SMR.1568- 13

SUMMER SCHOOL IN COSMOLOGY AND ASTROPARTICLE PHYSICS

28 June - 10 July 2004

Quantum field theory in the early Universe (II)

Richard F. HOLMAN
Department of Physics
Carnegie Mellon University
Pittsburgh, PA 15213-3890
U.S.A.

Please note: These are preliminary notes intended for internal distribution only.

Lecture III False Vacuum Decay in Extended Inflation

As we saw in the previous lecture, the parameter controlling the percolation of the true vacuum was $\epsilon(t) \equiv \lambda_0 / H^4(t)$. So far, we have assumed that the bubble nucleation rate λ_0 was constant throughout the extended inflationary era. However, it is not obvious that (i) this really is the case in the original La-Steinhardt model and (ii) if it is true for that model, whether $\lambda_0 = \text{const}$ necessarily holds for all extended inflation models. The problem concerns the fact that as the bubbles are nucleating, the JBD field Φ is continually evolving. This means that if we were to attempt to perform the full bubble nucleation calculation, including gravitational effects, we would naturally expect the time variation of Φ to reflect itself in a time variation of λ.

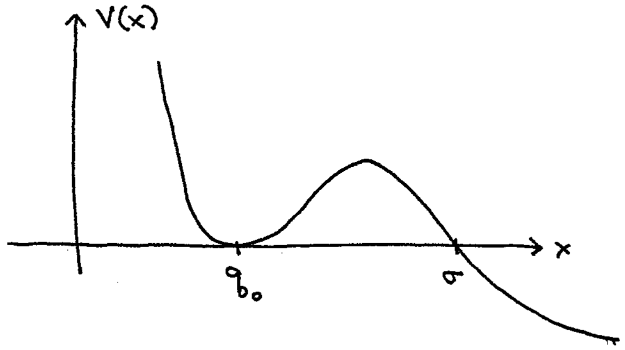
What we will do in this lecture is the following. First we will review the vacuum decay formalism set up by Callan and Coleman [24], first in the context of quantum mechanics and then in field theory. We will also discuss gravitational corrections to the decay rate as deduced by Coleman and DeLuccia [25].

After this review, we attempt to apply the above mentioned formalisms to the JBD model. We will see that new and subtle problems crop up that force us to make some approximations in order to produce answers. Our results show that the original extended inflation model does have $\lambda_0 = \text{const}$, at least at late times. However, we will see that in general, λ will be a function of the JBD Φ

field Φ. Furthermore, this dependence is typically ~~exponentially~~ strong!

Since we saw that one of the possible solutions to the large bubble problem was to make the nucleation rate turn on ~~due~~ near the end of inflation, understanding the time dependence of λ seems to be of critical importance. This variation will form the basis of a large number of models to be discussed in lecture IV.

Let us look at a one-dimensional quantum mechanical system with a potential:



As is well known, the state at $x = q_0$ is unstable to barrier penetration; thus it decays with a rate Γ. To find Γ, we note that if the energy "eigenvalues" have an imaginary part to them, so that $E_n = E_n^r - i\frac{\Gamma_n}{2}$, then the corresponding eigenstates will acquire the following time dependence:

$$\Psi_n(x,t) \sim \Psi_n(x) \exp(-iE_n^r) \exp(-\Gamma_n/2). \qquad (3.1)$$

The probability for finding the system in the state $\Psi_n(x,t)$, $|\Psi_n(x,t)|^2$ then decreases exponentially and Γ_n is identified as the rate. If, as ~~is~~ the usual case, we set up a wave-packet peaked at $x = q_0$, this is a superposition of Ψ_n's; at late enough times, only the ground state contributes, and $\Gamma \approx \Gamma_0$.

Thus, computing Γ is equivalent to looking for imaginary parts in the energy "eigenvalues". Note the quotes around "eigenvalues". Clearly, these are not the eigenvalues of a hermitian Hamiltonian with hermitian boundary conditions. The eigenvalue problem is really a quasi-stationary one, as opposed to a stationary problem.

We need an efficient technique for computing the ground state energy of the system. This is furnished us by the Feynman-Kac [26] formula. Consider the Green's function of the Schröedinger equation that propagates states at (y, t_0) to states at (x, t):

$$\langle x\, t | y\, t_0 \rangle = \langle x | \exp(-i\hat{H}(t-t_0)) | y \rangle . \tag{3.2}$$

Here, $\{|x\rangle\}$ are position eigenstates for the position operator $\hat{q}$ in the Schröedinger picture, while $|x t\rangle$ are the eigenstates of $\hat{q}(t) = e^{i\hat{H}t}\, \hat{q}\, e^{-i\hat{H}t}$. Inserting a complete set of $\hat{H}$ eigenstates $\{|n\rangle\}$, we find:

$$\begin{aligned} \langle x\, t | y\, t_0 \rangle &= \sum_n \langle x | n \rangle \langle n | y \rangle\, e^{-iE_n(t-t_0)} \qquad (3.3) \\ &= \sum_n \psi_n(x)\, \psi_n^*(y)\, e^{-iE_n(t-t_0)} \end{aligned}$$

For simplicity, we set $t_0 = 0$. What we want to do is to project out the ground state part of this sum. This can be done using the following prescription:

(i) rotate t into imaginary time: $t = -i\tau$

(ii) Compute $e^{E_0 \tau} \langle x\, -i\tau | x\, 0 \rangle$ and integrate over x (taking the trace of the propagator).

(iii) Take log of the expression in (ii)

Finally, (iv) let $\tau \to \infty$. If we implement this prescription, we find:

$$\ln\left\{e^{E_0\tau}\int dx\, \langle x\, -i\tau \mid x\ 0\rangle\right\} = \ln\left\{\sum_n \left(\int dx\, |\psi_n(x)|^2\right) e^{-\tau(E_n-E_0)}\right\}. \quad (3.4)$$

Using the fact that the $|\psi_n(x)|^2$ are normalized to one, and that $E_n - E_0 \geq 0$ (since E_0 is the ground state), we find, in the $\tau \to \infty$ limit,

$$E_0 = -\lim_{\tau\to\infty} \frac{1}{\tau} \ln\left[\int dx\, \langle x\, -i\tau \mid x\ 0\rangle\right]. \quad (3.5)$$

We are not done quite yet; if the above expression is to be of any use to us, we must find a way to compute the propagator $\langle x\, -i\tau \mid x\ 0\rangle$. This can be done via the path integral (PI) representation of the propagator [26]. The propator $\langle x\ t \mid y\ 0\rangle$ ~~can~~ be written as:

$$\langle x\ t \mid y\ 0\rangle = \lim_{N\to\infty}\int \prod_{i=1}^{N} dq_i \left(\frac{m}{2\pi i\epsilon}\right)^{\frac{N+1}{2}} \exp\left[i\epsilon \sum_{l=0}^{N}\left\{\frac{1}{2}m\left(\frac{q_{l+1}-q_l}{\epsilon}\right)^2 - V\left(\frac{q_{l+1}+q_l}{2}\right)\right.\right.$$

$$\equiv \int_{\substack{q(0)=y\\ q(t)=x}} \mathcal{D}q(\cdot)\, \exp\left(i\int_0^t dt'\left(\tfrac{1}{2}m\dot{q}^2 - V(q)\right)\right) = \int_{\substack{q(0)=y\\ q(t)=x}} \mathcal{D}q(\cdot)\ e^{iS[q(\cdot)]} \quad (3.6)$$

Here $x = x_{N+1}$, $y = x_0$, $\epsilon = \frac{t}{N+1}$ and $S[q(\cdot)] = \int_0^t dt'\left(\frac{1}{2}m\dot{q}^2 - V(q)\right)$ is the action of the path $q(\cdot)$. The quantity that enters in our expression for E_0 (eq. (3.5)) is $\int dx\, \langle x\, -i\tau \mid x\ 0\rangle$; this can be interpreted as the ~~[illegible]~~ trace (in the x-"index") of $\langle x\, -i\tau \mid y\ 0\rangle$. The rotation to imaginary time $t = -i\tau$ has the effect of changing

$$iS[q(\cdot)] = \int_0^t d(it')\left(\frac{1}{2}m\left(\frac{dq}{dt}\right)^2 - V(q)\right) = -\int_0^\tau d\tau'\left(\frac{1}{2}m\left(\frac{dq}{d\tau}\right)^2 + V(q)\right) \equiv -S_E[q(\cdot)]$$

$$(3.7)$$

Note the extremely important <u>relative</u> sign change between the kinetic and potential terms, as well as the <u>overall</u> sign change.

The calculation of $\int dx \langle x\ -i\tau | x\ 0\rangle$ has two parts to it; first compute $\langle x\ -i\tau | x\ 0 \rangle$ as the following (Euclidean) PI:

$$\langle x\ -i\tau | x\ 0\rangle = \int_{q_E(0)=q_E(\tau)=x} \mathcal{D}q_E(\cdot)\ \exp\left(-S_E[q_E(\cdot)]\right). \tag{3.8}$$

Next, the integral over x instructs us to consider any path that returns to the initial point by Euclidean time τ. Thus:

$$\int dx\ \langle x\ -i\tau | x\ .\ 0\rangle = \int_{q_E(0)=q_E(\tau)} \mathcal{D}q_E(\cdot)\ \exp\left(-S_E[q_E(\cdot)]\right). \tag{3.9}$$

The last thing we have to do is to evaluate this path integral. Unfortunately, if $V(q)$ contains terms larger than quadratic in order, there is no closed form expression for the path integral. However, a useful approximation scheme, the semiclassical expansion, does exist. This is the functional equivalent to the steepest descent approximation. Here we say that the greatest contribution to the integral comes from paths that are "close" to the path that extremizes $S_E[q_E(\cdot)]$, subject to the relevant boundary conditions. But this is just the classical path $\bar{q}_E(\tau)$:

$$\left.\frac{\delta S_E[q(\cdot)]}{\delta q_E}\right|_{q_E=\bar{q}_E} = 0\ . \tag{3.10}$$

We then Taylor expand $S_E[q_E(\cdot)]$ about the classical path to find (up to quadratic order):

$$S_E[q_E(\cdot)] \simeq S_E[\bar{q}_E] + (\text{no linear term}) + \frac{1}{2!}\int d\tau\, d\tau'\ \xi(\tau) \left.\frac{\delta^2 S_E[q_E]}{\delta q_E(\tau)\ \delta q_E(\tau')}\right|_{\bar{q}_E} \xi(\tau') + \cdots \tag{3.11}$$

where $\xi(\tau) \equiv q_E(\tau) - \bar{q}_E(\tau)$.

We can now change variables in the PI of eq. (3.9) from $q_E(\cdot)$ to $\xi(\cdot)$; since this is just a shift by $\bar{q}_E(\cdot)$ which is a fixed function (as opposed to an integration variable), the measure is unaffected. Doing this yields:

$$\int dx \, \langle x - i\tfrac{T}{2} | x, i\tfrac{T}{2} \rangle \doteq \exp\left(-S_E[\bar{q}_E]\right) \int_{\substack{\xi(-T/2)=0 \\ \xi(T/2)=0}} \mathcal{D}\xi(\cdot) \, \exp\left(-\tfrac{1}{2}\langle \xi | \mathcal{O}(\bar{q}_E) | \xi \rangle\right) \tag{3.12}$$

Note that we have made use of the time translational invariance of the system to place the boundary conditions at $\pm\tau/2$; this is just a matter of convenience. The operator $\mathcal{O}(\bar{q}_E)$ is defined as:

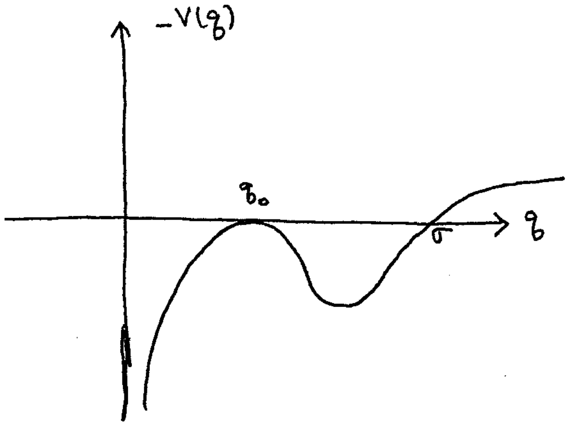
$$\mathcal{O}(\bar{q}_E)(\tau_1, \tau_2) \equiv \left. \frac{\delta^2 S_E[q_E]}{\delta q_E(\tau_1) \, \delta q_E(\tau_2)} \right|_{\bar{q}_E} = \left(-\frac{d^2}{d\tau^2} + V''(\bar{q}_E)\right) \delta(\tau_1 - \tau_2), \tag{3.13}$$

where we have set the particle mass to unity. The $\xi(\cdot)$ path integral is a Gaussian (in function space) and can be done exactly:

$$\int_{b.c.} \mathcal{D}\xi(\cdot) \, \exp\left(-\tfrac{1}{2}\langle \xi | \mathcal{O}(\bar{q}_E) | \xi \rangle\right) = N \det^{-1/2}\left(\mathcal{O}(\bar{q}_E)\right), \tag{3.14}$$

where N is a normalization factor and the determinant is defined via the product of the eigenvalues of $\mathcal{O}(\bar{q}_E)$.

