the vector potential in the A-B effect
- It depends on the geometry of the trajectory in parameter space, e.g. it is zero for a closed loop enclosing zero area

- The Berry connection is a function only of the eigenstates dependence on the control parameters but it is independent from the energy spectrum.
- The Berry phase keeps a memory of the path followed in parameter space
- The dynamic phase keeps a memory on how fast the path is followed

AN EXAMPLE: A SPIN ½ IN A MAGNETIC FIELD

$$H = \tfrac{1}{2}\, \boldsymbol{B}\cdot\boldsymbol{\sigma}$$

Eigenstates

$$|\uparrow>_B = e^{i\phi/2} \cos\theta/2\, |\uparrow>_z + e^{i\phi/2} \sin\theta/2\, |\downarrow>_z$$

$$|\downarrow>_B = e^{i\phi/2} \sin\theta/2\, |\uparrow>_z - e^{i\phi/2} \cos\theta/2\, |\downarrow>_z$$

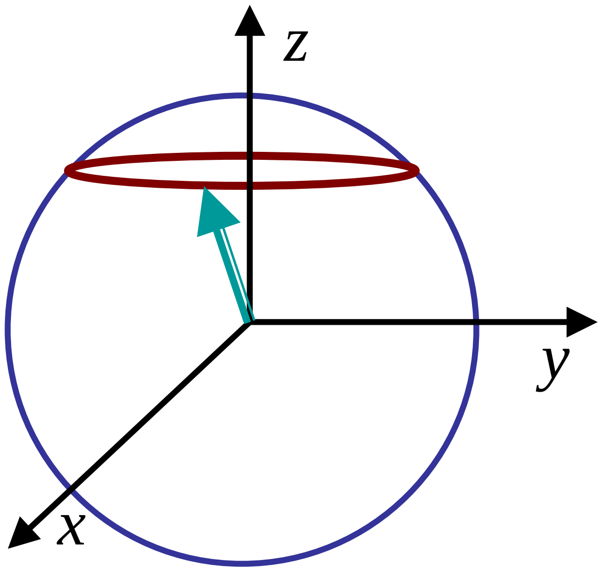
Independent from |B|

$$A_{\theta\uparrow} = A_{\theta\downarrow} = i <\uparrow_B|\partial/\partial\theta|\uparrow_B> = 0$$

$$A_{\phi\uparrow} = -A_{\phi\downarrow} = i <\uparrow_B|\partial/\partial\phi|\uparrow_B> = \tfrac{1}{2}\cos\theta$$

$$F_{\phi\theta\downarrow} = -F_{\phi\theta\uparrow} = \partial_\phi A_\theta - \partial_\theta A_\phi = \tfrac{1}{2}\sin\theta$$

