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SUMMER SCHOOL IN COSMOLOGY AND ASTROPARTICLE PHYSICS

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An introduction to CMB anisotropies (III)

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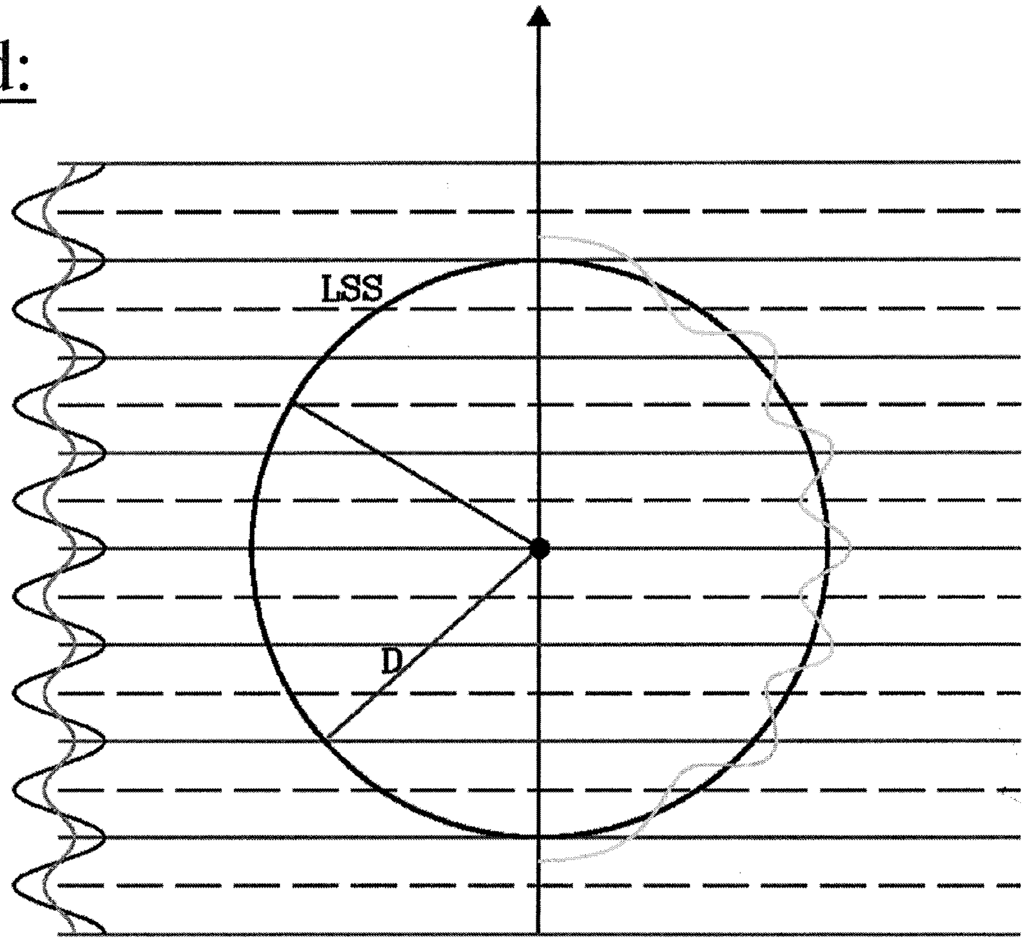
Please note: These are preliminary notes intended for internal distribution only.

How to proceed:

- Consider one Fourier mode at a time and calculate the state of the fluid at recombination
- Calculate the map it would produce
- superimpose the effect of all modes



$\Delta_T(k,\tau)\ \cos(kx)$

Before Decoupling

$\mathrm{Cos}(w\tau)\cdot\cos(k\tau)$

Perturbation Equations

Calculation of the state of the fluid at recombination

Note that one needs to choose a gauge

We will follow Ma & Bertschinger (1995). Purturbed metric:

$$ds^2 = a^2(\tau)\left\{-(1+2\psi)d\tau^2 + (1-2\phi)dx^i dx_i\right\} . \qquad (11)$$

Purturbed Einstein equations:

$$k^2\phi + 3\frac{\dot{a}}{a}\left(\dot{\phi} + \frac{\dot{a}}{a}\psi\right) = 4\pi G a^2 \delta T^0{}_0 , \qquad (12)$$

$$k^2\left(\dot{\phi} + \frac{\dot{a}}{a}\psi\right) = 4\pi G a^2(\bar{\rho} + \bar{P})\theta , \qquad (13)$$

$$\ddot{\phi} + \frac{\dot{a}}{a}(\dot{\psi} + 2\dot{\phi}) + \left(2\frac{\ddot{a}}{a} - \frac{\dot{a}^2}{a^2}\right)\psi + \frac{k^2}{3}(\phi - \psi) = \frac{4\pi}{3} G a^2 \delta T^i{}_i , \qquad (14)$$

$$k^2(\phi - \psi) = 12\pi G a^2(\bar{\rho} + \bar{P})\sigma . (15)$$

Perturbation Equations

Purturbed energy momentum tensor:

$$
\begin{aligned}
T^0{}_0 &= -(\bar\rho + \delta\rho)\,, \\
T^0{}_i &= (\bar\rho + \bar P)v_i = -T^i{}_0\,, \\
T^i{}_j &= (\bar P + \delta P)\delta^i{}_j + \Sigma^i{}_j\,, \qquad \Sigma^i{}_i = 0\,.
\end{aligned}
\tag{16}
$$

Perturbation Equations

Cold Dark Matter:

$$\dot{\delta}_c = -kv_c + 3\dot{\phi}\,,$$
$$\dot{v}_c = -\frac{\dot{a}}{a}v_c + k\psi\,. \tag{17}$$