To find the classical configuration $\bar{q}_E(\tau)$, we solve Newton's equations in imaginary time i.e. with a potential $-V(q_E)$. The boundary conditions are that $\bar{q}_E$ must return to its starting point when $\tau \to \infty$ and $\dot{\bar{q}}_E(\tau) \to 0$ as $\tau \to \infty$. This last condition ensures the finiteness of $S_E[\bar{q}_E]$. Let's apply our Newtonian intuition to the problem of motion in the potential $-V(q)$:



We consider configurations that start with $\bar{q}_E(-T/2) = q_0$ and end up at q_0 again at time $+T/2$ (remember that we will be taking the large T limit). One such configuration is just the constant one: $\bar{q}_E(\tau) = q_0$ for <u>all</u> Euclidean time τ.

Another, more interesting, configuration is given by the so-called bounce [27]. Here, the particle rolls down the hill, reaching the escape point σ at $\tau = 0$ and then rolling back to q_0, reaching it at $+T/2$ $(\to \infty)$. Now, the bounce contribution to E_0 is exponentially suppressed relative to the constant configuration (which has zero action since we have arranged $V(q_0) = 0$). Thus it might seem strange to keep this contribution around. However, the point is that the bounce contributions are the <u>leading</u> contributions to the <u>imaginary</u> part of E_0! This is why they're important enough to keep.

Thus we may write:

$$E_0 = -\lim_{T\to\infty} \frac{1}{T} \ln \left[\int_{q_E(-T/2) = q_E(T/2) = q_0} \mathcal{D}q_E(\tau)\, e^{-S_E[q_E]} \right] \simeq -\lim_{T\to\infty} \frac{1}{T} \ln \left[(\bar{q}_E = q_0) + (\bar{q}_E = \text{bounce}) \right]$$

$$= -\lim_{T\to\infty} \frac{1}{T} \left\{ \ln \left[(\bar{q}_E = q_0)\left(1 + \frac{(\bar{q}_E = \text{bounce})}{(\bar{q}_E = q_0)}\right) \right] \right\} \simeq \epsilon_0 + \left(-\lim_{T\to\infty} \frac{1}{T} \frac{(\bar{q}_E = \text{bounce})}{(\bar{q}_E = q_0)} \right) \quad (3.15)$$

where ϵ_0 is just the ground state energy of the harmonic oscillator of frequency $V''(q_0)$.

The time translational invariance of the system leads to some complications in computing the bounce contribution. First, it implies that $\mathcal{O}(\bar{q}_E = \text{bounce})$ has a zero mode, given by $\frac{d\bar{q}_E}{d\tau}$. Since we are computing the determinant of $\mathcal{O}$ in the space of fluctuations q that start and end at $q = 0$, $\frac{d\bar{q}_E}{d\tau}$ is clearly in this space, as a consequence of the finite action constraint. Furthermore:

$$\int d\tau_2 \; \mathcal{O}(\bar{q}_E = \text{bounce})(\tau_1, \tau_2) \frac{d\bar{q}_E}{d\tau_2}(\tau_2) = \left(-\frac{d^2}{d\tau_1^2} + V''(\bar{q}_E(\tau_1))\right) \frac{d\bar{q}_E(\tau_1)}{d\tau_1} =$$

$$-\frac{d}{d\tau}\left(\frac{d^2\bar{q}_E}{d\tau^2} - V'(\bar{q}_E)\right) = 0, \qquad (3.16)$$

where the last equality follows from the bounce equation of motion.

In essence, the time translation symmetry implies that fluctuations in the direction (in function space) of $\frac{d\bar{q}_E}{d\tau}$ cost no "energy", if we treat $\mathcal{O}(\bar{q}_E)$ as a Hamiltonian. Hence, this is a <u>flat</u> direction, and gives rise to a divergence in the integral. However, this is not the whole story concerning time translation invariance. If we call the time at which $\dot{\bar{q}}_E(\tau) = 0$ the <u>center</u> of the bounce, then configurations consisting of many bounces whose centers are widely separated are <u>also</u> approximate classical solutions; they must also be summed over. We will argue that performing this sum ~~in the same~~ ~~perfo~~ takes into account the effect of the flat direction in the integrand; this sum will yield a quantity proportional to $\mathcal{T}$.

To get a better understanding what is happening, expand $q(t)$ as

$$q(\tau) = \sum_{n \geq 0} c_n \chi_n(\tau) \qquad (3.17)$$

where the $\{x_n(\tau)\}$ are the eigenfunctions $\mathcal{O}(\bar{q}_E = q_b)$ (q_b is the bounce configuration). The path integral then becomes:

$$\int \prod_{n \geq 0} \frac{dc_n}{\sqrt{2\pi}} \; e^{-\frac{1}{2}\lambda_n c_n^2}, \tag{3.18}$$

where λ_n is the eigenvalue of $\mathcal{O}(q_b)$ corresponding to $x_n(\tau)$. We take $x_1 = \frac{\dot{q}_b}{\|\dot{q}_b\|}$ so that $\lambda_1 = 0$. We can then rewrite (3.18):

$$\int \prod_{n\geq 0} \frac{dc_n}{\sqrt{2\pi}} \; e^{-\frac{1}{2}\lambda_n c_n^2} = \left[\int \frac{dc_0}{\sqrt{2\pi}} \; e^{-\frac{1}{2}\lambda_0 c_0^2} \int \prod_{n>1} \frac{dc_n}{\sqrt{2\pi}} \; e^{-\frac{1}{2}\lambda_n c_n^2}\right] \int_{-\infty}^{\infty} \frac{dc_1}{\sqrt{2\pi}} \tag{3.19}$$

We have singled out the c_0 integral since it too will cause some problems.

Our goal is to isolate the effects of the divergence in the c_1 integral and to rewrite it as something involving the length T of the time interval we're allowing the bounce to live in. To do this let us write:

$$q_E(\tau) = q_b(\tau - \tau_0) + \sum_{n=1}^{\infty} q_n f_n(\tau - \tau_0),$$

$$\int_{-\infty}^{\infty} d\tau \; \dot{q}_b(\tau - \tau_0) \, f_n(\tau - \tau_0) = 0 \tag{3.20}$$

The reason for doing this is that motion in the $\dot{q}_b$ direction is now replaced by motion in the "time" parameter τ_0 ($\tau_0 \in (-T/2, T/2)$). Thus, we will change variables from the $\{c_n\}$'s to the set $\{t_0, q_n\}$. Using orthonormality of the c_n's:

$$c_m = \int d\tau \; x_m(\tau) \, q_b(\tau - \tau_0) + \sum_{1}^{\infty} q_n \int d\tau \; x_m(\tau) \, f_n(\tau - \tau_0). \tag{3.21}$$

We want to compute the Jacobian of the matrix $\left(\frac{\partial c_m}{\partial t_0}, \frac{\partial c_m}{\partial q_n}\right)_{\substack{m=0\ldots \\ n=1\ldots}}$:

$$\frac{\partial c_m}{\partial \tau_0} = -\int d\tau\, \chi_m(\tau)\, \dot{q}_b(\tau-\tau_0) + \sum_1^{\infty} q_n \int d\tau\, \chi_m(\tau)\, \dot{f}_n(\tau-\tau_0)$$

$$\frac{\partial c_m}{\partial q_n} = \int d\tau\, \chi_m(\tau)\, f_n(\tau-\tau_0) . \qquad (3.22)$$

Since we are only calculating quantities to leading order in the couplings (or $\hbar$, which is the same thing), we can neglect the dependence of the Jacobian J on the q_n's. Now $\left\{ \frac{\dot{q}_b(\tau-\tau_0)}{\|\dot{q}_b\|}, f_n(\tau-\tau_0) \right\}$ is an orthonormal basis,

$$\det\left(\int d\tau\, \chi_m(\tau) \frac{\dot{q}_b(\tau-\tau_0)}{\|\dot{q}_b\|} ,\ \int d\tau\, \chi_m f_n(\tau-\tau_0) \right) = 1 \qquad (3.23)$$

(similarity transformation on an orthogonal matrix). Thus, to this order

$$J = \|\dot{q}_b\| = \left[\int_{-\infty}^{\infty} d\tau\, \dot{q}_b^2(\tau-\tau_0) \right]^{1/2} = \left[\int_{-\infty}^{\infty} d\tau\, \dot{q}_b^2(\tau) \right]^{1/2} \qquad (3.24)$$

We can thus write

$$\prod_n dc_n = J\, d\tau_0 \prod_n' dq_n \implies$$

$$\prod_n \frac{dc_n}{\sqrt{2\pi}} = \frac{J}{\sqrt{2\pi}}\, d\tau_0 \prod_n' \frac{dq_n}{\sqrt{2\pi}} . \qquad (3.25)$$

The τ_0 integral is trivial since the integrand has no τ_0 dependence. This implies that the divergence in c_1 can be traded for the factor: $\left(\frac{B}{2\pi}\right)^{1/2} T$. Here B is the bounce action:

$$B = \int d\tau \left[\tfrac{1}{2} \dot{q}_b^2(\tau) + V(q_b) \right] = \left(\begin{matrix} \text{1st integral of} \\ \text{eqn of motion} \\ \text{w/ } V(q_0) = \dot{q}(-\infty) = 0 \end{matrix} \right) = \int d\tau\, \dot{q}_b(\tau)^2 = J^2 . \qquad (3.26)$$

Our next problem concerns the λ_0 integral. We first note that the zero mode $\dot{q}_b$ has a node since it stops at $q = \sigma$ and comes back to $q = q_0$. Thus there must be an eigenvalue of smaller than zero; this is λ_0 (note: $\dot{q}_b$ only has one node, so λ_0 is unique).

This implies that the c_0 integral is of the form:

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dc_0 \exp\left(-\tfrac{1}{2}(\lambda_0)c_0^2\right) \equiv I(\lambda_0). \qquad (3.27)$$

The only way to deal with $I(\lambda_0)$ for $\lambda_0 < 0$ is to analytically continue it to $\lambda_0 > 0$. Doing this yields:

$$I(\lambda_0) \rightarrow i \frac{1}{\sqrt{|\lambda_0|}}. \qquad (3.28)$$

We will not enter the question of whether this analytic continuation is the correct thing to do, since this is a rather thorny issue. We are not quite done with the calculation since we are missing a factor of $\frac{1}{2}$. For large negative values of c_0, the integral is not really Gaussian; as c_0 gets large & negative, the system is trying to return to the metastable state at ξ_0. Thus the original c_0 integral was only over the range $(0, \infty)$.