AN EXAMPLE: PRECESSION AROUND A PARALLEL

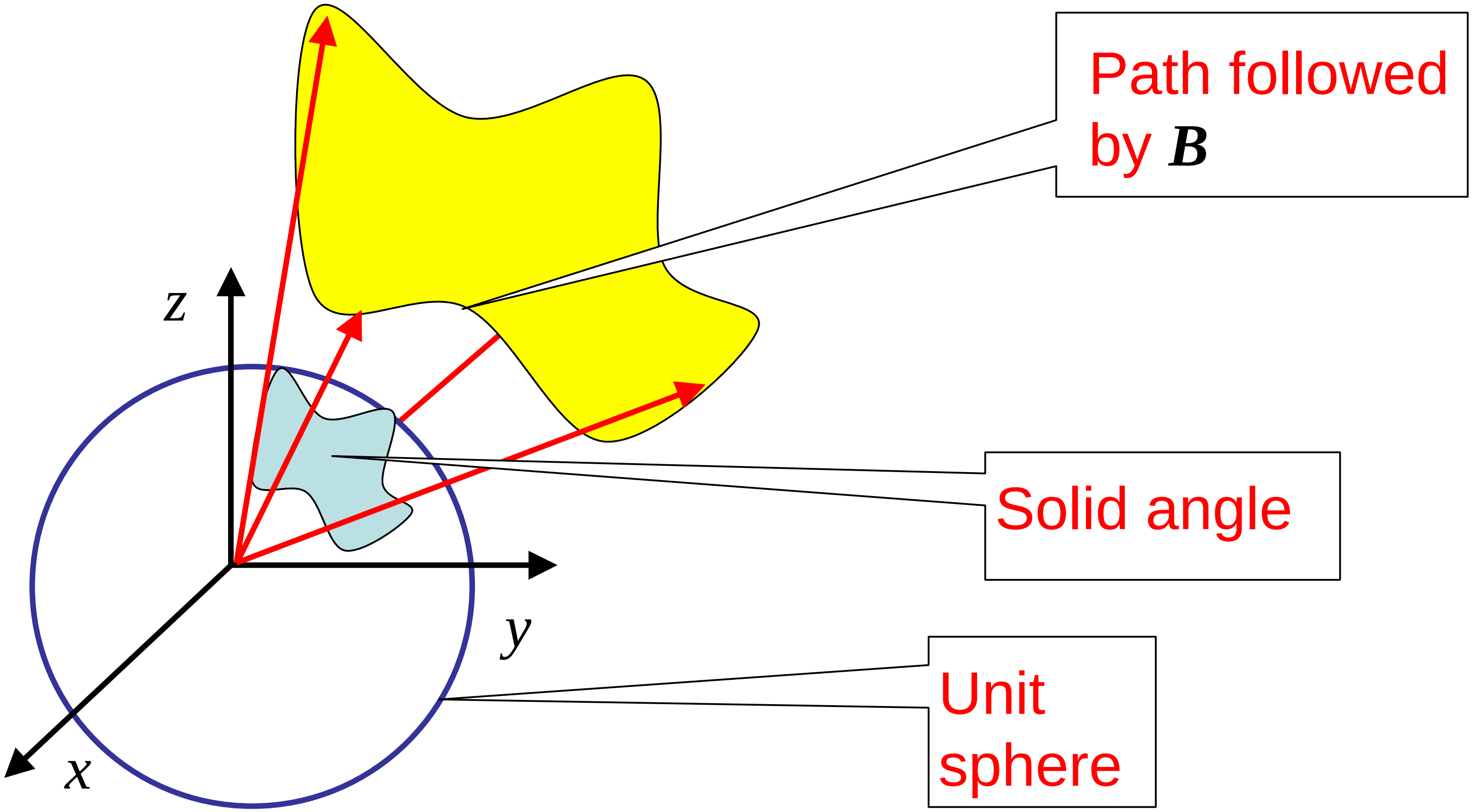


$\boldsymbol{B}$ precesses at an angle θ around the $\boldsymbol{z}$ axis with angular velocity Ω

The Berry phase is independent from $\Omega = 2\pi/T$

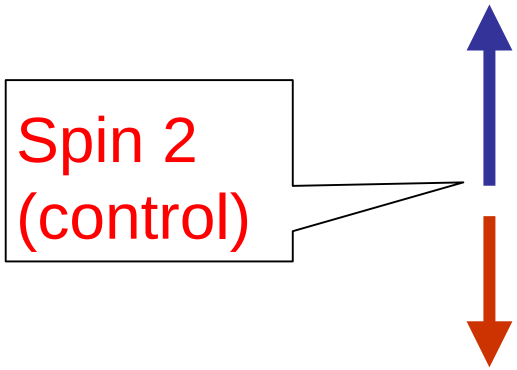
$$\gamma_{\uparrow} = -\gamma_{\downarrow} = -\int_0^{2\pi}\int_0^{\theta} ½ \sin\theta \, d\phi d\theta = \pi \cos\theta$$