Perturbation Equations

Photons and baryons are coupled thorugh Thomson scattering

For the baryons:

$$\dot{\delta}_b = -kv_b + 3\dot{\phi}\,,$$

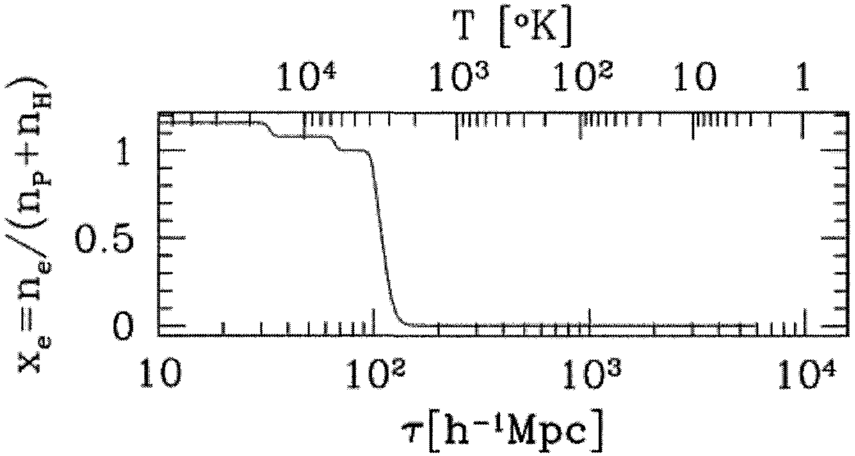
$$\dot{v}_b = -\frac{\dot{a}}{a}v_b + k\psi + c_s^2 k\delta_b + \frac{4\bar{\rho}_\gamma}{3\bar{\rho}_b} a n_e x_e \sigma_T (v_\gamma - v_b)\,. \tag{18}$$

For Photons:

$$\dot{\delta}_\gamma = -\frac{4}{3}kv_\gamma + 4\dot{\phi}\,,$$

$$\dot{v}_\gamma = k(\frac{\delta_\gamma}{4} - \sigma_\gamma) + k\psi + a n_e x_e \sigma_T (v_b - v_\gamma)\,. \tag{19}$$

Recombination



$$\lambda_T = (a n_e \sigma_T)^{-1} \approx 2 \text{ Mpc } x_e^{-1} \left[(1+z)/1000\right]^{-2}$$

Tight Coupling Approximation

At early times photons and baryons can be treated as a single fluid. Expand above equations in powers of $1/\lambda_T = 1/(an_e x_e \sigma_T)$ to get:

$$\ddot{\delta}_\gamma + \frac{\dot{R}}{(1+R)}\dot{\delta}_\gamma + k^2 c_s^2 \delta_\gamma = F$$

$$F = 4[\ddot{\phi} + \frac{\dot{R}}{(1+R)}\dot{\phi} - \frac{1}{3}k^2\psi]$$

$$\dot{\delta}_\gamma = -\frac{4}{3}kv_\gamma + 4\dot{\phi}$$

$$R \equiv \frac{3\rho_b}{4\rho_\gamma} \approx 0.63(\frac{\Omega_b h^2}{0.02})(\frac{1000}{1+z})(T_{cmb}/2.7K)^{-4} \quad (20)$$

Initial Conditions

Summary: for the growing mode of “adiabatic” initial conditions we get:

$$
\begin{aligned}
\phi &= \phi_* \\
\delta_\gamma &= -2\phi_* \\
\delta_c &= \delta_b = \frac{3}{4}\delta_\gamma \\
v_c &= v_b = v_\gamma = 0,
\end{aligned} \qquad (27)
$$

There is a growing and a decaying mode so even if the system starts in a random superposition of both modes, after a while the growing mode dominates.

Tight Coupling Approximation

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- + Initial conditions
- + Assume matter dominations ⟶ ϕ = constant

Solution for photon-baryon fluid

The solution for the photon-baryon perturbations is:

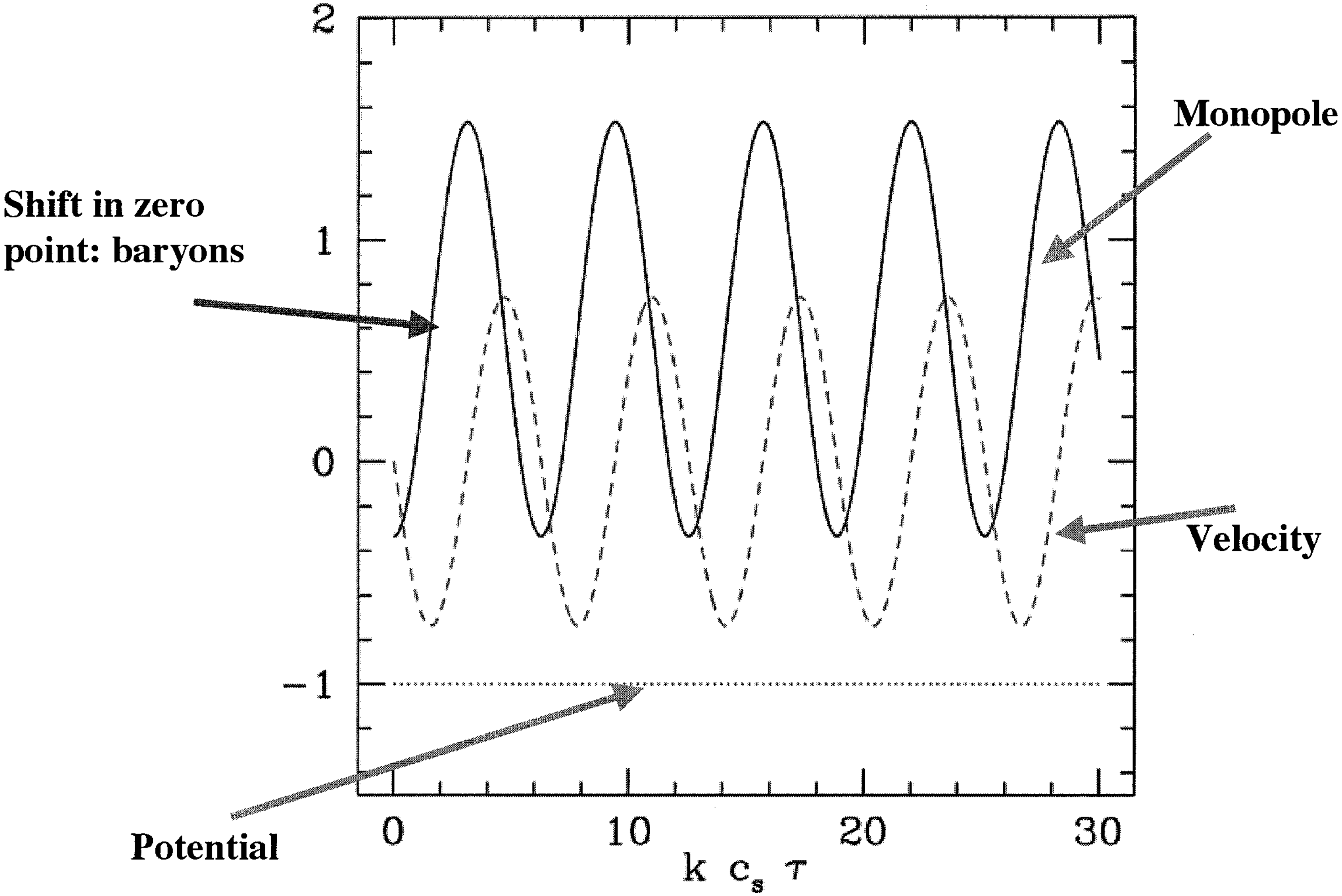
$$\begin{aligned} \frac{\delta_\gamma}{4} + \psi &= \frac{\phi_*}{3}(1+3R)\cos(kc_s\tau_R) - \phi_* R \\ v_\gamma &= -\phi_*(1+3R)c_s \sin(kc_s\tau_R). \end{aligned} \tag{28}$$