$\therefore$ the path integral over the fluctuations can be written as

$$\int \mathcal{D}\xi(\cdot)\, e^{-\frac{1}{2}(\xi\, \mathcal{O}\, \xi)} = \underbrace{\prod_{\lambda_n>0} \lambda_n^{-1/2}}_{+\text{'ve modes}} \cdot \underbrace{\frac{i}{2} |\lambda_0|^{-1/2}}_{-\text{'ve, unstable mode}} \cdot \underbrace{\left(\frac{B}{2\pi}\right)^{1/2} \cdot T}_{\text{zero mode.}} \qquad (3.29)$$

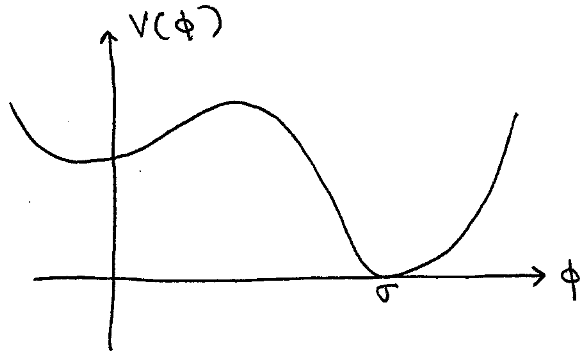
$$= i \cdot \frac{1}{2} \left[\det{}' \mathcal{O}\right]^{-1/2} \left(\frac{B}{2\pi}\right)^{1/2} \cdot T. \qquad ({}' \equiv \text{omit zero eigenvalue}).$$

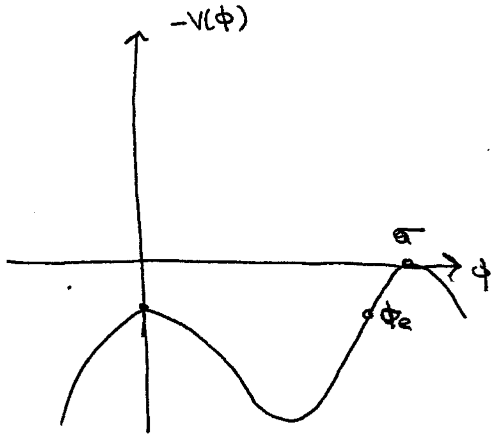
Using this in eq (3.15), we have:

$$\mathrm{Im}\, E_0 = -\lim_{T\to\infty} \cdot \frac{1}{T} \cdot T \cdot \frac{1}{2} \frac{|\det{}' \mathcal{O}_b|^{-1/2} (B/2\pi)^{1/2} e^{-B}}{|\det \mathcal{O}(\xi_0)|^{-1/2}},$$

$$\Gamma = -2\, \mathrm{Im}\, E_0 = \left(\frac{B}{2\pi}\right)^{1/2} e^{-B} \left(\left|\frac{\det{}' \mathcal{O}_b}{\det \mathcal{O}(\xi_0)}\right|\right)^{-1/2}. \qquad (3.30)$$

This is the quantum mechanical result. Let's find out how this works for a scalar field theory with the standard asymmetric double well as its potential:





Here the action is given by:

$$S = \int d^4x \left[\tfrac{1}{2} (\partial_\mu \phi)^2 - V(\phi) \right]. \tag{3.31}$$

Going through the same procedure as in Q.M. and noting that the bounce will now have <u>four</u> zero modes corresponding to translations in the four space time directions, we would find ($\Gamma = \frac{\text{prob of decay}}{\text{time- three-vol}}$).

$$\Gamma = \left[\left(\frac{B}{2\pi} \right)^{1/2} \right]^4 \left| \frac{\det' \left[-\partial^2 + V''(\phi_b) \right]}{\det \left[-\partial^2 + V''(\phi_{fv}) \right]} \right|^{-1/2} \exp(-B). \tag{3.32}$$

In order to arrive at eq. (3.32), we have made a some ~~series~~ ~~of~~ assumptions. First, $\phi_b(x)$ (in Euclidean space now) is the bounce configuration that goes to ϕ_e and then back to ϕ_{fv}. We have assumed that the bounce is $O(4)$ symmetric: Coleman, Glaser & Martin have shown that in the absence of gravity, these bounces are the ones of lowest action. Due to this $O(4)$ symmetry, the Jacobians for the zero modes are all equal; the zero modes are $\partial_\mu \phi_b$ and are normalized so that

$$\int d^4x \, (\partial_\mu \phi_b)^2 = B \qquad (\text{no sum on } \mu). \tag{3.33}$$

In the so-called "thin-wall" limit the bounce may be viewed as a bubble with true vacuum in the interior, a sharp transition region and then false vacuum on the outside. (Note that $B = S(\phi_b) - S(\phi_{fv})$)

Before we end this review, we bring up some questions of principle. The potentials we computed Γ for in QM and QFT were quite different. This was not by accident. If we had looked for false vacuum decay (i.e. $e^{-\Gamma t/2}$ behaviour) in the Q.M. case with the asymmetric double well, we would have found that $\mathrm{Im}\, E_0 = 0$. This should not come as a surprise; if the Hamiltonian is hermitian and the boundary conditions include both outgoing and reflected waves, the eigenvalues must be real! In the ~~two~~ true metastability case (i.e. just barrier, no reflecting wall), the boundary conditions corresponding to a purely outgoing wave; this is the source of the complex energies.

We conclude from this that the QFT situation must be quite different than that of Q.M. In particular, we expect that the existence of a continuum of open channels on the other side of the barrier allows the wave-functional to decohere and thus have a true decay, as opposed to the tunnelling back and forth that would occur in the QM case.

Coleman and De Luccia [25] (see also [9]) have incorporated the effects of classical gravity into the calculation above. Essentially as long as the bubble size is small compared to the radius of curvature of space, and all mass scales are less than M_p, no significant differences occur.

How does this all work in extended inflation? The first problem we note is that none of the formalism we've developed is really suited to a situation in which G_N is varying. Thus, we should do all our work in the Einstein frame and then transform back to the Jordan one. Having done that, we run into the greater problem of having to deal with three fields: the inflaton σ, the conformally transformed JBD field Ψ and the (imaginary time) scale factor $a(\tau)$. If this was not bad enough, we don't know what boundary conditions to impose on these fields. The reason is that Ψ is just rolling throughout all of the extended inflationary era; this means that the false and true vacua, as specified by (σ, Ψ) evolve in (real) time! Furthermore, there is a problem in that even if one could specify appropriate b.c.'s, the problem will be over-determined. As shown in [?], there is a point at which $a(\tau)=0$ during the Coleman-DeLuccia type bounce. Since $\dot{\Phi}(\tau)$ vanishes there also no singularities will occur; however, it is too much to expect that a, σ, Ψ will all vanish at the same point.

In order to compute the decay rate, we make the approximation of "freezing" out motion in the Ψ direction during the (imaginary time) bounce; note that Ψ will still roll in real time. This approximation is equivalent to $G_N \to 0$ (Einstein frame). This is because the Ψ kinetic term scales as $\Psi_0^2 \propto \frac{1}{G_N}$ (see eq. (2.29)). Note that since the Ricci scalar term in the action also scales with G_N^{-1}, both gravity and Ψ are "frozen-out" of the bounce.

We can now put some flesh on these bones. Let's take as our action in the Jordan frame:

$$S[g,\sigma,\Phi] = \int d^4x \sqrt{-g} \left\{ -\Phi R + \omega g^{\mu\nu} \frac{\partial_\mu \Phi \partial_\nu \Phi}{\Phi} + F(\Phi) \tfrac{1}{2} g^{\mu\nu} \partial_\mu\sigma\partial_\nu\sigma \right.$$
$$\left. - G(\Phi) V(\sigma) \right\}. \qquad (3.34)$$

Here we have allowed for a form that is more general than that of the original model (eqs.(2.2, 2.3)); this will prove useful when we discuss model building in the next lecture. Note that if $\Phi_0 = \frac{1}{16\pi G_N}$, $F(\Phi_0) = G(\Phi_0) = 1$.

Going to the Einstein frame as before yields the following new action:

$$\bar{S} = \int d^4x \sqrt{-\bar{g}} \left\{ -\frac{\bar{R}}{16\pi G_N} + \tfrac{1}{2} \bar{g}^{\mu\nu} \partial_\mu\psi\partial_\nu\psi - f(\tfrac{\psi}{\psi_0}) \tfrac{1}{2} \bar{g}^{\mu\nu} \partial_\mu\sigma\partial_\nu\sigma \right.$$
$$\left. - g(\psi/\psi_0) V(\sigma) \right\}, \qquad (3.35)$$

with $\bar{g}_{\mu\nu}$ and ψ as defined in eqs.(2.27, 2.29) and

$$f(\tfrac{\psi}{\psi_0}) \equiv \exp(-\tfrac{\psi}{\psi_0}) F(\Phi(\psi))$$
$$g(\tfrac{\psi}{\psi_0}) \equiv \exp(-2\tfrac{\psi}{\psi_0}) G(\Phi(\psi)). \qquad (3.36)$$

Implementing the approximation discussed above in Euclidean space leaves us with the following truncated action:

$$\bar{S}_E = \int d^4x \left[\tfrac{1}{2} f(\tfrac{\psi}{\psi_0}) \partial_\mu\sigma \partial^\mu\sigma + g(\tfrac{\psi}{\psi_0}) V(\sigma) \right]. \qquad (3.37)$$

Next rescale the coordinates to

$$\hat{x}^\alpha = \sqrt{\frac{g(\psi/\psi_0)}{f(\psi/\psi_0)}}\, x^\alpha, \qquad (3\text{-}38)$$

so that $\bar{S}_E$ becomes

$$\bar{S}_E = \frac{f^2(\Psi/\Psi_0)}{g(\Psi/\Psi_0)} \int d^4\hat{x} \left[\frac{1}{2} \hat{\partial}_\mu \sigma \hat{\partial}^\mu \sigma + V(\sigma) \right] \equiv \frac{f^2(\frac{\Psi}{\Psi_0})}{g(\frac{\Psi}{\Psi_0})} S_0 \qquad (3.39)$$

S_0 is clearly the Euclidean action when $\Psi = 0$. Thus, if σ_b is the bounce configuration we are looking for, then $\sigma_b(x) = \hat{\sigma}_B\left(\sqrt{\frac{g(\Psi/\Psi_0)}{f(\Psi/\Psi_0)}}\, x\right)$ where $\hat{\sigma}_B(\hat{x})$ is the bounce configuration when $\Psi = 0$. Furthermore,

$$\bar{B} = \frac{f^2(\Psi/\Psi_0)}{g(\Psi/\Psi_0)} B_0 . \qquad (3.40)$$

The last thing we need to do to compute the tunnelling rate in the Einstein frame is to compute the prefactor

$$A = \left\{ \left(\frac{C}{2\pi} \right)^{1/2} \right\}^4 \left| \frac{\det' \left[-f(\Psi/\Psi_0)\, \partial^2 + g(\Psi/\Psi_0)\, V''(\sigma_B) \right]}{\det \left[-f(\frac{\Psi}{\Psi_0})\, \partial^2 + g(\Psi/\Psi_0)\, V''(\sigma_{FV}) \right]} \right|^{-1/2} \qquad (3.41)$$

The quantities C are the normalization factors of the zero modes $\partial_\lambda \sigma_B$:

$$C = \int d^4x\, (\partial_\lambda \sigma_B)^2 = \int \underbrace{d^4\hat{x}}_{(\frac{g}{f})^{-2}} \left(\sqrt{\frac{g(\Psi/\Psi_0)}{f(\Psi/\Psi_0)}} \right)^2 (\hat{\partial}_\lambda \hat{\sigma}_B)^2 = \left(\frac{g(\Psi/\Psi_0)}{f(\Psi/\Psi_0)} \right)^{-1} \hat{C} \qquad (3.42)$$

Finally, we note that since if $\Psi_\theta(\hat{x})$ is an eigenstate of the operator $-\hat{\partial}^2 + V''(\hat{\sigma}_B)$ with eigenvalue θ, then $\Psi_\theta\left(\sqrt{\frac{g(\Psi/\Psi_0)}{f(\Psi/\Psi_0)}}\, x\right)$ is an eigenfunction of $-f(\Psi/\Psi_0)\, \partial^2 + g(\Psi/\Psi_0)\, V''(\sigma)$ with eigenvalue $g(\frac{\Psi}{\Psi_0})\theta$. Since the primed determinant has four fewer eigenvalues than the unprimed, we find:

$$A = \left(\frac{f(\Psi/\Psi_0)}{g(\Psi/\Psi_0)} \right)^2 g^2(\Psi/\Psi_0)\; \hat{A} = f^2(\Psi/\Psi_0)\, \hat{A} . \qquad (3.43)$$

Putting everything together yields

$$\bar{\lambda}(t) = f^2(\Psi/\Psi_0) \exp\left(-B_0 \left(\frac{f^2(\Psi/\Psi_0)}{g(\Psi/\Psi_0)} - 1 \right) \right) \lambda_0 , \qquad (3.44)$$

where λ_0 is the $\psi = 0$ value of the nucleation rate. Our final task is to convert this back to the Jordan frame. Since λ is the decay probability per unit proper 4-volume,

$$\lambda(t) \equiv \frac{dP}{\sqrt{-g}\, dt\, d^3x} = \frac{dP}{\sqrt{-\bar{g}}\, d\bar{t}\, d^3x} \frac{d\bar{t}}{dt} \cdot \frac{\sqrt{-\bar{g}}}{\sqrt{-g}} = \bar{\lambda}(\bar{t}) \left(\frac{\bar{a}^3(\bar{t})}{a^3(t)} \right) \frac{d\bar{t}}{dt}$$

$$= \Omega^{-4}(t)\, \bar{\lambda}(\bar{t}), \qquad (3.45)$$

where we have used eq(2.30). But $\Omega^2 \equiv \exp(-\psi/\psi_0)$ so that

$$\lambda(t) = \left[\exp(+\psi/\psi_0)\, f(\psi/\psi_0)\right]^2 \exp\left[-B_0\left(\frac{f^2(\psi/\psi_0)}{g(\psi/\psi_0)} - 1\right)\right] \lambda_0$$

$$= F^2(\Phi) \exp\left[-B_0\left(\frac{F^2(\Phi)}{G(\Phi)} - 1\right)\right] \lambda_0, \qquad (3.46)$$

We see that the promised exponential dependence on Φ is indeed present. If $F = G \equiv 1$, as in the original model, then $\lambda(t) = \lambda_0$ (Jordan frame). If $F(\Phi) = (16\pi G_N \Phi)^n$, $G(\Phi) = (16\pi G_N \Phi)^m$, as can occur in some models,

$$\lambda(t) = (16\pi G_N \Phi)^{2n} \exp\left[-B_0 (16\pi G_N \Phi)^{2n-m}\right] \lambda_0. \qquad (3.47)$$

We conclude with some comments on the validity of our approximation. We expect corrections of order G_N to be negligible as long as the effective Planck mass generated by Φ is much larger than that of the inflaton σ. In the original model, Φ grows in time, so our approximation should be best at late times. ~~We can also calculate~~.