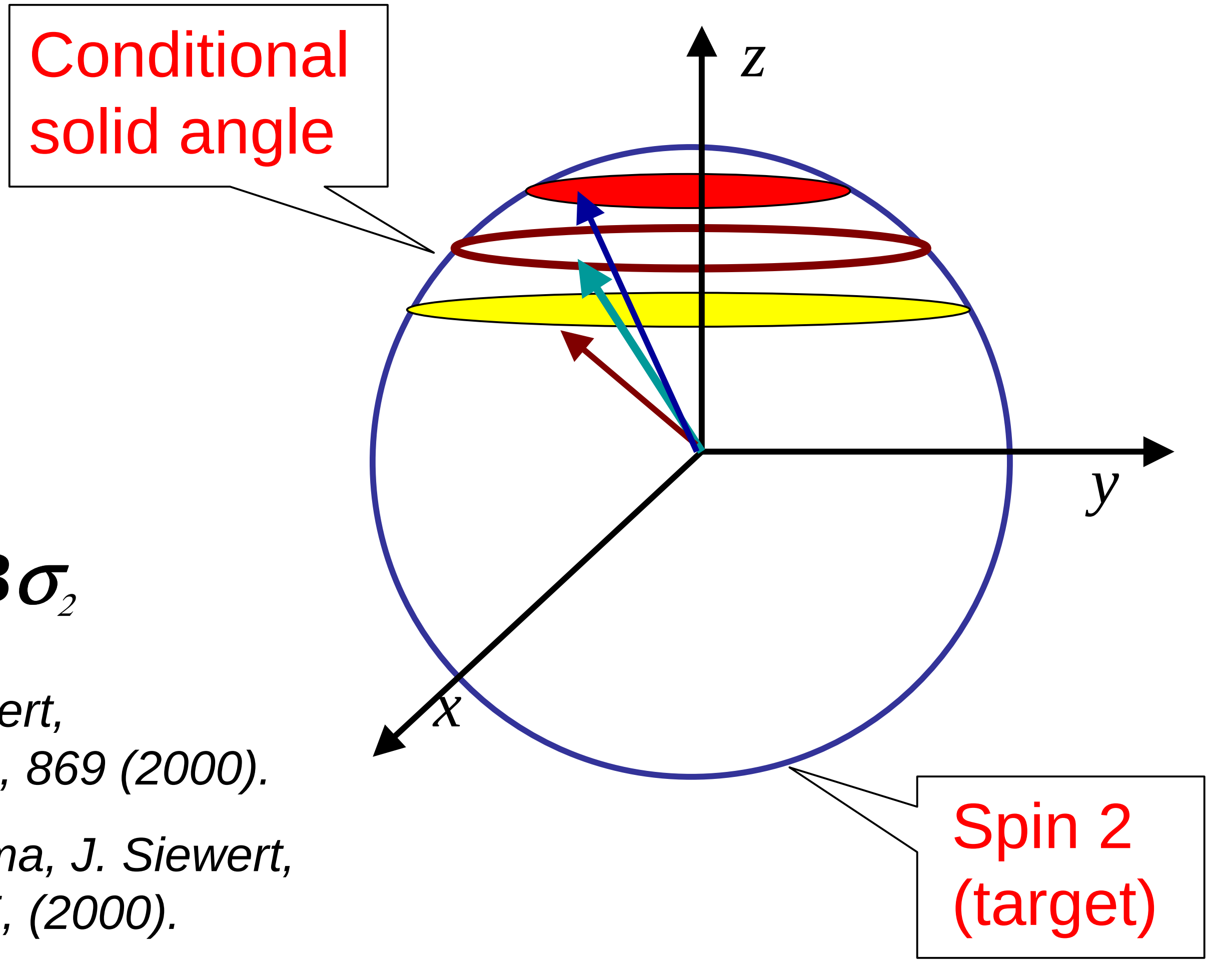
The Berry phase is equal to ½ the solid angle subtended by ***B*** at the degeneracy



CONDITIONAL PHASE SHIFT

$$H = + \tfrac{1}{2}\, \mathbf{B}_1 \cdot \boldsymbol{\sigma}_1 + \tfrac{1}{2}\, \mathbf{B}_2 \boldsymbol{\sigma}_2 + \Delta\, \mathbf{B} \boldsymbol{\sigma}_1 \boldsymbol{\sigma}_2$$





$$\mathbf{B}_{1Eff} = \tfrac{1}{2}\, \mathbf{B}_1 + \Delta\, \mathbf{B} \boldsymbol{\sigma}_2$$