where ϕ_* describes the initial seeds. We will assume that the initial ϕ is a **Gaussian random field**, which is fully specified by its power spectrum:

$$\langle \phi(\vec{k}_1)\phi(\vec{k}_2)\rangle = \delta^D(\vec{k}_1 + \vec{k}_2)P_\phi(k_1)$$

$$k^3 P_\phi(k) = Ak^{n-1} \qquad n \approx 1 \tag{29}$$

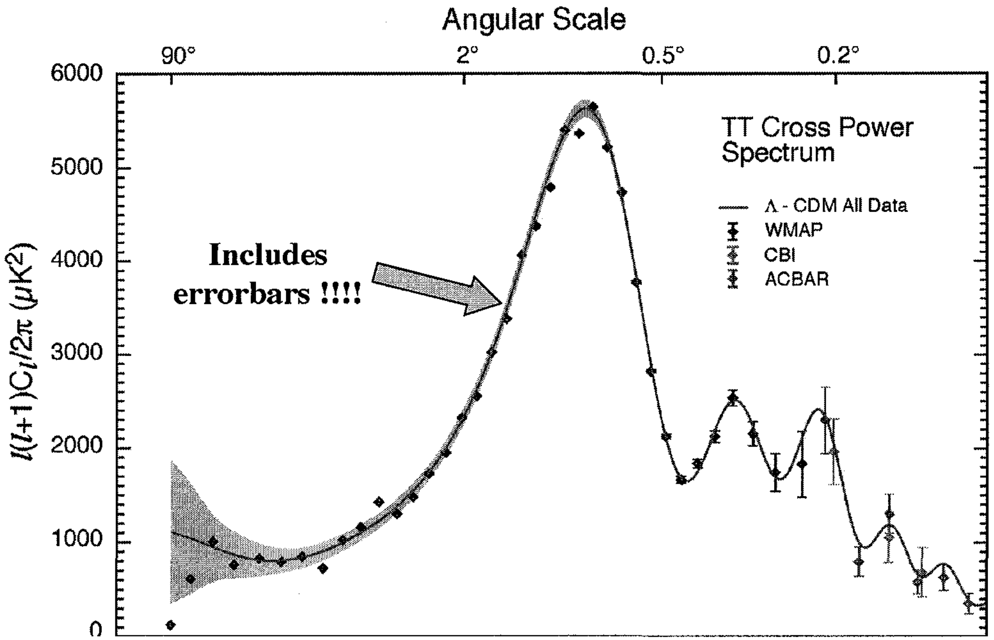
Acoustic Oscillations



WMAP Spectra

We now know what the state of the fluid was at recombination we still need to connect it with what we observe.

What is the relation of the peaks in the previous transparency to those in the observed C_l ?



The Boltzmann equation

Connecting to what we see on the sky

We need more detail in our description of the radiation field. Consider the distribution function $f = f(\vec{x}, \tau, p, \hat{n})$, which satisfies the Boltzmann equation,

$$\frac{Df}{D\tau} = 0$$

$$\frac{\partial f}{\partial \tau} + \frac{\partial x^i}{\partial \tau}\frac{\partial f}{\partial x^i} + \frac{\partial p}{\partial \tau}\frac{\partial f}{\partial p} + \frac{\partial n^i}{\partial \tau}\frac{\partial f}{\partial n^i} = 0 \tag{30}$$

From the Geodesic Equation:

$$\frac{1}{p}\frac{dp}{d\tau} = -\frac{\dot{a}}{a} + \dot{\phi} - \hat{n} \cdot \nabla\psi \tag{31}$$

The Boltzmann equation

Change of variables:

$$f = \frac{1}{e^{\alpha} - 1}; \quad \alpha = \frac{p}{k\bar{T}[1 + \Delta_T(\vec{x}, \hat{n}, \tau)]} \tag{32}$$

and to first order,

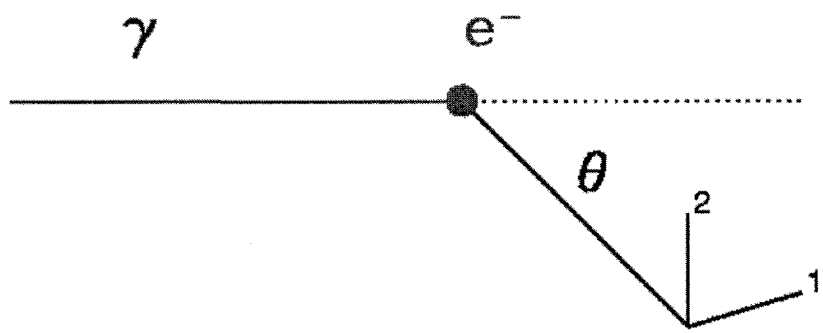
$$\frac{\partial \Delta_T}{\partial \tau} + n^i \frac{\partial \Delta_T}{\partial x^i} = \dot{\phi} - \hat{n} \cdot \nabla \psi. \tag{33}$$

$$\frac{d}{d\tau}(\Delta_T + \psi) = \dot{\phi} + \dot{\psi}$$

Conserved along the line of sight in the matter era

Source of the ISW effect

Thomson Scattering



$$I_1 \propto \sigma_{Th} \cos^2\theta$$

$$I_2 \propto \sigma_{Th}$$

The Boltzmann equation

We need to include Thomson scattering:

$$\frac{Df}{D\tau} = \frac{\partial f}{\partial \tau}|_{coll}$$

$$\frac{\partial f}{\partial \tau}|_{coll} = -an_e\sigma_T f + an_e\sigma_T \int P(\hat{n},\hat{n}')f(\hat{n}')\frac{d\Omega'}{4\pi} \quad (34)$$

In terms of ΔT,

$$[\frac{\partial}{\partial \tau} + n^i\frac{\partial}{\partial x^i} + \dot{\kappa}](\Delta_T + \psi) = (\dot{\phi} + \dot{\psi}) + \dot{\kappa}[\frac{\delta_\gamma}{4} + \psi + \hat{n}\vec{v}_b + \Pi/10] \quad (35)$$

where $\dot{\kappa} = an_e x_e \sigma_T$, $\delta_\gamma/4 = \Delta_{T0} = \int \frac{d\Omega'}{4\pi}\Delta_T(\hat{n})$ and Π involves the quadrupole moment of the radiation field and moments of the polarization components.