Non-Equilibrium Quantum Field Theory: Techniques and Applications [Wk4]

We are now ready to tackle non-equilibrium dynamics. What we have seen is that we can encode what type of expectation value we take by imposing different b.c. on fields in the path integral, as well as by choosing different contours in time.

So suppose we have general density matrix $\rho(t)$, which need not be an equilibrium one (though it may have been one originally). We know that ρ satisfies the Liouville equation

$$i \frac{\partial}{\partial t} \rho = [H, \rho], \qquad \rho(t_0) = \rho_0$$

We can solve this equation as $\rho(t) = U(t,t_0)\, \rho_0\, U^{-1}(t,t_0)$, where $i \frac{\partial}{\partial t} U(t,t_0) = H\, U(t,t_0)$, $U(t_0,t_0) = 1$ is the time evolution operator (which is not just $e^{-iH(t-t_0)}$ since H will contain time dependent pieces now).

Suppose ρ is not yet normalized, eg $\rho = e^{-\beta H}$ w/o the $\frac{1}{Z}$ normalization. Then let's take the trace of ρ in the position representation:

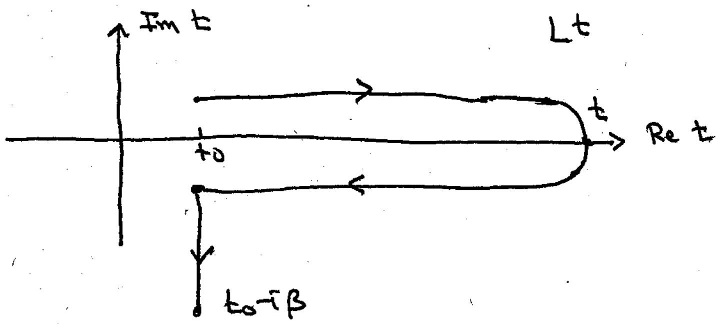
$$\mathrm{Tr}\, \rho = \int dx \, \langle x | U(t,t_0)\, \rho_0\, U^{-1}(t,t_0) | x \rangle$$

$$= \int dx\, dy\, dy' \, \langle x | U(t,t_0) | y \rangle \langle y | \rho_0 | y' \rangle \langle y' | U(t_0,t) | x \rangle$$

$$= \int dx\, dy\, dy' \, \langle y | \rho_0 | y' \rangle \langle y' | U(t_0,t) | x \rangle \langle x | U(t,t_0) | y \rangle$$

If we suppose (as we will do for most of the rest of the class) that the initial density matrix is thermal $\rho_0 = e^{-\beta_0 H_i}$ and that the full Hamiltonian is $H = \begin{cases} H_i & t \le t_0 \\ H_{evol} & t > t_0 \end{cases}$, then

$\langle y | \rho_0 | y' \rangle = \langle y | U(t_0 - i\beta, t_0) | y' \rangle$. The expression for the trace of ρ can be read as a road map in the complex t-plane: start at $(y, t_0) \longrightarrow (x, t) \longrightarrow (y', t_0) \longrightarrow (y, t_0 - i\beta)$



Thus the paths we should consider are those define on the above closed time path, together with the periodic b.c. between $t_0, t_0 - i\beta$. Let's tighten this up. Consider an operator $\mathcal{O}$ whose time-dependent expectation value we want to compute. By definition:

$$\langle \mathcal{O} \rangle(t) = \frac{\mathrm{Tr}\,[\rho(t)\,\mathcal{O}]}{\mathrm{Tr}\,[\rho(t)]} = \frac{\mathrm{Tr}\,[U(t,t_0)\,\rho_0\,U^{-1}(t,t_0)\,\mathcal{O}]}{\mathrm{Tr}\,[\rho_0]}$$

$$= \frac{\mathrm{Tr}\,[\rho_0\,U^{-1}(t,t_0)\,\mathcal{O}\,U(t,t_0)]}{\mathrm{Tr}\,[\rho_0]}.$$ Now assume thermal initial conditions ~~on~~ ρ_0 and chose an arbitrary initial time $T < 0$ so that ~~$U(t_0,T)$~~ $U(T) = e^{-iTH_i}$ (we take $t_0 = 0$) and $e^{-\beta H_i} = U(T - i\beta, T)$. Inserting $U^{-1}(T)\,U(T) = I$ and commuting ρ_0 w/ $U^{-1}(T)$, we have

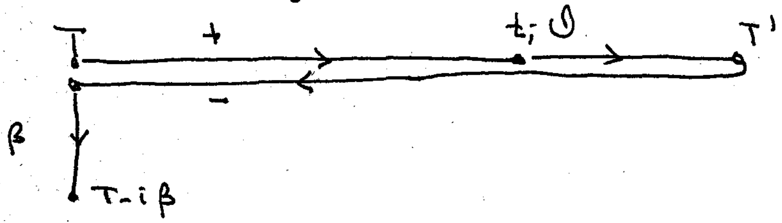
$\langle \mathcal{O} \rangle(t) = \mathrm{Tr}\,[U(T-i\beta,T)\,U^{-1}(T)\,U(T)\,U^{-1}(t)\,\mathcal{O}\,U(t)] / \mathrm{Tr}\,[U(T-i\beta,T)] =$

$\mathrm{Tr}\,[U(T-i\beta,T)\;U(T,t)\,\mathcal{O}\,U(t,T)] / \mathrm{Tr}\,[U(T-i\beta,T)]$.

Now introduce an arbitrary large positive time T' and write

$$\langle \mathcal{O} \rangle(t) = \mathrm{Tr}\left[U(T-i\beta,T)\, U(T,T')\, U(T',t)\, \mathcal{O}\, U(t,T)\right] / \mathrm{Tr}\left[U(T-i\beta,T)\right]$$

What this says to do is:



Finally we take $T \to -\infty$, $T' \to \infty$ to get the entire contour.
Thus we have closed time path on which our fields are to be defined. There is a time ordering, where times on the + path are earlier than those on the − path which in turn are earlier than those on the β path.

Instead of defining fields on the entire path path we think of doubling (actually tripling) the field content; $\Phi^{\pm}$ defined on the $\pm$ parts of the contour, and Φ^{β} on the β path.

The operator $\mathcal{O}$ can be inserted via functional differentiation with respect to sources $J^{\pm}, J^{\beta}$ defined on the contour. We then define the generating functional:

$$Z[J^+; J^-; J^\beta] = \mathrm{Tr}\left[U(T-i\beta, T; J^\beta)\, U(T,T'; J^-)\, U(T',T; J^+)\right]$$

in the $T \to -\infty$, $T' \to +\infty$ limit. This has a path integral representation:

$$Z[J^+, J^-, J^\beta] = \int \mathcal{D}\Phi^+ \mathcal{D}\Phi^- \mathcal{D}\Phi^\beta \, e^{\, i\int_T^{T'} d^4x \left[\mathcal{L}^{J^+}[\Phi^+] - \mathcal{L}^{J^-}[\Phi^-]\right]}$$
$$\times \, e^{\, i\int_T^{T-i\beta} d^4x \, \mathcal{L}^{J^\beta}[\Phi^\beta]}$$

w/ the b.c. $\Phi^+(T') = \Phi^-(T')$, $\Phi^-(T) = \Phi^\beta(T)$ & $\Phi^+(T) = \Phi^\beta(T-i\beta)$.

In the limit $T \to -\infty$, it can be shown that cross correlations between $\Phi^{\pm}$, and Φ^{β} vanish (using the Riemann-Lebesgue lemma). The contour β essentially serves to fix the initial density matrix as thermal. Thus the generating functional for computing <u>real time</u> correlations is

$$Z[J^+, J^-] = e^{i\int_T^{T'} d^4x \left[\mathcal{L}_{int}\left(-i\frac{\delta}{\delta J^+}\right) - \mathcal{L}_{int}\left(-i\frac{\delta}{\delta J^-}\right)\right]} \quad Z_0[J^+, J^-]$$

where, as before

$$Z_0[J^+, J^-] = \exp\left[\frac{i}{2}\int_T^{T'} d^4x_1 \int_T^{T'} d^4x_2 \; J^a(x_1)\, G_{ab}(x_1, x_2)\, J^b(x_2)\right]$$

$a, b = \pm$. There are now <u>four</u> Green's functions to deal with. However, these can be obtained much in the same way that we dealt with Green's functions before. ~~We tea~~ For the free theory, we have to solve:

$$(\Box_x + m^2)\, G(x, x') = \delta_C^{(4)}(x - x') \quad , \quad C \equiv \text{contour } \delta \text{ fcn}$$

We can use space translation invariance to decompose $G(x, x')$ as

$$G(x, x') = \int \frac{d^3k}{(2\pi)^3}\, e^{-i\vec{k}\cdot(\vec{x}-\vec{x}')}\, G_k(t, t')$$

The function $G_k(t, t')$ obeys the equation:

$$\left[\frac{d^2}{dt^2} + k^2 + m^2\right] g_k(t, t') = \delta_C(t - t')$$

As before we write $g_k(t, t') = \Theta_C(t - t')\, g_k^{>}(t, t') + \Theta_C(t' - t)\, g_k^{<}(t, t')$

where Θ_C are <u>contour</u> Θ fcns, where times on the β contour are later than those on the $-$ contour, which in turn are later than those on $+$ contour.

Imposing the equation of motion, we obtain the usual conditions:

- Continuity: $g_k^>(t,t) = g_k^<(t,t)$

- δ-fcn jump: $\dot{g}_k^>(t,t') - \dot{g}_k^<(t,t')\Big|_{t'=t} = 1$

- periodicity: $g_k^>(T-i\beta, t') = g_k^<(T,t')$ for any t' (KMS)

- $\left[\frac{d^2}{dt^2} + \omega_k^2\right] g_k^{\gtrless}(t,t') = 0$, $\omega_k^2 = k^2+m^2$

Then $g_k^{\gtrless}(t,t') = A_k^{\gtrless}(t')\, e^{i\omega_k t} + B_k^{\gtrless}(t')\, e^{-i\omega_k t}$; imposing the above conditions we have:

$$g_k^>(t,t') = +\frac{i}{2\omega_k}\left[n(\omega_k)\, e^{i\omega_k(t-t')} + (n(\omega_k)+1)\, e^{-i\omega_k(t-t')}\right]$$

$$g_k^<(t,t') = +\frac{i}{2\omega_k}\left[(n(\omega_k)+1)\, e^{i\omega_k(t-t')} + n(\omega_k)\, e^{-i\omega_k(t-t')}\right]$$

Note that in this situation, since ω_k^2 is constant, we have time translation invariance, which shows up in the form of $g_k^{\gtrless}(t,t')$; this is <u>not</u> the usual situation.

We see the effects of the initial thermal condition in the distribution $n(\omega_k)$ appearing in $g_k^{\gtrless}$; however, unlike the imaginary time case, there is still evolution in <u>real</u> time.

