SMR.1568- 3

SUMMER SCHOOL IN COSMOLOGY AND ASTROPARTICLE PHYSICS

28 June - 10 July 2004

Non-perturbative phenomena in gauge theories (III)

Valerie A. RUBAKOV
Russian Academy of Sciences
Institute for Nuclear Research
117 312 Moscow
RUSSIAN FEDERATION

Please note: These are preliminary notes intended for internal distribution only.

II-1

(3+1) DIMENSIONS

Simplest and physically relevant example:

SU(2) THEORY WITH HIGGS doublet.
(SU(2) part of Standard Model; comment on U(1) later).

Notations:

⊛ gauge field, matrix, anti-Hermitean

$$A_\mu = -ig \frac{\tau^a}{2} A_\mu^a \quad , \; a = 1,2,3$$

$\frac{\tau^a}{2}$: generators of SU(2).

Field strength:

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]$$

⊛ Higgs doublet

$$\phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \quad , \; \phi_1(x), \phi_2(x) = \text{complex}.$$

⊛ $D_\mu \phi = \partial_\mu \phi + A_\mu \phi \equiv (\partial_\mu + A_\mu)\phi$

⊛ Gauge transformations:

gauge function $\omega(x) \in SU(2)$ at every x^μ.

$$A_\mu(x) \to \omega A_\mu \omega^{-1} + \omega \partial_\mu \omega^{-1}$$

$$\phi(x) \to \omega(x)\, \phi(x).$$

ACTION (Minkowski signature)

$$S = \int \left[\frac{1}{2g^2} \operatorname{Tr} F_{\mu\nu} F^{\mu\nu} + (D_\mu \phi)^+ (D_\mu \phi) - V(\phi) \right] d^4x$$

$- \frac{1}{4} F^a_{\mu\nu} F^a_{\mu\nu}$ in components.

$$V(\phi) = \frac{\lambda}{2} \left(\phi^+ \phi - v^2 \right)^2$$

Classical vacua $A_0 = 0$ gauge

(1) TRIVIAL: $A_i = 0$

$$\phi = \phi_0 = \begin{pmatrix} 0 \\ v \end{pmatrix}$$

Perturbations about this ~~(and any other~~ vacuum (and any gauge copy of it):

- vector bosons, massive, $m_V = \frac{1}{\sqrt{2}} g v$

- Higgs boson, $m_H = \sqrt{2\lambda} \cdot v$

(2) OTHER vacua: gauge copies of trivial one,

$$A_i = \omega \partial_i \omega^{-1}$$

$$\phi = \omega \phi_0$$

$\omega(\vec{x})$, time-independent, belongs to $SU(2)$ at each point $\vec{x}$.

Compactify our space to 3-sphere S^3

$\Updownarrow$

REQUIRE $\vec{A}=0$, $\Phi=\Phi_0$ at $|\vec{x}|\to\infty$

Then $\omega\to\mathbb{1}$ at $|\vec{x}|\to\infty$

Space = topologically 3-sphere.

$\omega(\vec{x})$: mapping $S^3_{space}\to SU(2)$.

Characterized by integer topological #.

Case of gauge group $SU(2)=S^3_{\text{gauge group}}$

$\Uparrow$

$\omega=\sigma_\mu v_\mu$

$\sigma_0=1$, $\sigma_i=i\tau_i$

$\sum_{\mu=0}^{3} v_\mu v_\mu=1$

One-to-one

ω: $S^3_{space}\to S^3_{\text{gauge group}}$

n = degree of the map.

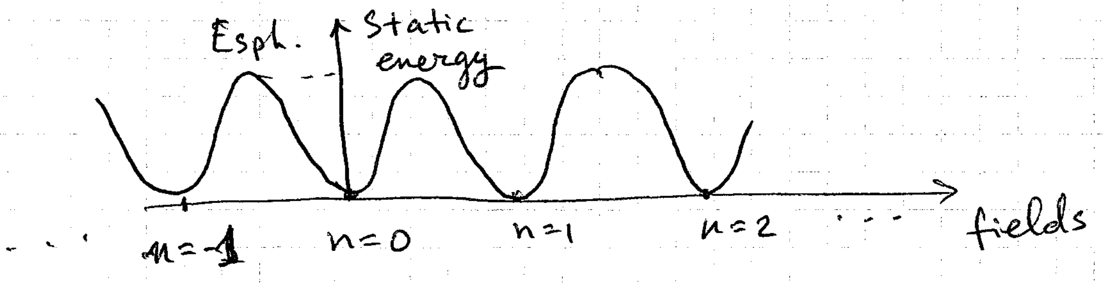
— o —

<u>NB</u>: Generally for simple compact Lie group G : $\pi_3(G)=\mathbb{Z}$,

— o —

mappings $S^3\to G$ divide into disjoint sets labeled by integer $n\in\pi_3(G)$.