Integral Solution

The Boltzmann equation can be solved explicitly to give:

$$\Delta_T(\vec{x}_0, \tau_0, \hat{n}) + \psi(\vec{x}_0, \tau_0) = \int d\tau e^{-\kappa(\tau_0,\tau)}(\dot{\phi} + \dot{\psi})$$

Visibility Function

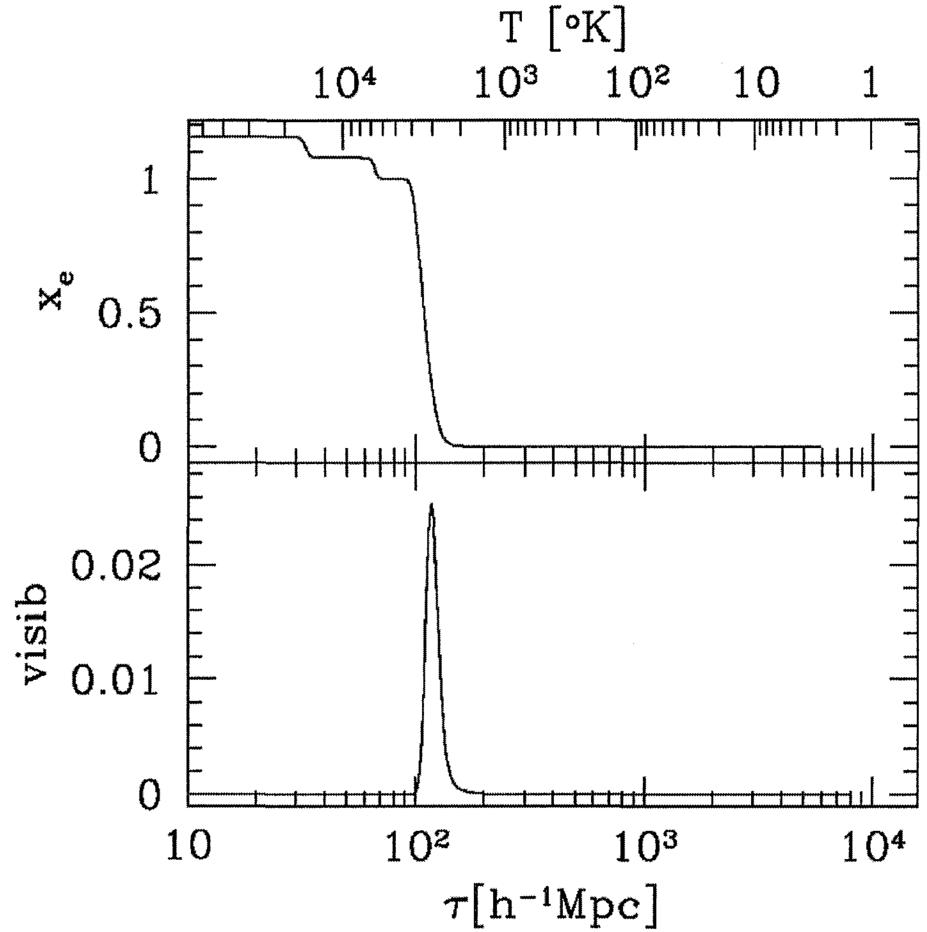
$$+ \int d\tau \dot{\kappa} e^{-\kappa(\tau_0,\tau)} S(\tau)$$

$$S(\tau) = (\delta_\gamma/4 + \psi) + \hat{n} \cdot \vec{v}_b + \Pi/10$$

$$\kappa(\tau_0, \tau) = \int_\tau^{\tau_0} d\tau' \dot{\kappa}(\tau') \qquad (36)$$

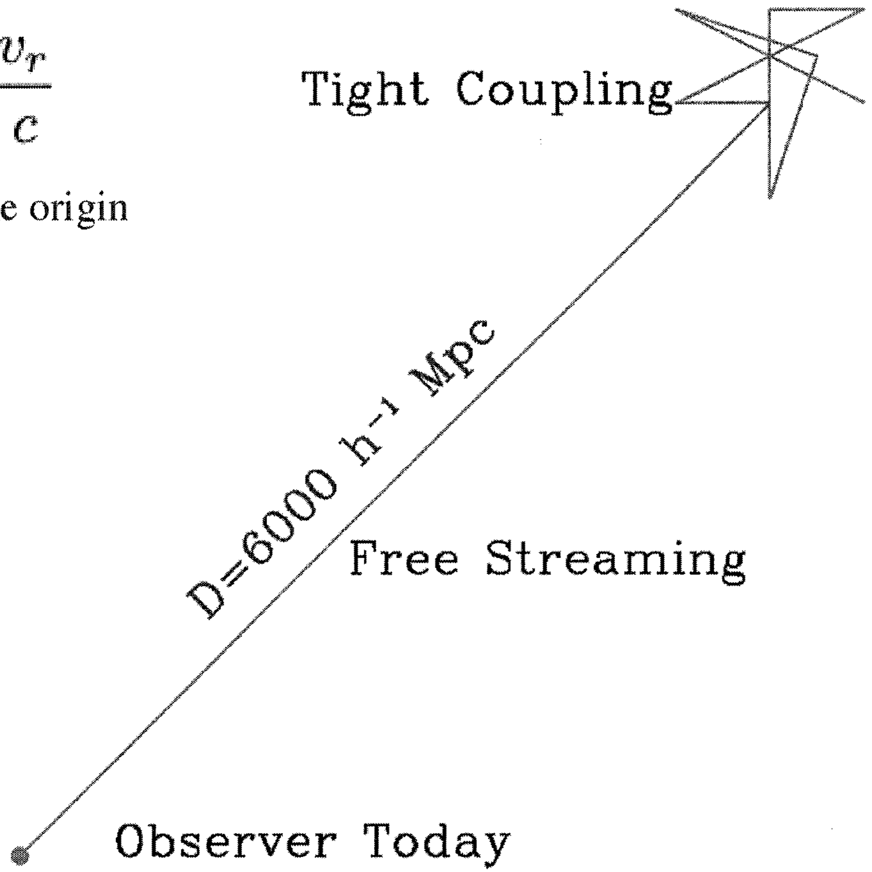
Seljak & MZ

Recombination



$$\frac{\delta T}{T} = \phi + \frac{\delta_\gamma}{4} + \frac{v_r}{c}$$

All 3 effects have the same origin



Integral Solution

We need the solution for the source $S(\tau)$ to replace into the integral solution.