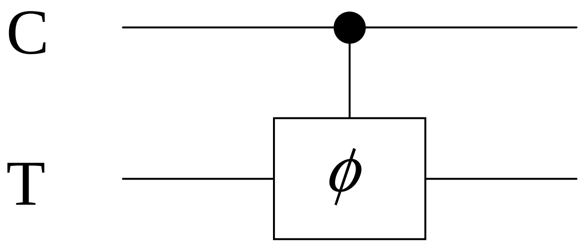
J. Jones, V. Vedral, A.K. Ekert, C.Castagnoli, NATURE, 403, 869 (2000).

G. Falci, R. Fazio, G.M. Palma, J. Siewert, V.Vedral, NATURE,407, 355, (2000).

GEOMETRIC QUANTUM PHASE GATE

The phase shift on the “target” spin (qubit) depends on the value of the “control” spin (qubit).

Can be used to implement a quantum conditional phase gate (a universal two-qubit gate)

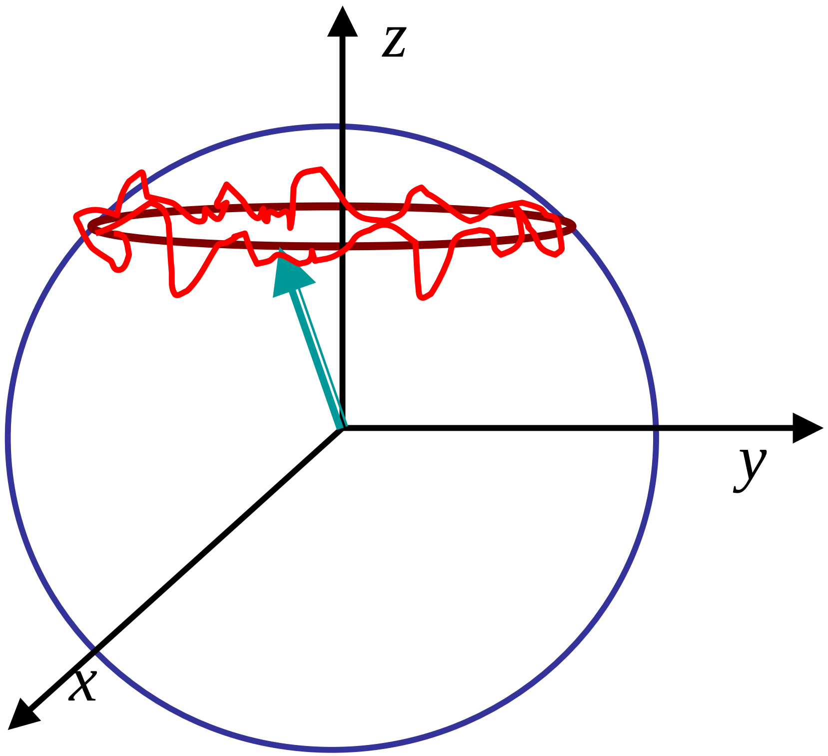


$$\begin{pmatrix} e^{i\gamma_{\uparrow}/2} & 0 & 0 & 0 \\ 0 & e^{-i\gamma_{\uparrow}/2} & 0 & 0 \\ 0 & 0 & e^{i\gamma_{\downarrow}/2} & 0 \\ 0 & 0 & 0 & e^{-i\gamma_{\downarrow}/2} \end{pmatrix}$$

With a suitable choice of path in parameter space it is possible to fix $\gamma_{\uparrow}$