Same structure as in (1+1)-dim. Abelian Higgs.



NB: TRIVIAL VACUUM STRUCTURE FOR $U(1)$ THEORY IN (3+1)-dim's
$S^3 \to U(1)$ always TOPOLOGICALLY TRIVIAL

Sphaleron: STATIC unstable solution (lump) of classical field eqs.

$$E_{sph} = \text{height of barrier.}$$

Found only numerically.

* Let us estimate E_{sph} by dimensional analysis

$$E_{static} = \int d^3x \left[-\frac{1}{2g^2} \mathrm{Tr} F_{ij}^2 + (D_i\phi)^\dagger (D_i\phi) + \frac{\lambda}{2}(\phi^\dagger\phi - v^2)^2 \right]$$

Idea: get rid of as many parameters in the integrand as possible, by rescaling fields and coordinates.

$$\phi(\vec{x}) = v\, f(\vec{y}) \qquad y^i = gv\, x^i$$

$$A(\vec{x}) = gv\, a(\vec{y})$$

Then
$$F_{ij} = \partial_i^{(x)} A_j - \partial_j^{(x)} A_i + [A_i, A_j]$$
$$= (gv)^2 \left(\partial_i^{(y)} a_j - \partial_j^{(y)} a_i\right) + (gv)^2 [a_i, a_j]$$
$$= (gv)^2 \cdot a_{ij}$$

$$D_i^{(x)}\phi = gv\, D_i^{(y)}(vf) = gv^2\, D_i^{(y)} f$$
$$\uparrow \; \partial_i^{(y)} + a_i$$

$$V(\phi) = \frac{\lambda}{2} v^4 (f^\dagger f - 1)^2$$
$$d^3x = \frac{1}{(gv)^3} d^3y$$

Combine:

$$E_{static} = g^2 v^4 \cdot \frac{1}{g^3 v} \int d^3y \left[-\frac{1}{2} \mathrm{Tr}\, a_{ij}^2 + |D_i^{(y)} f|^2 + \frac{\lambda}{2g^2} (f^+ f - 1)^2 \right]$$

$\Downarrow$ at saddle point $a \sim 1,\ f \sim 1,\ y \sim 1$ for $\lambda \sim g^2$.

$$E_{sph} = \frac{v}{g} \mathcal{F}\left(\frac{\lambda}{g^2}\right)$$

Unknown. Need actual solution

Recall $m_V \sim g v$

$\Downarrow$

$$E_{sph} = \frac{m_V}{g^2} \mathcal{F}\left(\frac{\lambda}{g^2}\right)$$

Traditionally parametrized as (since $\frac{m_H}{m_V} \sim \frac{\sqrt{\lambda}}{g}$)

$$\boxed{E_{sph} = \frac{2 m_V}{\alpha_V} B\left(\frac{m_H}{m_V}\right)} \quad ; \quad \alpha_V = \frac{g^2}{4\pi}$$

Numerically: B changes from 1.56 ($m_H \ll m_V$) to 2.27 ($m_H \gg m_V$)

NB: $E_{sph} \gg m_V, m_H$ for $m_H \sim m_V$

ELECTROWEAK THEORY: $m_V = 80$ GeV

$$\alpha_V = \frac{g^2_{SU(2)}}{4\pi} \approx \frac{1}{30}$$

$$E_{sph} \simeq 10 \text{ TeV}.$$

Fermions :

Simplest representation of $SU(2)$: doublet.

Same story as in (1+1)-dim. Abelian Higgs model :

$$\Delta N_L = n_{final} - n_{initial}$$ FOR LEFT-HANDED FERMIONS

$$\Delta N_R = - \Delta N_L$$ RIGHT-HANDED.

- The same for any $SU(N)$ for fermions in fundamental representation

— o —

- QCD: u_L, u_R, d_L, d_R, etc. have THE SAME COLOR REP. (TRIPLETS)

SELECTION RULE IN CHIRAL LIMIT $m_u = m_d = m_s = 0$

$$\Delta N_{u_L} = - \Delta N_{u_R} = \Delta N_{d_L} = - \Delta N_{d_R} = \Delta N_{s_L} = - \Delta N_{s_R}$$

Baryon # $= N_{u_L} + N_{u_R} + \dots + N_{s_L} + N_{s_R}$

CONSERVED: $\Delta(N_{u_L} + N_{u_R}) = 0$, ETC.