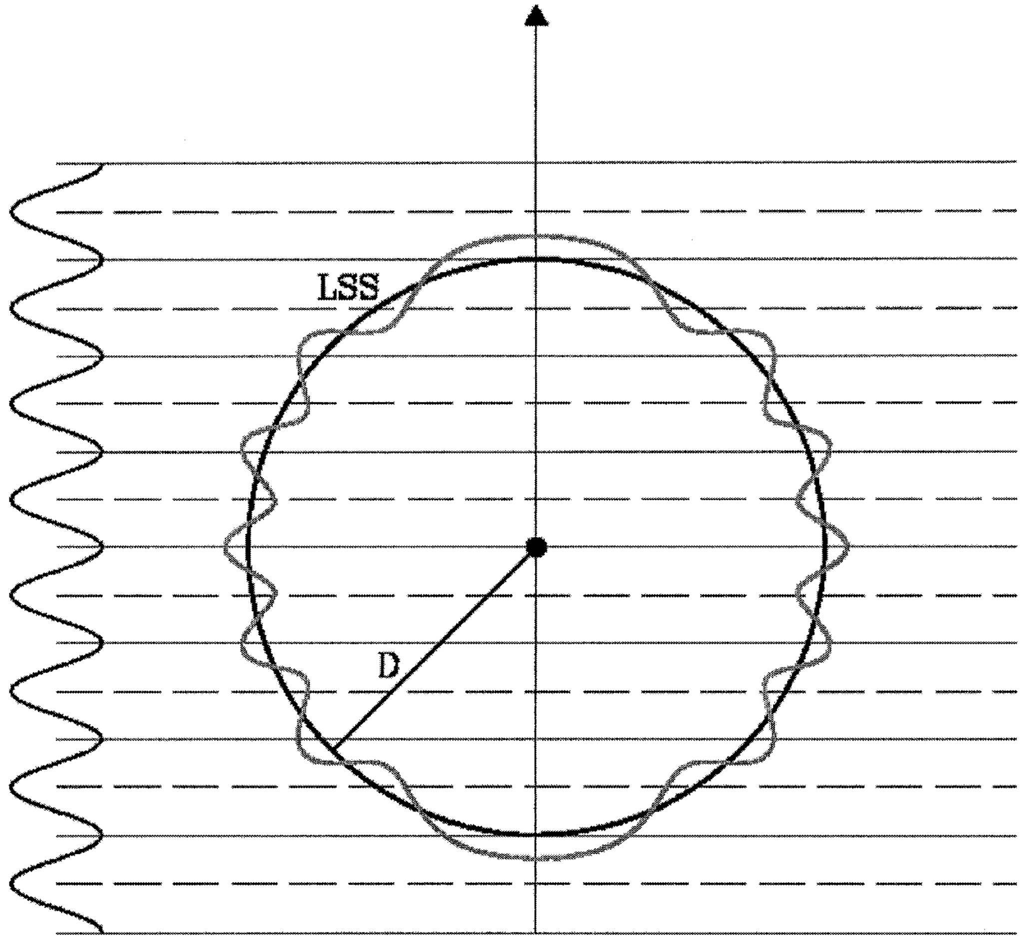
$$\frac{\delta_\gamma}{4} + \psi = \frac{\phi_*}{3}(1+3R)\cos(kc_s\tau_R) - \phi_* R$$

$$v_\gamma = -\phi_*(1+3R)c_s \sin(kc_s\tau_R). \qquad (37)$$

Equation (37) is the solution for a single Fourier mode. All quantities have an additional spatial dependence ($e^{i\boldsymbol{k}\cdot\boldsymbol{x}}$) which we have not included to make the notation more compact. With that additional term the solution (37) becomes,

$$\Delta_T(\hat{n}) = e^{ikD_{LSS}\cos\theta} S$$

$$S = \phi_* \frac{(1+3R)}{3}\left[\cos(kc_s\tau_R) - \frac{3R}{(1+3R)} - i\sqrt{\frac{3}{1+R}}\cos\theta \sin(kc_s\tau_R)\right], \qquad (38)$$