FAULT TOLERANT QUANTUM COMPUTATION

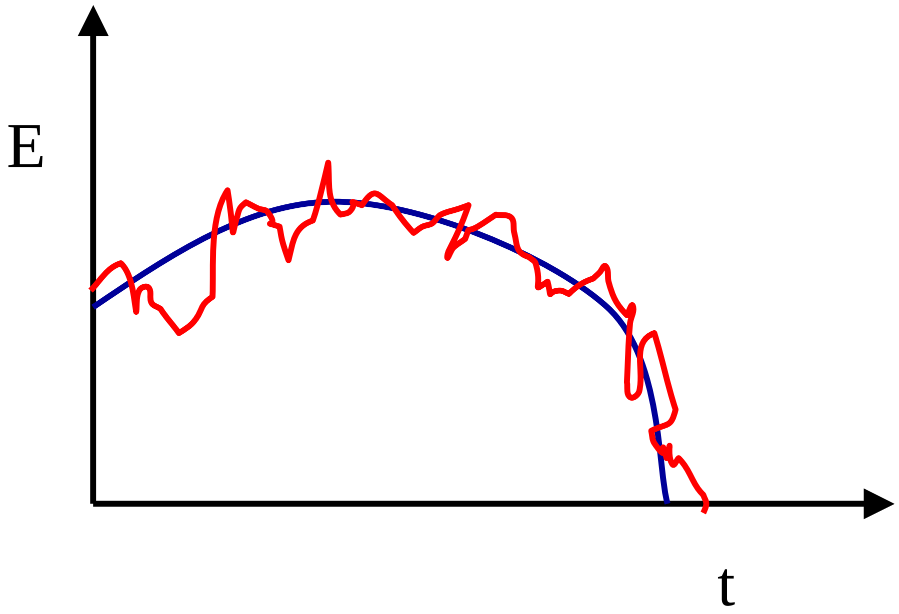
Geometric quantum computation is believed to be intrinsically more robust against random errors



As the geometric phase is proportional to the overall area traced on the unit sphere i.e. to a global property of the path in parameter space, errors with zero time average should not introduce errors

OBJECTION

- Dynamic phase $\delta = \int_0^T E(t)dt$

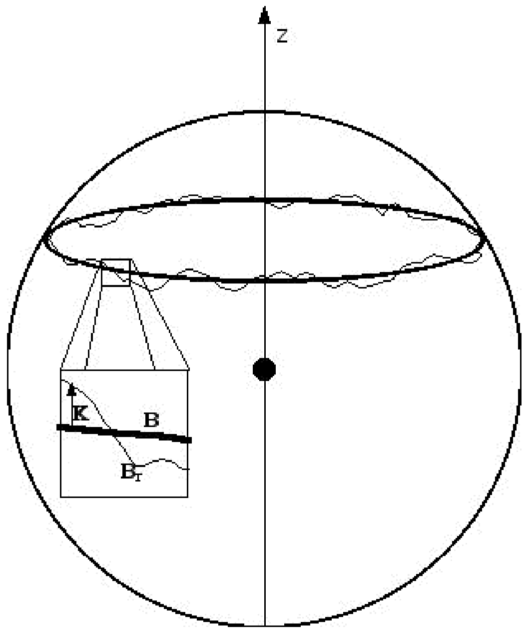


- The dynamical phase is proportional to the area of $E(t)$ vs. t
- Dynamic phase fluctuations are known to introduce decoherence

Do fluctuations play a different role in geometric and in dynamic phases?

THE NOISE MODEL

G.DeChiara G.M.Palma, Phys.Rev.Lett, ***91****, 090404 (2003*) also *quant-ph/0303155*



$$H = -\tfrac{1}{2}\,\mathbf{B}_T\cdot\boldsymbol{\sigma}$$

$$\mathbf{B}_T = \mathbf{B} + \mathbf{K}$$

Control field

Fluctuating field

A spin ½ interacting with a classical magnetic field with a small fluctuating component to model fluctuations in the control parameters

NOISE PROPERTIES

- $K \ll B$
- K is assumed to be a Ornstein –Uhlenbeck process with zero average and variance σ^2. It is therefore:
  - Gaussian
  - Markovian
  - Stationary

FIRST ORDER CORRECTIONS

First order correction the connection

$$A_\phi(\theta) \;\; \cong A_\phi(\theta_0) + \partial/\partial\theta\, A_\phi(\theta)\, \delta\theta$$
$$= ½\,(1 - \cos\theta_0 \; + \delta\theta \sin\theta_0)$$