The free field generating functional is then:

$$Z_0[J_c] = e^{\frac{i}{2}\langle J_c G^c J_c\rangle}$$

Let's open up $\langle J_c G_k^c J_c\rangle$. ~~By space translation invariance, this is just $\int \langle J_{ck} G_k^c J_{ck}\rangle\, d^3k\, e^{-i\vec{k}(\vec{x}-\vec{x}')}\, d^3k/(2\pi)^3$, and~~

We have to take which contour the time variables are on into account. We will set J^β to zero since we are not concerned with equilibrium, imaginary time, phenomena and write the integrals in Z_0 as:

$$\left(\int_T^{T'} d\tau + \int_{T'}^{T} d\tau \right)\left(\int_T^{T'} d\tau' + \int_{T'}^{T} d\tau' \right) J_c(\tau)\; G^c(\tau,\tau')\; J_c(\tau')$$

$$= \int_T^{T'} d\tau \int_T^{T'} d\tau' \left(J^+ G_{++} J^+ + J^- G_{--} J^- \right) - \int_T^{T'} d\tau \int_T^{T'} d\tau' \left(J^+ G_{+-} J^- + J^- G_{-+} J^+ \right)$$

where $+,-$ indicate the contour on which the relevant time variables live on.

From this and the definition of the (contour) 2-pt function, we read off:

$$i \langle T_+(\phi(x)\,\phi(x'))\rangle = G_{++}(x,x'), \quad i\langle T_-(\phi(x)\,\phi(x'))\rangle = G_{--}(x,x')$$

$$i\langle \phi(x_+)\,\phi(x_-)\rangle = -G_{+-}(x_+,x_-), \quad i\langle\phi(x_-)\,\phi(x_+)\rangle = -G_{-+}(x,x')$$

We are now ready to tackle some problems.

The first question we will ask is the following. Consider a field Φ with potential $V(\Phi) = \frac{1}{2} m^2 \Phi^2 + \frac{\lambda}{4!}\Phi^4$. We would like to define a time dependent expectation value of this field and follow its evolution in time, at least within some approximation.

We can do this by setting $\Phi^\pm = \phi(t) + \psi^\pm(\vec{x},t)$, where $\phi(t)$ is the "zero (momentum) mode" and $\psi^\pm$ are fluctuations about it on the $\pm$ contours. How do we get the equation of motion for $\phi(t)$?

Having shifted the field, we note that the term linear in $\psi^\pm$ in the shifted action is just $\left\langle \frac{\delta S^\pm}{\delta\Phi}\Big|_{\Phi=\phi} \psi^\pm \right\rangle$. ~~This implies that~~

Suppose we were to compute $\langle \psi^{\pm} \rangle$ in this shifted theory, to lowest order in perturbation theory. Then we would have:

$-i \frac{\delta}{\delta J^{\pm}_x} Z[J^{\pm}]\Big|_{J=0} = \langle \psi^{\pm}_x \rangle$; if we neglect the interaction terms such as $\psi^{\pm 3}$, $\psi^{\pm 4}$ for now we would find that $Z[J] = e^{i/2 \langle \tilde{J} G \tilde{J} \rangle}$ with $\tilde{J}^{\pm}_x = J^{\pm}_x - \vartheta^{\pm}_x$ where $\vartheta^{\pm}_x \equiv \dot{\phi} \frac{\delta S}{\delta \Phi^{\pm}}\Big|_{\Phi^{\pm}=\phi}$ (note that $\vartheta^{\pm}$ will actually be contour independent if ϕ is).

Taking the relevant functional derivative yields $\langle G_{\pm b}(x,w) \tilde{J}_b \rangle_w$ evaluated at $J=0$. But this is just $\langle G_{\pm +} \vartheta_w \rangle + \langle G_{\pm -} \vartheta_w \rangle$. From this we see that setting $\langle \psi^{\pm}_x \rangle = 0$ yields $\vartheta_x = 0$ which is the equation of motion for ϕ at this order. It can be shown (tadpole method) that this is true to <u>all</u> orders.

Let's now do this in $\frac{\lambda}{4!}\Phi^4$ theory. Shifting the fields, defined on the contour, by ϕ yields:

$$\Phi_c^{2} \to \phi^2 + 2\phi\psi_c + \psi_c^2$$

$$\Phi_c^4 \to \phi^4 + 4\phi^3\psi_c + 6\phi^2\psi_c^2 + 4\phi\psi_c^3 + \psi_c^4$$

The shifted <u>contour</u> action is then; after the relevant integrations by parts:

$$S[\psi_c; \phi] = \int_C d^4x \Big[-\psi_c(\ddot{\phi} + m^2\phi + \frac{\lambda}{3!}\phi^3) - \frac{1}{2}\psi_c(\Box + m^2)\psi_c + J_c\psi_c - \frac{\lambda}{4!}(4\phi\psi_c^3 + 6\phi^2\psi_c^2 + \psi_c^4) \Big]$$

We assume that $\frac{1}{2}m^2 \gg \frac{\lambda}{4}\phi^2$, so that we don't have to adjoin the $\lambda\phi^2$ term to the mass in the ψ_c propagator.

Let's now compute $\langle \psi_c \rangle$ as described above. First we write:

$$Z[J_c] = \exp\left[-i\lambda\int_C d^4z \left[\frac{1}{6}\,\phi\left(-i\frac{\delta}{\delta J_c(z)}\right)^3 + \frac{\phi^2}{2}\left(-i\frac{\delta}{\delta J_c(z)}\right)^2 + \frac{1}{4!}\left(-i\frac{\delta}{\delta J(z)}\right)^4\right]\right] Z_0[J_c]$$

w/ $Z_0[J_c] = \exp\left[\frac{i}{2}\langle \tilde J_c G_c \tilde J_c\rangle\right]$ where G_c is the contour Green's function obtained earlier. Now note that we can trade $\frac{\delta}{\delta J_c}$'s for $\frac{\delta}{\delta \tilde J_c}$ (recall, $\tilde J_c = J_c - \mathcal{O}$, $\mathcal{O} = \ddot\phi + m^2\phi + \frac{\lambda}{3!}\phi^3$).

$\mathcal{O}(\lambda^0)$: $-i\frac{\delta}{\delta J_c(x)} Z_0[J_c] = -i\frac{\delta}{\delta\tilde J_c(x)} Z_0[J_c] = \langle G_c(x,y)\,\tilde J_c(y)\rangle_y \Big|_{J=0} =$

$-\langle G_c(x,y)\,\mathcal{O}_y\rangle$

$\mathcal{O}(\lambda^1)$: $-i\frac{\delta}{\delta J_c(x)}\left[-i\lambda\int d^4z\left\{\frac{\phi}{6}\left(-i\frac{\delta}{\delta J_c(z)}\right)^3 + \frac{\phi^2}{2}\left(-i\frac{\delta}{\delta J_c(z)}\right)^2+\frac{1}{4!}\left(-i\frac{\delta}{\delta J_c(z)}\right)^4\right\}\right] e^{i/2\langle\tilde J_c G_c \tilde J_c\rangle}\Big|_{J=0}$

Look at $\lambda\frac{\phi_z^2}{2}\left(-i\frac{\delta}{\delta J_c(z)}\right)^2\left(-i\frac{\delta}{\delta J_c(x)}\right)e^{i/2\langle\tilde J_c G_c\tilde J_c\rangle}$

$= i\lambda\frac{\phi_z^2}{2}\left(-i\frac{\delta}{\delta J_c(z)}\right)^2\langle i\, G_c\tilde J_c\rangle\, e^{i/2\langle\tilde J G_c\tilde J\rangle}$

$= -\lambda\frac{\phi_z^2}{2}\left[\frac{\delta}{\delta J_c(z)}\right]\left[G_c(x,z) + i\langle G_c(z,w)\tilde J_w\rangle\langle G_c(x,w')\tilde J_{w'}\rangle\right]e^{i/2\langle\tilde J G\tilde J\rangle}$

$= -i\lambda\frac{\phi_z^2}{2}\Big[G_c(z,z)\langle G_c(x,w')\tilde J_{w'}\rangle + \langle G_c(z,w)\tilde J_w\rangle G_c(x,z)$

$+ \langle G_c(z,w'')\tilde J_c(w'')\rangle\left[G_c(x,z) + i\langle G_c(z,w)\,G_c(x,w)\,\tilde J_w\tilde J_{w'}\rangle\right]\Big]e$

~~at J=0. Now note that the difference between the full eqn of~~ motion and the $\mathcal{O}(\lambda^0)$ version is at least of order λ, i.e. the difference between J & $\tilde J = J - \mathcal{O}^{(0)} = J - (\mathcal{O} + \lambda\bar{\mathcal{O}})$

λ^0 eqn of motion full eqn of motion $\mathcal{O} = \mathcal{O}^0 + \lambda\bar{\mathcal{O}}$

is at best of order λ

Thus, setting $J=0$ when the full eqn of motion is valid, i.e. when $\mathcal{O}=0$, is equivalent, to the <u>given order</u> in λ to setting $\tilde{J}=0$. Thus the above term vanishes. In fact, only terms with an even total # of derivations will survive. To $\mathcal{O}(\lambda)$ this leaves:

$$-i\frac{\delta}{\delta J_c(x)}\left(\frac{-i\lambda}{6}\int d^4z\ \phi_z\left(-i\frac{\delta}{\delta J_c}\right)^3\right)e^{i/2\langle \tilde{J}_c G_c \tilde{J}_c\rangle}\Big|_{J=0} =$$

$$-\frac{i\lambda}{6}\int d^4z\ \phi_z\ \frac{\delta^3}{\delta\tilde{J}_c(z)^3}\ \frac{\delta}{\delta\tilde{J}_c(x)}\left(\frac{i}{2}\right)^2\frac{1}{2!}\langle \tilde{J}_c G_c \tilde{J}_c\rangle^2\Big|_{\tilde{J}=0}$$

$$\frac{\delta}{\delta\tilde{J}_c(x)}\ \langle \tilde{J}_c G_c \tilde{J}_c\rangle^2 = 2\cdot 2\ \langle G_c(x,w)\tilde{J}_w\rangle\ \langle \tilde{J}_c G_c\tilde{J}_c\rangle$$

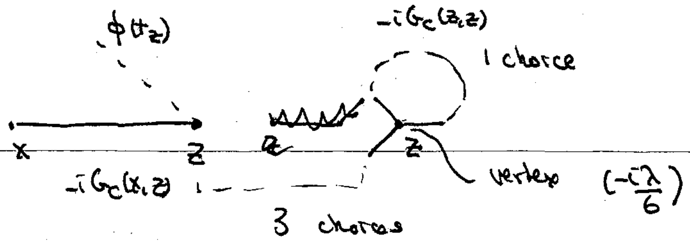
$$\frac{\delta}{\delta\tilde{J}_c(z)}:\ 4\left[G_c(x,z)\langle\tilde{J}G\tilde{J}\rangle + 2\langle G_c(x,w)\,G_c(z,w')\tilde{J}_w\tilde{J}_{w'}\rangle\right.$$

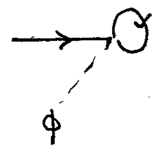
$$\frac{\delta}{\delta\tilde{J}_c(z)}:\ 4\Big[2G_c(x,z)\langle G_c(z,w)\tilde{J}_w\rangle + 2\,G_c(x,z)\langle G_c(z,w)\tilde{J}_w\rangle$$
$$+\,2\,G_c(z,z)\ \langle G_c(x,w)\tilde{J}_w\rangle\Big]$$

$$\frac{\delta}{\delta\tilde{J}(z)}:\ 8\cdot 3\left[G_c(x,z)\ G_c(z,z)\right]$$

$$\therefore\ -\frac{i\lambda}{6}\cdot\frac{(i)^2}{8}\cdot 8\cdot 3\int d^4z\ G_c(x,z)\,G_c(z,z)\ \phi(t_z)$$

$$= -\frac{i\lambda}{2}\int d^4z\ (-iG_c(x,z))\,(-iG_c(z,z))\ \phi(t_z)$$

Diagramatically 

$$-\frac{i\lambda}{6}\cdot 3\int d^4z\ \phi(t_z)\ (-iG_c(x,z))\,(-iG_c(z,z)) \equiv$$ 

Thus to this order in λ, the equation of motion takes the form

$$\left\langle -G_c(x,z)\left[\mathcal{O}_z + i\frac{\lambda}{2}\, G_c(z,z)\, \phi(t_z)\right]\right\rangle_z = 0$$

or $\ddot{\phi} + m^2\phi + \frac{\lambda}{3!}\phi^3 + \frac{\lambda}{2}\phi\,\left(-i\,G_c(z,z)\right) = 0.$

Now $-iG_c(z,z) = \int \frac{d^3k}{(2\pi)^3}\frac{1}{2\omega_k}\underbrace{\left(2n(\omega_k)+1\right)}_{\coth \beta\frac{\omega_k}{2}} = \underbrace{\int\frac{d^3k}{(2\pi)^3}\frac{n(\omega_k)}{\omega_k}}_{T\text{- dep; finite}} + \underbrace{\int\frac{d^3k}{(2\pi)^3}\frac{1}{2\omega_k}}_{T\text{ indep; divergent}}$

We see here the first need to perform renormalization; the $T=0$ part of the integral is divergent. If we perform it with a cutoff Λ and extract the high Λ behavior, we have

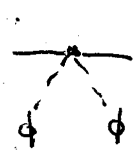
$$\int_\Lambda \frac{d^3k}{(2\pi)^3}\frac{1}{2\omega_k} = \frac{1}{4\pi^2}\int_0^\Lambda dk\,\frac{k^2}{\sqrt{k^2+m^2}} = (\Lambda\to\infty) = \frac{1}{4\pi^2}\left[\frac{\Lambda^2}{2} - \frac{1}{4}m^2\ln\frac{\Lambda^2}{m^2}\right]$$

Since this term is linear in ϕ as is the $m^2\phi$ term, we should group these together to define the renormalized mass. The eqn of motion becomes:

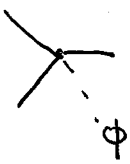
$$\ddot{\phi} + m_R^2\phi + \frac{\lambda}{3!}\phi^3 + \frac{\lambda}{2}\phi\int\frac{d^3k}{(2\pi)^3}\frac{n(\omega_k)}{\omega_k} + \underbrace{\left\{\int_\Lambda\frac{d^3k}{(2\pi)^3}\left[\frac{1}{2\omega_k}\right] - \frac{1}{4\pi^2}\left(\frac{\Lambda^2}{2} - \frac{1}{4}m^2\ln\frac{\Lambda^2}{m^2}\right)\right\}}_{T\text{-indep; finite as } \Lambda\to\infty.}$$

Having seen this, we expect that the λ term will have to be renormalized as well; thus, we compute terms w/ ϕ^3 type interactions at $\mathcal{O}(\lambda^2)$.