$$kD = 6\pi$$

Expand in $\cos(l\phi)$

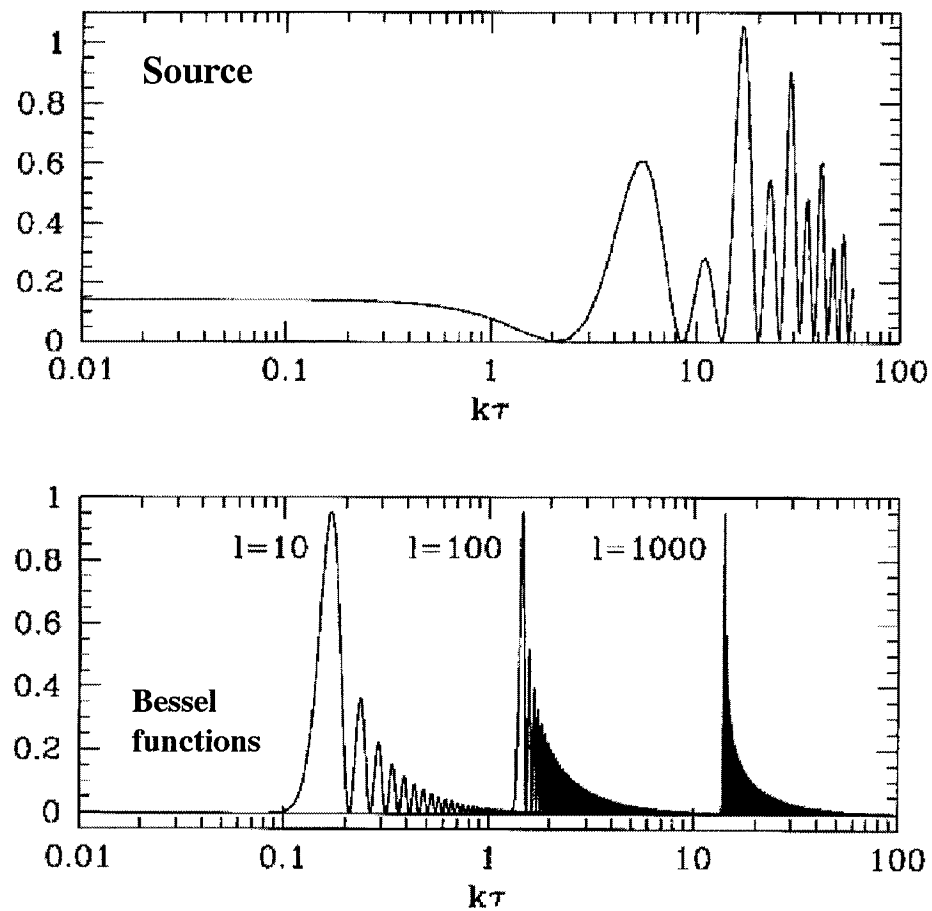
$$l \sim kD$$

$$\int_0^{2\pi} \cos(l\phi)\,\cos(kD\cos\phi) \sim J_l(kD)$$

Projection on the sky

$$
\begin{aligned}
a_{lm} &= \int d\Omega Y^*_{lm}(\hat{n})\Delta_T(\hat{n}) \\
&= \delta_{m0}\phi_*[(\frac{(1+3R)}{3}\cos(kc_s\tau_R) - R)j_l(kD_{LSS}) \\
&\quad +(1+3R)c_s\sin(kc_s\tau_R)j'_l(kD_{LSS})] \\
&\equiv S^m j_l(kD_{LSS}) + S^v j'_l(kD_{LSS}), \qquad (39)
\end{aligned}
$$

Projection Kernels



Final C_l calculation

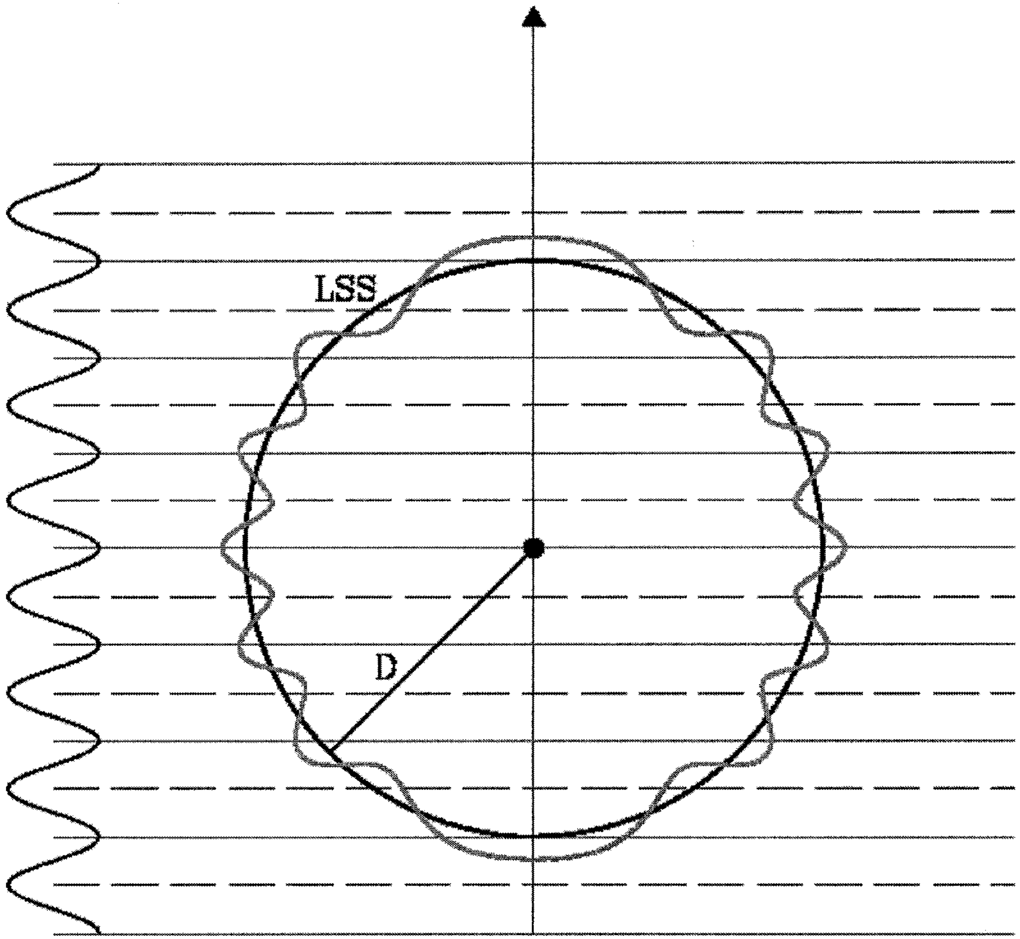
$$C_{Tl} = \int d^3k P_\phi(k) [S^m j_l(kD_{LSS}) + S^v j_l'(kD_{LSS})]|^2 \tag{40}$$

$$\begin{aligned} C_{Tl} &\approx [S^m(k^*)]^2 \int d^3k P_\phi j_l^2(kD_{LSS}) + [S^v(k^*)]^2 \int d^3k P_\phi j_l'^2(kD_{LSS}) \\ &+ 2S^m(k^*)S^v(k^*) \int d^3k P_\phi j_l(kD_{LSS}) j_l'(kD_{LSS}) \end{aligned} \tag{41}$$

with $k^* \approx l/D_{LSS}$.

$$\int \frac{dx}{x} j_l^2(x) = \frac{1}{2l(l+1)} \qquad \int \frac{dx}{x} j_l'^2(x) = \frac{1}{6(l-1)(l+2)}$$

$$\int \frac{dx}{x} j_l(x) j_l'(x) = \frac{\pi}{2(2l-1)(2l+1)(2l+3)} \tag{42}$$



$kD = 6\pi$

Expand in $\cos(l\phi)$

$l \sim kD$

$$\int_0^{2\pi} \cos(l\phi)\,\cos(kD\cos\phi) \sim J_l(kD)$$

Final C_l calculation

In this approximation we obtain,

$$l^2 C_{Tl} \propto \{[\frac{(1+3R)}{3}\cos(k^* c_s \tau_R) - R]^2 + \frac{(1+3R)^2}{3} c_s^2 \sin^2(k^* c_s \tau_R)\}$$

$$k^* = \frac{l}{D_{LSS}}$$

$$c_s^2 = \frac{1}{3(1+R)}$$

$$R \approx 0.63(\frac{\Omega_b h^2}{0.02})(\frac{1000}{1+z})(T_{cmb}/2.7K)^{-4} \qquad (43)$$