$(T_3)_{VECTOR} = (N_{u_L} + N_{u_R}) - (N_{d_L} + N_{d_R})$ CONSERVED

$(T_3)_{AXIAL} = (N_{u_L} - N_{u_R}) - (N_{d_L} - N_{d_R})$ conserved

AXIAL $U(1) = (N_{u_L} - N_{u_R}) + (N_{d_L} - N_{d_R}) + (N_{s_L} - N_{s_R})$

VIOLATED (BY 3 UNITS)

Axial $U(1)$, $j^5_\mu = \sum_{\text{flavors}} \bar{q}\gamma^\mu\gamma^5 q$

Naively a symmetry. In fact not.

Consequence: 8 Goldstone bosons ~~insted~~ instead of 9.

TRUE IN NATURE ($\pi^0, \pi^\pm, K^0, \bar{K}^0, K^\pm, \eta$ are light
η' is heavy).

— o —

! • ELECTROWEAK Theory

Left-handed doublets

$$\left.\begin{matrix} \begin{pmatrix} u \\ d \end{pmatrix}_L \times 3 \text{ COLOR} \\ \\ \begin{pmatrix} \nu_e \\ e \end{pmatrix}_L \end{matrix}\right\} \times 3 \text{ families}$$

No right-handed doublets.

Baryon number changes by (recall $B = \frac{1}{3} N_{\text{quarks}}$)

$$\Delta B = \frac{1}{3} \times 3 \times 3 \times (n_{fin} - n_{in}) = 3\,(n_{fin} - n_{in})$$

def. B, color, family

Lepton number of each family

$$\Delta L_e = 1 \times (n_{fin} - n_{in}).$$

etc. for every family

Selection rules

$$\Delta L_e = \Delta L_\mu = \Delta L_\tau = \frac{1}{3}\,\Delta B \left(= n_{fin} - n_{in}\right).$$

* $(B - L)$ is conserved $\quad (L = L_e + L_\mu + L_\tau)$
* $(L_e - L_\mu)$ is conserved
* $(L_e - L_\tau)$ is conserved.

Three quantum numbers are conserved out of naive four.

Baryon number and Lepton number are not conserved separately.

RATE AT HIGH TEMPERATURES.

Relatively low temperatures: naive estimate

$$\Gamma = f\left(\alpha_W, \frac{m_H}{m_W}, T\right) \cdot T^4 \, e^{-\frac{E_{sph}}{T}}$$

rate per unit time per unit volume

Boltzmann suppression.

Too naive.

$$E_{sph} = \frac{2 m_W}{\alpha_W} \, B\left(\frac{m_H}{m_W}\right)$$

$$m_H \approx \sqrt{\lambda}\, v \qquad ; \qquad \frac{m_H}{m_W} \simeq \frac{\sqrt{\lambda}}{g_W}$$
$$m_V \approx g_W \cdot v$$

Point (lectures by Holman)

v depends on temperature

[g_W and λ too, but only logarithmically].

Less naive estimate

$$\boxed{\Gamma \propto T^4 e^{-\frac{E_{sph}(T)}{T}}}$$

$$E_{sph}(T) = \frac{2 m_W(T)}{\alpha_W} \, B\left(\frac{m_H}{m_W}\right)$$

Actually, fairly reasonable order-of-magnitude estimat

Electroweak phase transition:
(loosely speaking) at $T > T_c \Rightarrow m_W(T) = 0$
since
(lectures by Holman). $v(T) = 0$

Rate of B-violation high above T_c,
and even just below T_c

$$\Gamma \gg T^3 \, H(T)$$

$H(T)$ = (universe's lifetime)$^{-1}$ = expansion rate.

T^3 = typical (3-volume)$^{-1}$

(TRUE IN STANDARD MODEL AND IN MOST OF PARAMETER SPACE OF MSSM).

NO GENERATION OF BARYON ASYMMETRY AT ELECTROWEAK EPOCH

SINCE

AFTER PHASE TRANSITION BARYON # VIOLATION IS STILL IN EQUILIBRIUM.

NO ASYMMETRIES CAN BE GENERATED IN THERMAL EQUILIBRIUM.

$$\boxed{\text{BOTTOM LINE: ANOMALOUS ELECTROWEAK BARYON \# VIOLATION IS FAST AT } T \gtrsim 100 \text{ GeV, BUT BY ITSELF CANNOT GENERATE BARYON ASYMMETRY (in SM AND MOST LIKELY IN MSSM).}}$$

II - 11

Why baryon number is so well conserved in vacuo (i.e., at $T=0$, today)?
at low energies

Tunneling

Need instanton, solution to Euclidean field equations connecting vacua with $n=0$ $(t=-\infty)$ and $n=1$ $(t=+\infty)$.