~~We can now proceed diagrammatically. The vertices are~~

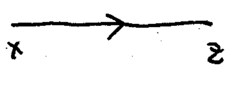


$= \frac{-i\lambda}{2}$



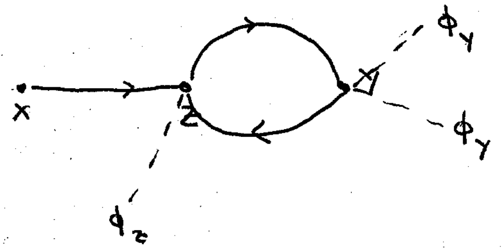
$= \frac{-i\lambda}{6}$

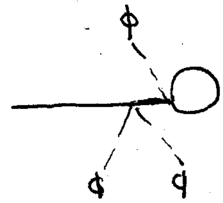

$-i\frac{\lambda}{4!}$



$= -i\,G_c(x,z).$

To order λ^2 we have, one-loop diagrams of the form:

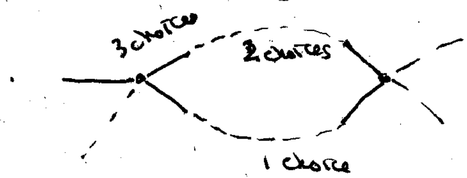


There are also diagrams of the form  ; we don't include these since they are not one particle irreducible; they can be made disconnected by cutting a line (between vertex and bubble). These can be shown not to appear in effective equation of motion.

Our new contribution to the equations of motion ~~lies~~ is

$$\left(\frac{-i\lambda}{6}\right)\left(\frac{-i\lambda}{2}\right) S \int_C d^4z\, d^4y\, (-i G_c(x,z))\,(-i G_c(z,y))\,(-i G_c(y,z))\,\phi(t_y)^2\,\phi(t_z)$$

where S is the combinatoric factor:



$\Rightarrow$ $S = 2\cdot 3 = 6$

$$\therefore\ \ddot{\phi} + m_R^2\phi + \frac{\lambda}{3}\phi^3 + \frac{\lambda}{2}\phi\,(-iG_c(z,z)_R) + (+i)\frac{\lambda^2}{4}\phi\int_C d^4y\ \phi^2(t_y)(-iG_c(z,y))(-iG_c(y,z)) = 0$$

Let's analyze the last term. We take the time t_z to be on the + contour. Then

$$\int_C dt_y\, (-iG_c(z,y))(-iG_c(y,z)) = \int_+ dt_y\, [(-iG^{++}(z,y))(-iG^{++}(y,z))] + \int_- dt_y\, [(-iG^{+-}(z,y))(-iG^{-+}(y,z))] = \int_+ dt_y\, \left[(-iG^{++}(z,y))^2 - (-iG^{+-}(z,y))^2\right]$$

Now use

$G^{++}(z,y) = \Theta(t_z - t_y)\, G^{>}(z,y) + \Theta(t_y - t_z)\, G^{<}(z,y)$ as well as

$G^{+-}(z,y) = G^{<}(z,y)$ and the fact the $\Theta(t)^2 = \Theta(t)$, $\Theta(t)\Theta(-t) = 0$

to find:

$$G^{++}(z,y)^2 - G^{+-}(z,y)^2 = \Theta(t_z - t_y)\, G^{>}(z,y)^2 + \Theta(t_y - t_z)\, G^{<}(z,y)^2 - G^{<}(z,y)^2$$

$$= \Theta(t_z - t_y)\left(G^{>}(z,y)^2 - G^{<}(z,y)^2\right).$$

We can evaluate the difference of squares by factoring it:
$G^{>2} - G^{<2} = (G^{>} + G^{<})(G^{>} - G^{<})$. Let's use the propagators found earlier to write, after Fourier transforming.

$$G^{>}_k(z,y) = \frac{i}{2\omega_k}\left[n_k\, e^{i\omega_k(t_z - t_y)} + (n_k+1)\, e^{-i\omega_k(t_z - t_y)}\right]$$

$$G^{<}_k(z,y) = \frac{i}{2\omega_k}\left[n_k\, e^{-i\omega_k(t_z - t_y)} + (n_k+1)\, e^{i\omega_k(t_z - t_y)}\right]$$

$$G^{>}_k + G^{<}_k = \frac{i}{2\omega_k}\left[e^{i\omega_k(t_z - t_y)} + e^{-i\omega_k(t_z - t_y)}\right](2n_k+1)$$

$$= \frac{i}{\omega_k}\cos\omega_k(t_z - t_y)\,(2n_k+1)$$

$$G^{>}_k - G^{<}_k = \frac{-i}{2\omega_k}\left[e^{i\omega_k(t_z - t_y)} - e^{-i\omega_k(t_z - t_y)}\right] = \frac{1}{\omega_k}\sin\omega_k(t_z - t_y)$$

$$G^{>2}_k - G^{<2}_k = \frac{i}{2\omega_k^2}\sin 2\omega_k(t_z - t_y)\quad(2n_k+1)$$

The contribution to the equations of motion is then

$$i\frac{\lambda^2}{4}\,\frac{i}{2}\,\phi(t)\int dt'\,\Theta(t-t')\,\phi^2(t')\int\frac{d^3k}{(2\pi)^3}\,\frac{1}{2\omega_k^2}\sin 2\omega_k(t-t')\,\underbrace{(2n_k+1)}_{1 \text{ when } T=0}$$

$$= -\frac{\lambda^2}{8}\,\phi(t)\int dt'\,\Theta(t-t')\,\phi^2(t')\,\bar{K}(t-t')$$

w/ $\bar{K}(t-t') \equiv \int\frac{d^3k}{(2\pi)^3}\,\frac{1}{2\omega_k^2}\sin 2\omega_k(t-t')$.

Write $\bar{K}(t-t') = \frac{d}{dt'} K(t-t')$ where $K(t-t') = \int \frac{d^3k}{(2\pi)^3} \frac{1}{4\omega_k^3} \cos 2\omega_k(t-t')$

and integrate by parts. The contribution in the equations of motion becomes:

$$-\frac{\lambda^2}{8}\phi(t)\left\{\left[K(t-t')\,\theta(t-t')\,\phi^2(t')\right|_{-\infty}^{\infty} - \int dt' \left\{\delta(t-t')\,\phi^2(t') + \theta(t-t')\,2\phi(t')\dot{\phi}(t')\right\} K(t-t')\right\}$$

$$= +\frac{\lambda^2}{8}\left[+\phi^2(t)\,K(0) + 2\int_{-\infty}^{t} dt'\,\phi(t')\,\dot{\phi}(t')\,K(t-t')\right]\phi(t) = \left(\begin{array}{c}\text{surface term}\\ \text{vanishes}\end{array}\right)$$

$$= \frac{\lambda^2}{8}K(0)\,\phi^3(t) + \frac{\lambda^2}{4}\phi(t)\int_{-\infty}^{t} dt'\,\phi(t')\,\dot{\phi}(t')\,K(t-t')$$

The first term is proportional to ϕ^3 just as the $\frac{\lambda}{3!}\phi^3$ term and serves to renormalize the quartic coupling λ:

$\frac{\lambda_B}{3!} + \frac{\lambda_B^2}{8}K(0) = \frac{\lambda_R}{3!}$ or $\lambda_R = \lambda_B + \frac{3}{4}\lambda_B^2 K(0)$. Note that

$K(0) = \int \frac{d^3k}{(2\pi)^3}\frac{1}{4\omega_k^3}$ and is logarithmically divergent; if we again use a cutoff Λ to regulate the high k behavior we have

$$K(0) = \frac{1}{8\pi^2}\int_0^{\Lambda} dk\,\frac{k^2}{(k^2+m^2)^{3/2}} =$$

Thus, to this order in λ_R, our equation of motion becomes:

$$\ddot{\phi} + m_R^2\phi + \frac{\lambda_R}{3!}\phi^3 + \frac{\lambda_R^2}{4}\phi\int_{-\infty}^{t} dt'\,\phi(t')\,\dot{\phi}(t')\,K(t-t') = 0.$$

The amazing thing about this equation is the <u>first</u> order derivative in the last term. This is a <u>dissipative</u> effect. What has happened is that the system has been made open by "tracing" out the other degrees of freedom ~~for~~ than ϕ. Now ϕ can lose its energy into an infinite number of modes.

The Failure of Naive Perturbation Theory

Our effective equation of motion takes the form

$$\ddot{\phi} + m_R^2 \phi + \frac{\lambda_R}{3!}\phi^3 + \frac{\lambda_R^2}{4}\phi \int_{-\infty}^{t} dt' \, \phi(t') \, \dot{\phi}(t') \, K(t-t') = 0$$

Now, this equation was derived within the context of both perturbation theory in λ_R and within the amplitude expansion $\phi / \left(\frac{m_R^2}{\lambda_R}\right) \ll 1$.

If we wanted to solve this equation, we would technically have to expand ϕ as $\phi = \phi_0 + \lambda_R \phi_1 + \cdots$, with all of the ϕ_i small. To see what will happen, let's look at the following, far simpler example. Consider the damped harmonic oscillator,

$$\ddot{x} + \Gamma \dot{x} + x = 0,$$

where Γ is supposed to be small, say of order a small parameter ε. We can then write $x = x_0 + \varepsilon_1 x_1 + \cdots$ and substitute:

$$\ddot{x}_0 + x_0 = 0$$
$$\ddot{x}_1 + x_1 = -\Gamma \dot{x}_0$$
$$\vdots$$

The first equation has as a solution $x_0 = A_0 \cos(t - t_0)$, A_0, t_0 integration parameters. The second equation becomes

$$\ddot{x}_1 + x_1 = +\Gamma A_0 \sin(t - t_0) = \Gamma A_0 [\sin t \cos t_0 - \cos t \sin t_0]$$

Try an ansatz of the form $x_1 = A(t)\sin t + B(t)\cos t$:

$$\dot{x}_1 = \dot{A}\sin t + A\cos t + \dot{B}\cos t - B\sin t$$

$$\ddot{x}_1 = \ddot{A}\sin t + 2\dot{A}\cos t - A\sin t + \ddot{B}\cos t - 2\dot{B}\sin t - B\cos t$$
$$= (\ddot{A} - A - 2\dot{B})\sin t + (\ddot{B} + 2\dot{A} - B)\cos t$$

$\ddot{x}_1 + x_1 = (\ddot{A} - 2B)\sin t + (\ddot{B} + 2\dot{A})\cos t = (\Gamma A_0 \cos t_0)\sin t + (-\Gamma A_0 \sin t_0)\cos t.$

Matching the coefficients of the independent functions yields $\ddot{A} - 2B = \Gamma A_0 \cos t_0$, $\ddot{B} + 2\dot{A} = -\Gamma A_0 \sin t_0$. These can be solved for A and B. The important thing to note is that $A(t)$, $B(t)$ have terms <u>linear</u> in t: $A(t) = -\frac{1}{2}(\Gamma A_0 \sin t_0)t + (\text{const.} + \text{trig})$, $B(t) = -\frac{1}{2}(\Gamma A_0 \cos t_0)t + \text{const.} + \text{trig}$. Thus the solution for x_1 has the form

$$x_1 = -\tfrac{1}{2}\Gamma A_0 (\sin t \sin t_0 + \cos t \cos t_0)\, t + \text{const.} + \text{trig}$$
$$= -\tfrac{1}{2}\Gamma A_0 t \cos(t - t_0) + \cdots$$

This means that x_1 will grow, eventually to be larger than x_0 i.e. the expansion in the small parameter breaks down! Note that the actual solution is $x = e^{-\Gamma t/2}\left[A\, e^{t/2\sqrt{\Gamma^2-4}} + B\, e^{-t/2\sqrt{\Gamma^2-4}}\right]$. If $\Gamma \ll 1$, this can be expanded as:

$$x \approx \left(1 - \tfrac{\Gamma}{2}t\right)\left[A\, e^{it} + B\, e^{-it}\right] \approx (\cos t, \sin t) - \tfrac{\Gamma}{2}(\cos t, \sin t) + \cdots$$

just as we found. However, for large times, $\Gamma t \gg 1$, this expansion misses all the important behavior of $x(t)$.