Summary

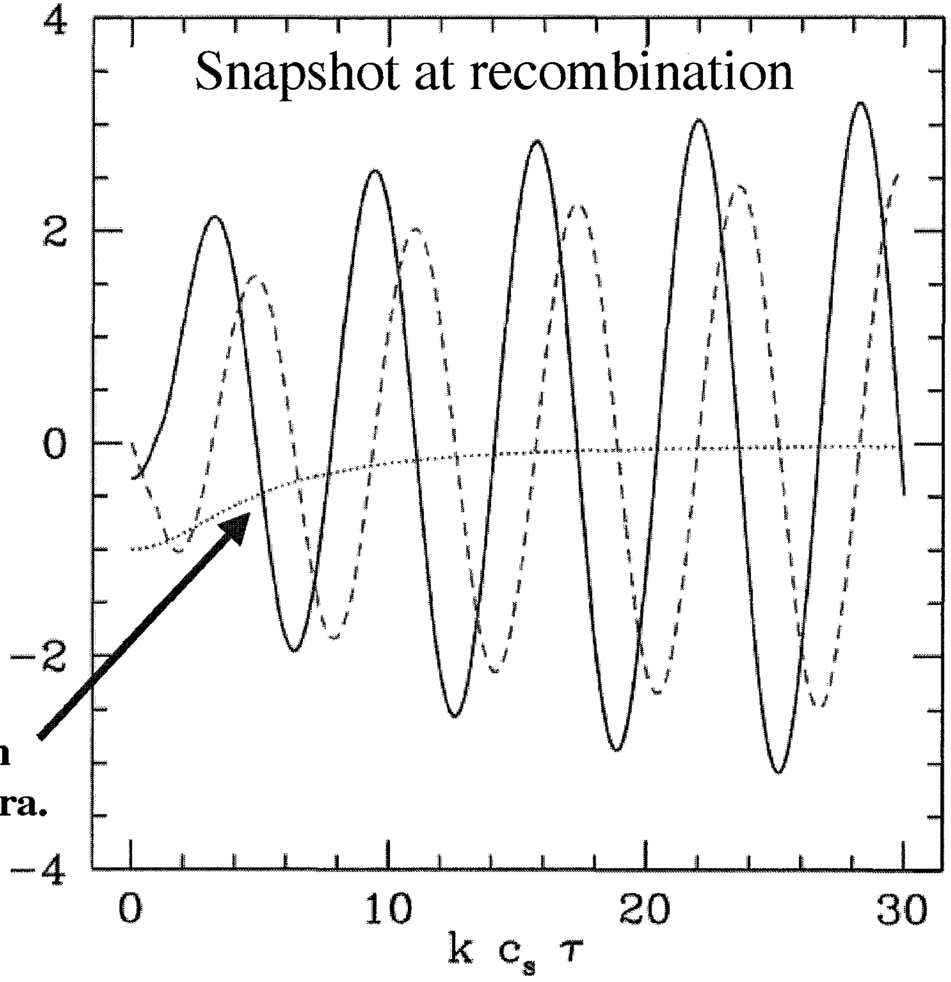
- The C_l peaks reflect the phase in the oscillation of the modes.
- There are 2 contributions that are "out of phase" one coming from the density and the other from the velocity.
- Because of the presence of baryons (R), the density one dominates.
- The density contribution peaks for $k^* c_s \tau_R = \pi, 2\pi, 3\pi....$ or equivalently $l_{peak} = n\pi D_{LSS}/c_s\tau_R$.
- The monopole term term has two contributions with opposite signs.Thus the peaks where $\cos(k^* c_s \tau_R)$ is positive will be smaller than those in which is negative. The difference in heights is $(1+2R)^2 - 1$, it measures $\Omega_b h^2$.
- The physics of the accoustic peaks only depends on the densities of the different fluids, ie. $(\Omega_m h^2, \Omega_b h^2)$. At this level Λ and K only enter through D_{LSS}.

Summary

Physical Effects I neglected:

- There is a radiation dominated era so the potential is not always constant and “drives” the oscillations.
- Tight coupling is not perfect, there is diffusion (or Silk damping).
- The last scattering surface has finite width.
- Thomson scattering polarizes the radiation.
- The gravitational potentials can change with time late in the evolution of the universe and create fluctuations through the ISW effect.

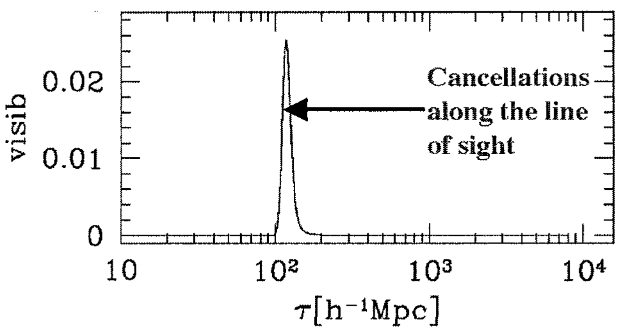
Neglected Effects: Matter radiation equality