First order correction the line element

$$\delta\phi = \phi'\, dt \cong (\phi'_0 + \delta\phi')\, dt$$

For a precession around the z *axis* $\phi'_0 = 2\pi/T$

First order correction to the Berry phase

$$\gamma \;\; = \;\; \int_0^T (A_\phi(\theta_0) + \; \delta A_\phi)(\phi'_0 + \delta\phi')\, dt$$

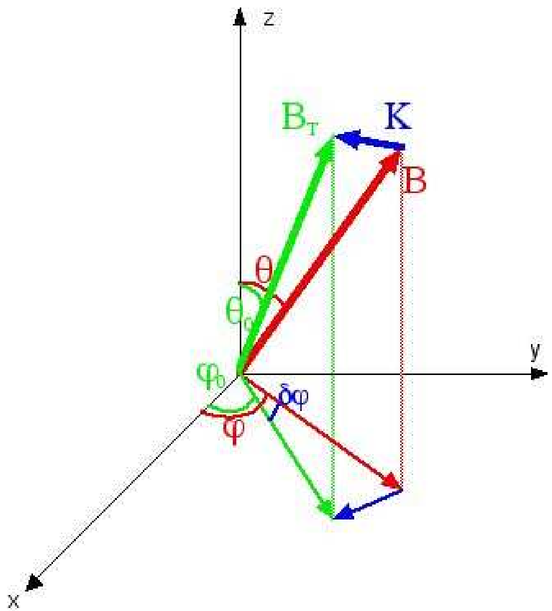
$$\cong \gamma_0 \; + 2\pi/T \int_0^T \delta A_\phi \, dt \; + A_\phi(\theta_0) \int_0^T \delta\phi' dt$$

$$\cong \; \gamma_0 \; + 2\pi/T \int_0^T \sin\theta_0 \delta\phi'\, dt \; + A_\phi(\theta_0)\; \delta\phi(T)$$

The connection fluctuates

The path does not close

APPROXIMATE EXPRESSIONS



$$\cos(\theta_0 + \delta\theta) \cong \cos\theta_0 - \delta\theta \sin\theta_0$$
$$= B_3/B + K_3/B - \mathbf{B}\cdot\mathbf{K}\, B_3/B^3$$

$$\gamma = \gamma_0 + 2\pi/T \int_0^T (K_3/B - \mathbf{B}\cdot\mathbf{K}\, B_3/B^3)dt$$

The Berry phase can be evaluated in terms of the Cartesian component of ***K***, each of which is a Gaussian random process with its own (generally different) variance.

It can be shown that the non cyclic corrections do not contribute.

ENERGY FLUCTUATIONS

The fluctuating field ***K*** introduces also fluctuations in the energy eigenvalues

$$H = -\tfrac{1}{2}\,(\mathbf{B} + \mathbf{K})\cdot\boldsymbol{\sigma}$$

To first order $$E = \pm\, 1/2\,(B + \mathbf{B}\cdot\mathbf{K} / B)$$

$$\begin{aligned}\delta &= \delta_0 + \int_0^T \mathbf{B}\cdot\mathbf{K} / B\, dt \\ &= \delta_0 + \int_0^T \delta E(t)\, dt\end{aligned}$$

DECOHERENCE

$|\Psi>_0 = a|\uparrow> + b|\downarrow>$ ϕ has a probability distribution $P(\phi)$

$|\Psi>_T = e^{i\phi} a|\uparrow> + e^{-i\phi} b|\downarrow>$

In our case the joint probability distribution for the dynamic + geometric phase is gaussian

$$\rho = \int |\Psi(\phi)><\Psi(\phi)| d\phi = \begin{pmatrix} |a|^2 & ab^* e^{i2\alpha} \exp\{-2\sigma^2\} \\ a^*b\, e^{i2\alpha\uparrow} \exp\{-2\sigma^2\} & |b|^2 \end{pmatrix}$$