Gauge-invariant characteristic:

$$Q = -\frac{1}{16\pi^2}\int d^4x \; Tr\left[\frac{1}{2}\varepsilon_{\mu\nu\lambda\rho} F_{\mu\nu} F_{\lambda\rho}\right] = -\frac{1}{16\pi^2}\int d^4x \, Tr F_{\mu\nu}$$

— o —

Sketch of derivation:

(1) $Tr(F_{\mu\nu}\tilde{F}_{\mu\nu}) = \partial_\mu K_\mu$

$$K_\mu = \varepsilon_{\mu\nu\lambda\rho}\, Tr\left(F_{\nu\lambda}A_\rho - \frac{2}{3}A_\nu A_\lambda A_\rho\right)$$

(2) $$Q = -\frac{1}{16\pi^2}\int d^4x \; \partial_\mu K_\mu$$

$$= -\frac{1}{16\pi^2}\int_{\substack{S^3 \\ (x^\mu)^2\to\infty}} K_\mu \, d\sigma^\mu$$

(3) At $(x^\mu)^2 \to \infty$: $F_{\mu\nu} = 0$

Choose gauge $A_0 = 0$

$$K_i = 0 \qquad i = 1, 2, 3$$

$$K_0 = -\frac{2}{3}\,\varepsilon_{ijk}\ \mathrm{Tr}\left(A_i A_j A_k\right)$$

$$\Downarrow$$

$$Q = \frac{1}{24\pi^2}\left[\int \varepsilon_{ijk}\ \mathrm{Tr}\left(A_i A_j A_k\right) d^3x\right]_{t=-\infty}^{+\infty}$$

(4) At $t = \pm\infty$ A_i is pure gauge (vacuum).

$A_i(t = -\infty) = 0$, trivial vacuum

$A_i(t = +\infty) = \omega\,\partial_i\,\omega^{-1}$

$$\Downarrow$$

$$Q = \frac{1}{24\pi^2}\int \varepsilon_{ijk}\ \mathrm{Tr}\left(\omega\partial_i\omega^{-1}\cdot\omega\partial_j\omega^{-1}\cdot\omega\partial_k\omega^{-1}\right) d^3x$$

THIS IS TOP. NUMBER OF ω.

(5) ω has topological number n

$$\Downarrow$$

$$Q = n$$

—o—

Thus, WANT TO FIND minimum of ACTION WITH $Q = 1$.

BPST TRICK

$$-\operatorname{Tr}\left(F_{\mu\nu} - \tilde{F}_{\mu\nu}\right)^2 \geq 0$$

$$/\!/$$

$$-2\operatorname{Tr}\left(F_{\mu\nu}^2\right) + 2\operatorname{Tr}\left(\tilde{F}_{\mu\nu}^2\right) \geq 0$$

$$S_{gauge} = \int -\frac{1}{2g^2}\operatorname{Tr} F_{\mu\nu}^2 \, d^4x$$

$$\geq -\frac{1}{2g^2}\int \operatorname{Tr}\left(F_{\mu\nu} \cdot \tilde{F}_{\mu\nu}\right) d^4x$$

$$= \frac{8\pi^2}{g^2}\left(-\frac{1}{16\pi^2}\int \operatorname{Tr}\left(F_{\mu\nu}\tilde{F}_{\mu\nu}\right) d^4x\right) = \frac{8\pi^2}{g}Q$$

$$S_{gauge} \geq \frac{8\pi^2}{g^2}Q$$

Equality : $F_{\mu\nu} = \tilde{F}_{\mu\nu}$

First-order equation!

(unlike original second order $D_\mu F_{\mu\nu} = 0$)

Pure gauge theory:

Solve $F_{\mu\nu} = \tilde{F}_{\mu\nu} \Rightarrow$ find instanton.

Gauge - Higgs theory : a bit trickier.

Solution (in components) in pure Yang-Mills

$$A_\mu^a = \frac{2}{g} \frac{1}{x^2+\rho^2} \eta_{\mu\nu a} x_\nu \qquad \Big\{ \; A_\mu = -ig\frac{\tau^a}{2} A_\mu^a$$

$x^2 = x_\mu x_\mu$, Euclidean signature

ρ: arbitrary parameter.

$$S_{inst} = \frac{8\pi^2}{g^2} = \frac{2\pi}{\alpha} \qquad \alpha = \frac{g^2}{4\pi}$$

Tunneling probability

$$\Gamma \propto e^{-2S_{inst}} = e^{-\frac{4\pi}{\alpha}}$$

In Electroweak Theory

$$\alpha = \alpha_W \simeq \frac{1}{30}$$

$$\Downarrow$$

$$\Gamma \propto 10^{-160}$$

Negligibly small.

THE ESTIMATE $\Gamma \propto e^{-2S_{inst}}$ VALID AT LOW ENERGIES ONLY ($E \ll E_{sph} \sim 10\,\mathrm{TeV}$ in EW THEORY)

WHAT HAPPENS IN COLLISIONS OF HIGHLY ENERGETIC PARTICLES, $E \gtrsim E_{sph}$?