This is exactly what will happen if a strictly perturbative solution of our equation is attempted; at long enough times the solution will break down and perturbation theory in λ_R along with it.

An approximation beyond strict perturbation theory is what we call <u>one-loop</u> <u>resummed</u>. This involves keeping the equation as is, i.e. perturbative in λ_R and ϕ, but <u>not</u> expanding ϕ in a λ_R power series. This is equivalent to some resummation of the fully perturbative series, but it is somewhat unclear what terms are being neglected in doing this, and whether they truly are negligible.

Relaxation of Field Configurations.

We are now in a position to understand, at least perturbatively how a given field configuration relaxes through couplings to other fields.

Thus, suppose we have two scalar fields Φ and σ, such that the expectation value of Φ is in motion, but that of σ is at rest at $\sigma = 0$. The Lagrangean density takes the form;

$$\mathcal{L} = -\frac{1}{2}\Phi(\Box + m_\Phi^2)\Phi - \frac{1}{2}\sigma(\Box + m_\sigma^2)\sigma - \frac{1}{2} g \sigma^2 \Phi^2 + \text{quartic}$$

self couplings.

What we want to do is to look at the equation of motion for $\phi(t)$ in the presence of σ. In fact, we will be more general, and allow for spatial dependence in $\langle\Phi\rangle$ i.e allow slight deviations from homogeneity.

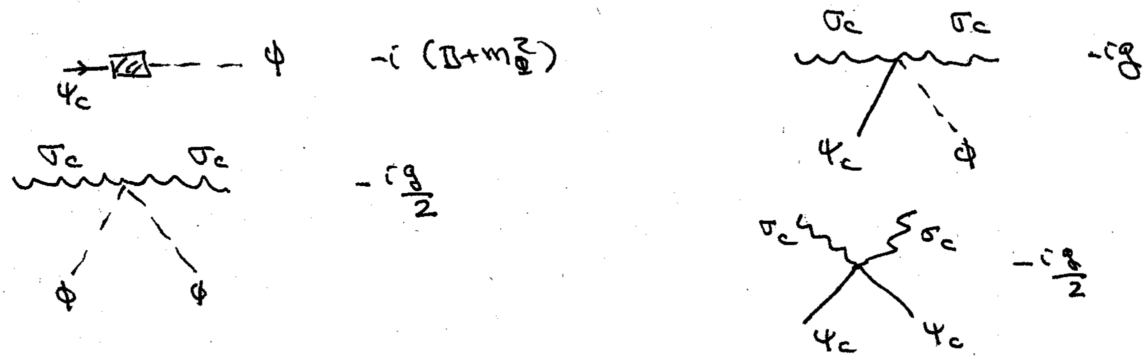
Thus write $\langle\Phi\rangle(x,t) = \phi(x,t) = \int \frac{d^3k}{(2\pi)^3} e^{i\vec{k}\cdot\vec{x}} \delta_{\vec{k}}(t)$, and

decompose Φ as $\Phi = \phi(x,t) + \Psi_c(x,t)$ along the contour. Doing this yields

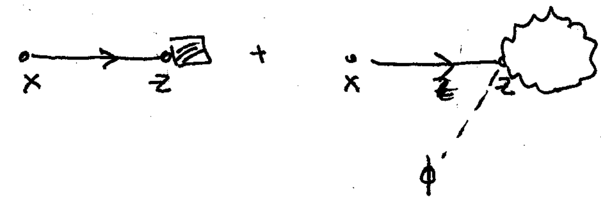
$$\mathcal{L}_c = -\frac{1}{2}\Psi_c(\Box + m_\Phi^2)\Psi_c - \frac{1}{2}\sigma_c(\Box + m_\sigma^2)\sigma_c - \cancel{\tfrac{1}{2}}\,\Psi_c(\Box + m_\Phi^2)\phi$$

$$-\frac{1}{2} g \sigma_c^2 \phi^2 - g\sigma_c^2 \phi \Psi_c - \frac{1}{2} g \sigma_c^2 \Psi_c^2 + \text{quartic etc.}$$

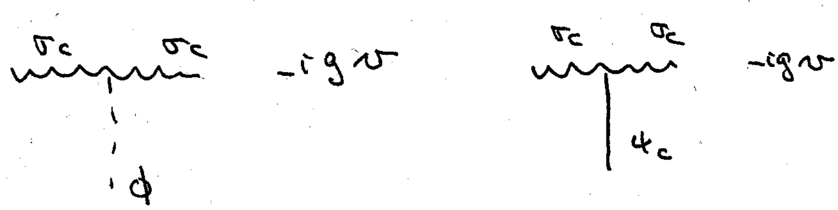
As before, we will compute $\langle \psi_c \rangle$, and set it to zero. The vertices are



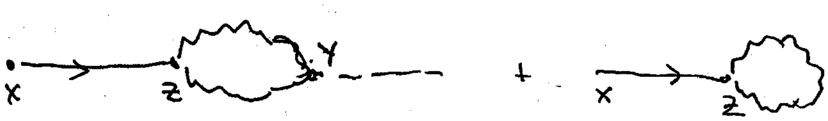
To leading order in the coupling g, and in the amplitude expansion, we have:

$\langle \psi_c \rangle =$ 

This is true in the case where there is <u>no</u> spontaneous symmetry breaking. If there IS SSB and the field is oscillating about the true minimum at $\Phi = v$, write $\Phi = v + \phi(x,t) + \psi$. This will give rise to extra vertices:



as well as a shift in the σ mass $m_\sigma^2 \to m_\sigma^2 + g v^2$. In this situation, we also have graphs as:



Let's look at the case with symmetry breaking first: the full contribution is given by:

$$\int_C d^4z \; (-i\,G_c^{\psi}(x,z))\Big[-i(\Box+m_\phi^2)\phi + (-ig)\,v\,(-i\,G_c^{\sigma}(z,z)) + (-ig)\,\phi_z\,(-i G_c^{\sigma}(z,z))$$

$$+ 2(-ig\,v)^2 \int_C d^4y \; (-i\,G_c^{\sigma}(z,y))\,(-i\,G_c^{\sigma}(y,z))\,\phi_y\Big] = 0 \quad \text{or}$$

$$(\Box+m_\phi^2)\phi + (g)v\,(-i\,G_c^{\sigma}(z,z)) + (g)\,(-i\,G_c^{\sigma}(z,z))\,\phi$$

$$+ 2ig^2v^2 \int_c d^4y \;\phi_y\; G_c^{\sigma}(z,y)\,G_c^{\sigma}(y,z) = 0.$$

Let's shift to momentum space; before we do this, though, we note that the term $(g)(-iG_c^{\sigma}(z,z))\phi$ serves to renormalize the ϕ mass^2: $m_\phi^2 \to M_\phi^2 = m_\phi^2 + g\,(-iG_c^{\sigma}(z,z))$. We also note that the constant term (which is divergent as well), $g v\,(-iG_c^{\sigma}(z,z))$ can be absorbed into a shift of the vacuum expectation value v. Having done this, we now go to momentum space:

$$\ddot{\delta}_k + \omega_k^2\,\delta_k + 2ig^2v^2 \int_c dt'\,\delta_k(t')\underbrace{\left[\int \frac{d^3p}{(2\pi)^3}\, g_p^{\sigma}(t,t')\, g_{k+p}^{\sigma}(t',t)\right]}_{K(t-t')} = 0$$

Let's analyze the kernel $K(t-t')$. Just as in the Φ^4 example, we arrive at a retarded quantity when integrating only along the + contour.

$$K(t-t') = \Theta(t-t')\,\Sigma_k^{ret}(t-t')$$

$$\Sigma_k^{ret}(t-t') = \int \frac{d^3p}{(2\pi)^3}\Big[g_p^{\sigma>}(t,t')\,g_{k+p}^{\sigma>}(t,t') - g_p^{\sigma<}(t,t')\,g_{k+p}^{\sigma<}(t,t')\Big]$$

Now let's use our explicit expressions for $g_p^{\sigma\gtrless}(t,t')$, $g_{k+p}^{\sigma\gtrless}(t,t')$ to reduce this expression:

There is a common factor of $-\frac{1}{4\Omega_p\Omega_{k+p}}$, where $\Omega_k^2 = k^2 + M_\sigma^2$, and

There will be terms proportional to $e^{\pm i(\Omega_p+\Omega_{k+p})(t-t')}$, $e^{\pm i(\Omega_p-\Omega_{k+p})(t-t')}$.

$e^{+i(\Omega_k+\Omega_{k+p})\Delta t}$: $n_p n_{k+p} - (n_p+1)(n_{k+p}+1)$

$e^{-i(\Omega_k+\Omega_{k+p})\Delta t}$: $(n_p+1)(n_{k+p}+1) - n_p n_{k+p}$

This combination adds to $2i\sin(\Omega_p+\Omega_{k+p})\Delta t\,[n_p n_{k+p} - (n_p+1)(n_{k+p}+1)]$

$e^{+i(\Omega_k-\Omega_{k+p})\Delta t}$: $n_p(n_{k+p}+1) - (n_p+1)n_{k+p}$

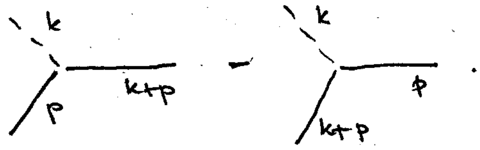
$e^{-i(\Omega_k-\Omega_{k+p})}$: $(n_p+1)n_{k+p} - n_p(n_{k+p}+1)$,

so we get $2i\sin(\Omega_p-\Omega_{k+p})\Delta t\,[n_p(n_{k+p}+1) - n_{k+p}(n_p+1)]$

The terms involving the occupation numbers can be interpreted in terms of creation and annihilation events: $n_k n_{k+p} - (n_k+1)(n_{k+p}+1)$ can be thought of as

, while

$n_p(n_{k+p}+1) - n_{k+p}(n_p+1)$ looks like:



These are the type of terms that appear in a Boltzmann equation describing the rates of creation & annihilation of δ_k.

We arrive at:

$$\Sigma_k^{ret}(t-t') = \int \frac{d^3p}{(2\pi)^3}\,(2ig^2v^2)\,\frac{(-2i)}{4\Omega_p\Omega_{k+p}}\Big[-\sin(\Omega_p+\Omega_{k+p})\Delta t\,(1+n_p+n_{k+p}) + \sin(\Omega_p-\Omega_{k+p})\Delta t\,(n_p-n_{k+p})\Big]$$

$$= g^2v^2\int\frac{d^3p}{(2\pi)^3}\frac{1}{\Omega_p\Omega_{k+p}}\Big[\sin(\Omega_p-\Omega_{k+p})\Delta t\,(n_p-n_{k+p}) - \sin(\Omega_p+\Omega_{k+p})(1+n_p+n_{k+p})\Big]$$

Let's now specialize to the homogeneous, $\vec{k}=\vec{0}$ case. We also take the temperature to zero, so that all the n's $\to 0$.