Phase fluctuations generate decoherence

RESULTS

The variance grows linearly with time

In the adiabatic limit the variance is due to the dynamic contribution

The effect of noise is different in the geometric and in dynamic phases

$$\gamma \cong \gamma_0 + 2\pi/T \int_0^T \delta A_\phi \, dt$$

$$\delta \cong \delta_0 + \int_0^T \delta E(t) dt$$

A MODEL OF QUANTUM NOISE: THE SPIN-BOSON HAMILTONIAN

$$H = \tfrac{1}{2}\, \mathbf{B}(t)\cdot \boldsymbol{\sigma} + \Sigma_k\, g_k\, (a^\dagger_k + a_k)\, \sigma_x + \Sigma_k \omega_k a^\dagger_k a_k$$

- The spin is weakly coupled to a quantum environment modeled as a bath of harmonic oscillators
- The direction of the magnetic field B changes adiabatically in time

THE MOVING FRAME

$$H = \Sigma_n E_n(t)|n_t><n_t| \qquad\qquad U = \Sigma_n |n_0><n_t|$$

The time evolution of the state $/\psi'> = U/\psi>$
is generated by

$$H' = UHU^\dagger - iU\partial_t U^\dagger =$$

$$= \Sigma_n E_n(t)|n_0><n_0| - i\,\Sigma_{n,m} |n_0><m_0|\,<n_t|\,\partial_t\,|m_t>$$

THE ADIABATIC HAMILTONIAN

Adiabatic approximation: assume that transitions are negligible $\langle n_t | \partial_t | m_t \rangle = 0$

$$H_{ad} = \Sigma_n \left(E_n(t) - i \langle n_t | \partial_t | n_t \rangle\right) | n_0 \rangle \langle n_0 |$$

At the end of a cyclic evolution of period T

$$| n(t) \rangle = exp\ i \int_0^T \left(E_n(t) - i \langle n_t | \partial_t | n_t \rangle\right) | n(0) \rangle$$

$$= exp\{ i \delta \} exp\{ i \gamma \} | n(0) \rangle$$

THE DECAY CONSTANTS

$$\gamma_{||} = \pi \Sigma_k |g_k|^2 \delta(\omega_k)$$

$$\gamma_{\perp} = \pi \Sigma_k |g_k|^2 \delta(\omega_k - \omega_o)$$

THE ADIABATIC SPIN BOSON HAMILTONIAN

G.DeChiara, A.Łozinski & G.M.Palma quant-ph/0410183

For a slowly precessing field at an angle θ, choosing as $\boldsymbol{z}$ axis the direction of $\boldsymbol{B}$

$$H_{ad} = ½\, \omega_o \sigma_z + \Sigma_k \omega_k a^\dagger{}_k a_k$$

$$+ \Sigma_k\, g_k\, (a^\dagger{}_k + a_k)\, (\cos\theta\, \sigma_x + \sin\theta\, \sigma_z\,)$$

$$\omega_o = B + \Omega \cos\theta(0)$$

The quantum noise acts both in the “parallel” and in the “orthogonal” direction of $\boldsymbol{B}$

THE LAMB SHIFT

$$\lambda = \Sigma_k |g_k|^2 \{ P/(\omega_o - \omega_k) + P/(\omega_o + \omega_k) \}$$

$$\sim \Sigma_k |g_k|^2 \{ P/(B - \omega_k) + P/(B + \omega_k) \} +$$

$$- \Omega \cos\theta \Sigma_k |g_k|^2 \{ P/(\omega_o - \omega_k)^2 + P/(\omega_o + \omega_k)^2 \}$$

The Lamb shift is due to virtual transitions due to the σ_x term in the interaction Hamiltonian

$$|\Psi_\downarrow\rangle = |\downarrow\rangle_z |vac\rangle + |\uparrow\rangle_z \Sigma_k \cos\theta\, g_k |1_k\rangle\ 1/(\omega_o + \omega_k)$$

$$|\Psi_\uparrow\rangle = |\uparrow\rangle_z |vac\rangle + |\downarrow\rangle_z \Sigma_k \cos\theta\, g_k |1_k\rangle\ 1/(\omega_o - \omega_k)$$