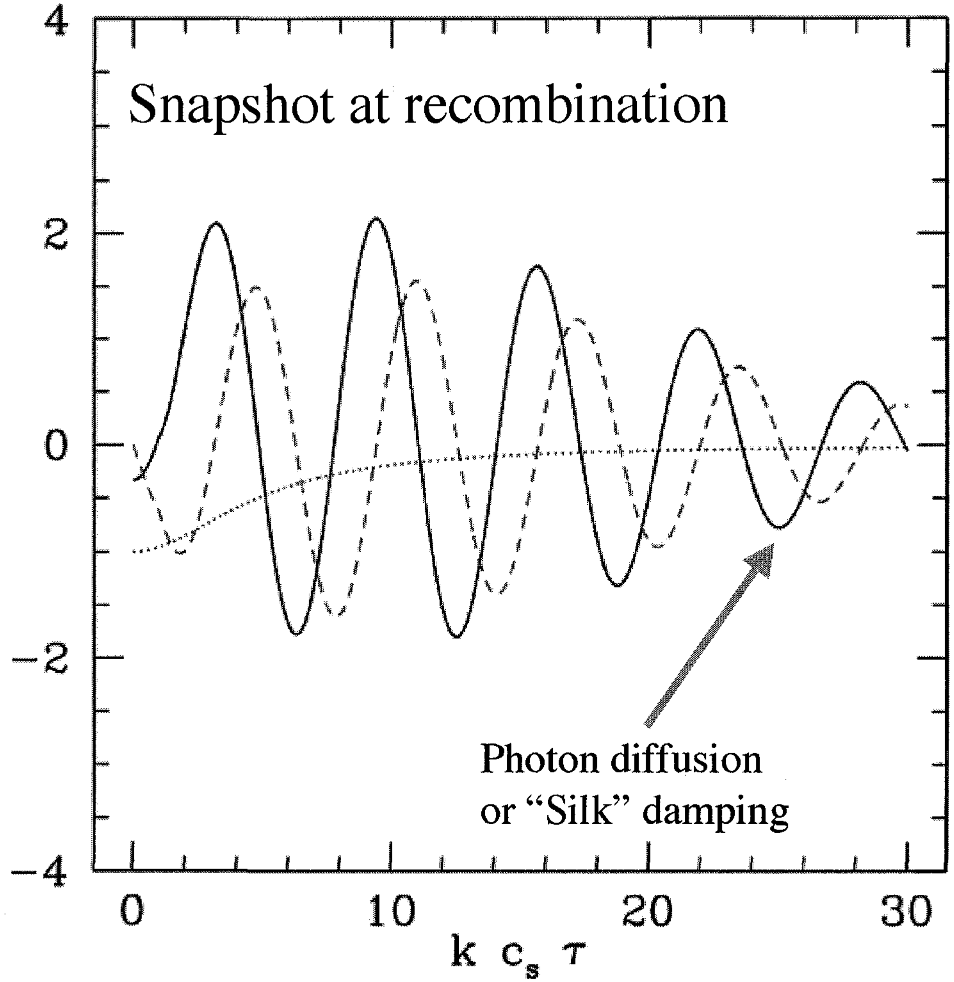


Potential decays inside the horizon in the radiation era.

Neglected Effects:Damping

There is extra damping due to the finite width of the last scattering surface





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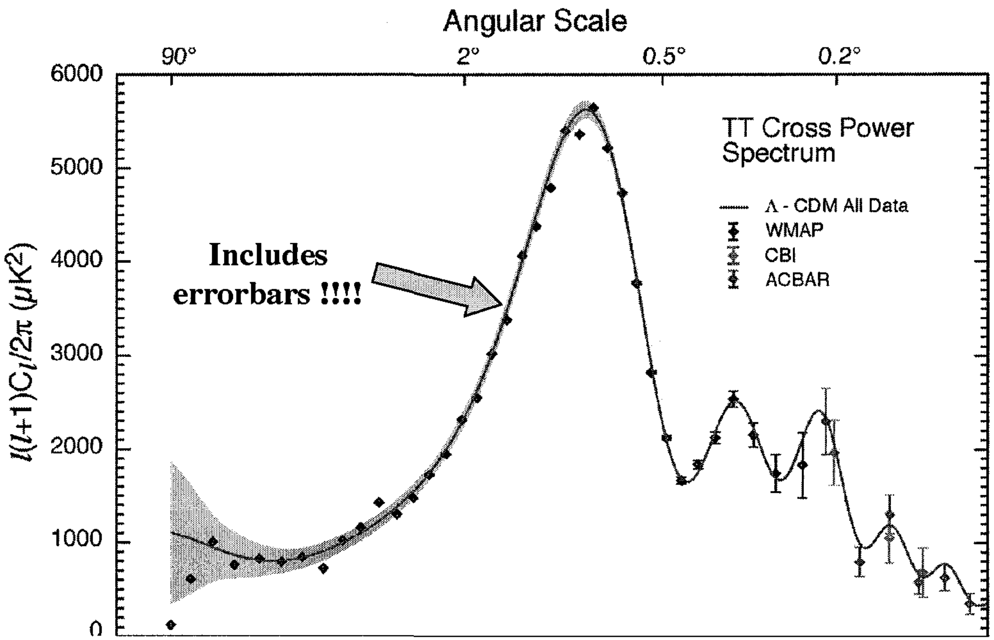

Topics for future lectures

- CMB polarization: Origin and information it encodes
- Secondary anisotropies
- Other probes of structure formation. Summary about what they tell us about the parameters of the cosmological model.
- Origin of the perturbations: inflation

Initial Conditions

We need the initial conditions. Combine first and third of the purturbed Einstein equations to get:

$$\ddot{\phi} + 3(1+\bar{c}_s^2)\eta\dot{\phi} + 3(\bar{c}_s^2 - w)\phi + k^2\bar{c}_s^2\phi = 4\pi G a^2 \bar{\rho_m} c_s^2 \sigma$$

$$\delta\sigma = 3/4\delta_\gamma - \delta_m \qquad (21)$$

Where we have defined $\bar{c}_s^2 = \dot{p}/\dot{\rho}$, $w = p/\rho$ and $\eta = \dot{a}/a$ and we ignored shear and took $\phi = \psi$. We have also introduced the **entropy fluctuations** $\delta\sigma$ and for its definition assumed that we have only two fluids, cold dark matter and radiation. The entropy fluctuations satisfy a very simple equation:

$$\delta\dot{\sigma} = -k(v_c - v_\gamma) \qquad (22)$$

Initial Conditions

For simplicity assume:

- There are no initial entropy fluctuations.
- We are considering a universe in the matter dominated era $(\vec{c}_s^2 = w = 0, \eta = 2/\tau)$.

Then

$$\ddot{\phi} + \frac{6}{\tau}\dot{\phi} = 0 \tag{23}$$

Which leads to two solutions:

$$\phi(\tau) = \begin{cases} \phi_* & \text{“GrowingMode''}, \\ \phi_*(\tau/\tau^*)^{-5} & \text{“DecayingMode''}, \end{cases} \tag{24}$$

Initial Conditions

To obtain the initial δ we use:

$$k^2\phi + 3\frac{\dot{a}}{a}\left(\dot{\phi} + \frac{\dot{a}}{a}\psi\right) = -4\pi G a^2 \bar{\rho}\delta \tag{25}$$

so,

$$\delta = -2\phi \tag{26}$$

and $\delta = \delta_c = 3/4\delta_\gamma$ in the matter era.