The overall phase difference between the parallel and antiparallel energy eigenstates

$$\Phi_T = \Phi_D + \Phi_G$$

$$\Phi_D = [B + \cos^2\theta \, \Sigma_k \, |g_k|^2 (P/(\omega_o - \omega) + P/(\omega_o + \omega))] \, T$$

$$\Phi_G = 2\pi \cos\theta [1 \; -$$

$$- \Omega \cos^2\theta \, \Sigma_k \, |g_k|^2 (P/(\omega_o - \omega)^2 + P/(\omega_o + \omega)^2)]$$

- During a virtual transition the state acquires an opposite geometric phase
- The Berry phase is diminished by the probability of a virtual transition

DISSIPATION INDUCED BERRY PHASE

A change in perspective: is it possible to generate a Berry phase by measn of a dissipative dynamics

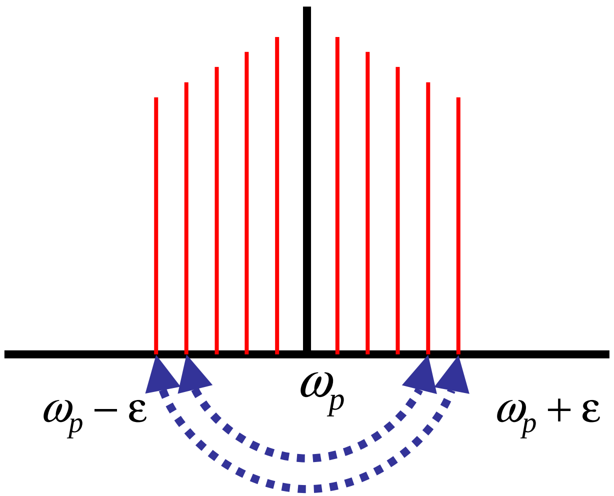
$$\partial_t \rho = -\tfrac{1}{2}\gamma(\rho R^\dagger R + R^\dagger R\rho - R^\dagger \rho R)$$

BROADBAND SQUEEZED VACUUM

$|vac_{sq}> = U|vac>$

Squeezed states are generated in nonlinear optical media by a strong pump field of frequency $2\omega_p$

$$U = \exp\sum_{\varepsilon}\{\eta_{\varepsilon} a^{\dagger}(2\omega_p - \varepsilon) a^{\dagger}(2\omega_p + \varepsilon) - \eta_{\varepsilon}^{*} a(2\omega_p - \varepsilon) a(2\omega_p + \varepsilon)\}$$



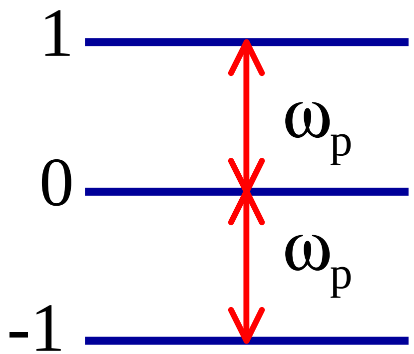
sideband modes around ω_p are paiwise entangled

$$<a^{\dagger}(\omega_k) a(\omega_{k'})> = \sinh r_{\omega}\, \delta(\omega_k - \omega_{k'})>$$

$$<a^{\dagger}(\omega_k) a^{\dagger}(\omega_{k'})> = \sinh r_{\omega} \cosh r_{\omega}\, e^{i\phi}\, \delta(2\omega_p - \omega_k - \omega_{k'})>$$

$$\eta_{\omega} = r_{\omega} e^{i\phi}$$

THE EQUILIBRIUM STATE



$$|\Psi_{eq}> = (\cosh\eta_\omega / \sinh 2\eta_\omega)|1> + (\sinh\eta_\omega / \sinh 2\eta_\omega)|1>$$

CONCLUSIONS

G.M.Palma and P.L. Knight Phys. Rev.A 39, 1962, (1989)
A.K.Ekert, G.M.Palma, S.M.Barnett and P.L.Knight,
Phys.Rev.A 39, 6026, (1989)