Then we can write:

$$\Sigma^{ret}_{k=0}(t-t') = -g^2 v^2 \int \frac{d^3p}{(2\pi)^3} \frac{1}{\Omega_p^2} \sin 2\Omega_p \Delta t .$$

We can extract the divergences present in $\Sigma^{ret}_{k=0}$ by remembering that $\frac{d}{dt'} \frac{\cos 2\Omega_p \Delta t}{2\Omega_p} = \sin 2\Omega_p \Delta t$, integrating by parts under the dt' integral. We find our equation to be:

$$\ddot{\delta} + \left(M_\phi^2 + g^2 V^2 \int \frac{d^3p}{(2\pi)^3} \frac{1}{2\Omega_p^3}\right)\delta + g^2 V^2 \int_{-\infty}^{t} dt' \, \dot{\delta}(t') \int \frac{d^3p}{(2\pi)^3} \frac{\cos 2\Omega_p \Delta t}{2\Omega_p^3} = 0$$

We now have to renormalize the ϕ mass^2 once again, this time absorbing the log divergent term. Finally the fully renormalized equation of motion is:

$$\ddot{\delta} + M_\phi^2 \delta + g^2 V^2 \int_{-\infty}^{t} dt' \, \dot{\delta}(t') \int \frac{d^3p}{(2\pi)^3} \frac{\cos 2\Omega_p \Delta t}{2\Omega_p^3} = 0$$

To solve this equation, we use the Laplace transform: define $\varphi(s) = \int_0^\infty dt \, e^{-st} \delta(t)$. Then

$$s^2 \varphi(s) - s\,\delta(0) - \dot{\delta}(0) + M_\phi^2 \varphi(s) + (s\varphi(s) - \delta(0))\, g^2 v^2 \int \frac{d^3p}{(2\pi)^3} \frac{1}{2\Omega_p^3} \frac{s}{s^2 + 4\Omega_p^2}$$

$$= \left(s^2 + g^2 v^2 \int \frac{d^3p}{(2\pi)^3} \frac{s^2}{s^2 + 4\Omega_p^2} \frac{1}{2\Omega_p^3} + M_\phi^2\right)\varphi(s) =$$

$$\dot{\delta}(0) + s\,\delta(0)\left(1 + g^2 v^2 \int \frac{d^3p}{(2\pi)^3} \frac{1}{s^2 + 4\Omega_p^2}\right) \Rightarrow$$

$$\varphi(s) = \frac{\dot{\delta}(0) + s\,\delta(0)\left(1 + g^2 v^2 \int \frac{d^3p}{(2\pi)^3} \frac{1}{s^2 + 4\Omega_p^2} \frac{1}{2\Omega_p^3}\right)}{s^2\left(1 + g^2 v^2 \int \frac{d^3p}{(2\pi)^3} \frac{1}{s^2 + 4\Omega_p^2} \frac{1}{2\Omega_p^3}\right) + M_\varphi^2}$$

We take $\dot{\delta}(0) = 0$ so that

$$\varphi(s) = \frac{\delta(0)}{s}\left[1 - \frac{M_\phi^2}{s^2 + M_\phi^2 + s^2 G\, b(s)}\right]$$, where we have defined

$$G \equiv \frac{g^2 v^2}{(2\pi)^2}, \quad b(s) \equiv \int dp\; p^2 \frac{1}{(p^2 + M_\sigma^2)^{3/2}} \frac{1}{s^2 + 4M_\sigma^2 + 4p^2}$$

The inverse Laplace transform is given by

$$\delta(t) = \int_{-i\infty+\epsilon}^{i\infty+\epsilon} \frac{ds}{2\pi i}\, e^{st}\, \varphi(s)$$

where the contour runs parallel to the imaginary s axis, and runs to the right of all singularities of $\varphi(s)$. The singularities of $\varphi(s)$ are all encoded in the analytic behavior of $s^2 + M_\phi^2 + s^2 G\, b(s)$, so we need to calculate $b(s)$.

Define $q = p/M_\sigma$ so that we can rewrite $b(s)$ as:

$$b(s) = \frac{1}{4M_\sigma^2} \int dq\, \frac{q^2}{(q^2+1)^{3/2}} \frac{1}{q^2 + 1 + \frac{s^2}{4M_\sigma^2}}.$$

Now

$$\frac{1}{(q^2+1)(q^2+1+a^2)} = \frac{A}{q^2+1} + \frac{B}{q^2+1+a^2} \Rightarrow 1 = A(q^2+1+a^2) + B(q^2+1)$$ or

$A + B = 0 \qquad A(1+a^2) + B = 1$ or $Aa^2 = 1$, $A = \frac{1}{a^2}$, $B = -\frac{1}{a^2}$ so

Thus

$$\int dq\, \frac{q^2}{(q^2+1)^{3/2}} \frac{1}{(q^2+1+a^2)} = \frac{1}{a^2}\int dq\, \frac{q^2}{(q^2+1)^{1/2}}\left[\frac{1}{(q^2+1)} - \frac{1}{(q^2+1+a^2)}\right]$$

where we intend to set $a^2 = \frac{s^2}{4M_\sigma^2}$. Next, note: $\frac{q^2}{q^2+b} = 1 - \frac{b}{q^2+b}$

so we can now write $b(s)$ as:

$$b(s) = \frac{1}{s^2} \int_0^\infty dq \, \frac{1}{(q^2+1)^{1/2}} \left[1 - \frac{1}{q^2+1} - 1 + \frac{q^2+1}{q^2+1+a^2} \right] =$$

$$\frac{1}{s^2} \int_0^\infty dq \, \frac{1}{(q^2+1)^{1/2}} \left[\frac{a^2+1}{q^2+1+a^2} - \frac{1}{q^2+1} \right].$$

$$= \frac{1}{s^2} \int_0^\infty dq \, \frac{1}{(q^2+1)^{1/2}} \left[\frac{\frac{s^2}{4M_\sigma^2}+1}{q^2+1+\frac{s^2}{4M_\sigma^2}} - (\text{same at } s=0) \right]$$

Doing the integral (look it up), we find:

$$b(s) = \frac{2}{s^2} \left\{ \frac{1}{2} \sqrt{1+\frac{4M_\sigma^2}{s^2}} \ln \left[\frac{1 + \frac{1}{\sqrt{1+4M_\sigma^2/s^2}}}{1 - \frac{1}{\sqrt{1+4M_\sigma^2/s^2}}} \right] - 1 \right\}, \text{ and}$$

$$G s^2 b(s) = 2G \left\{ \frac{1}{2} \sqrt{1+\frac{(2M_\sigma)^2}{s^2}} \ln \left[\frac{1+(1+(2M_\sigma)^2/s^2)^{-1/2}}{1-(1+(2M_\sigma)^2/s^2)^{-1/2}} \right] - 1 \right\} \equiv l(s)$$

Now we can start looking at the singularity structure of $\varphi(s)$. First there is a pole at $s=0$, with residue 0, since $l(0)=0$. At weak coupling, i.e. $G \ll 1$, there must be a pole of $\varphi(s)$ near $\pm i M_\phi$ (within G of this). Set the location of these poles at $\pm i\omega_0$ so that

$$-\omega_0^2 + M_\phi^2 + l(\pm i\omega_0) = 0$$

Now

$$l(\pm i\omega_0) = 2G \left[\frac{1}{2} \sqrt{1-\frac{4M_\sigma^2}{\omega_0^2}} \ln \left[\frac{1+\frac{1}{\sqrt{1-4M_\sigma^2/\omega_0^2}}}{1-\frac{1}{\sqrt{1-4M_\sigma^2/\omega_0^2}}} \right] - 1 \right]; \text{ for}$$

$$\frac{4M_\sigma^2}{\omega_0^2} > 1, \quad l(\pm i\omega_0) = 2G \left[\sqrt{\frac{4M_\sigma^2}{\omega_0^2}-1} \; \underbrace{\frac{1}{2i} \ln \left[\frac{1+i/\sqrt{}}{1-i/\sqrt{}} \right]}_{\tan^{-1} \frac{1}{\sqrt{}}} - 1 \right].$$

$$\left(\sqrt{} \equiv \sqrt{\frac{4M_\sigma^2}{\omega_0^2}-1} \right)$$

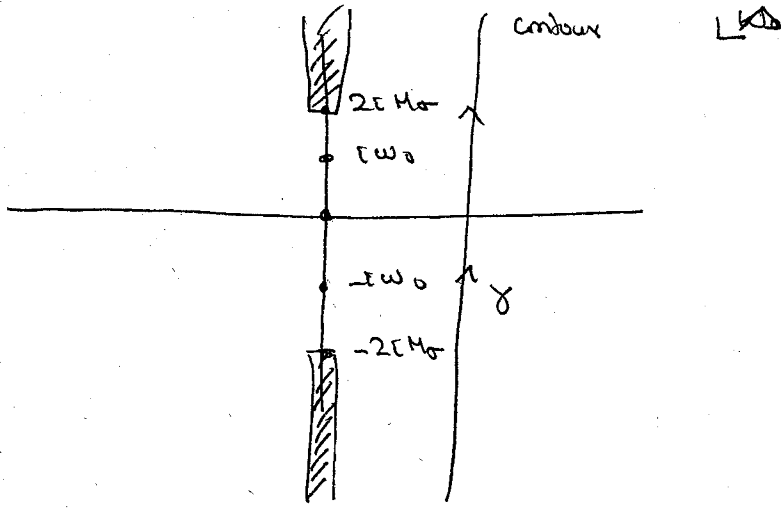
To $O(G)$, write $\omega_0^2 = M_\phi^2 + 2GW$ and substitute into $-\omega_0^2 + M_\phi^2 + \ell(\pm i\omega)$ $=0$ to find:

$$-2GW + 2G\left[\sqrt{\frac{4M_\sigma^2}{M_\phi^2}-1}\;\tan^{-1}\frac{1}{\sqrt{\frac{4M_\sigma^2}{M_\phi^2}-1}} - 1\right] = 0 \quad \text{or}$$

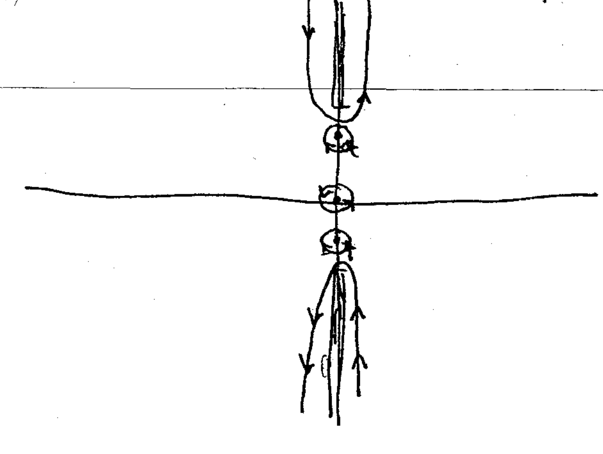
$$W = \sqrt{\frac{4M_\sigma^2}{M_\phi^2}-1}\;\tan^{-1}\frac{1}{\sqrt{\frac{4M_\sigma^2}{M_\phi^2}-1}},$$

These are all the poles; what about cuts?

For $\omega_0^2 < 4M_\sigma^2$, the square root has a cut; but $\sqrt{\ }\ \ln\left[\frac{1+\frac{1}{\sqrt{\ }}}{1-\frac{1}{\sqrt{\ }}}\right]$ only has <u>even</u> powers of $\sqrt{\ }$ so no cut. For $\omega_0^2 > 4M_\sigma^2$, $\sqrt{\ }$ has positive argument, but $\frac{1}{\sqrt{\ }} > 1$ so ln has negative argument, and hence cuts. They start at $\pm 2iM_\sigma$ and run up & down to $\pm i\infty$ resp.



Deform the contour to catch pole & cut contributions



$$\frac{1}{2\pi i}\int_{\gamma} \varphi(s)\, e^{st}\, ds = \sum(\text{residues}) + \left(\int_{2im+\varepsilon}^{i\infty+\varepsilon} - \int_{2im-\varepsilon}^{i\infty-\varepsilon}\right) +$$

$$\left(\int_{-i\infty+\varepsilon}^{-2im+\varepsilon} ds - \int_{-i\infty-\varepsilon}^{-2im-\varepsilon} ds\right) \qquad (m \equiv M_\sigma)$$

Change variables to $\omega = -is$ and write the cut contribution as

$$\int_{2m}^{\infty} \frac{d\omega}{2\pi}\left[\varphi(s=i\omega+\varepsilon) - \varphi(s=i\omega-\varepsilon)\right] e^{i\omega t} +$$

$$\int_{-\infty}^{-2m} \frac{d\omega}{2\pi}\left[\varphi(s=i\omega+\varepsilon) - \varphi(s=i\omega-\varepsilon)\right] e^{i\omega t}$$

$$= \int_{2m}^{\infty}\frac{d\omega}{2\pi}\left\{e^{i\omega t}\left[\varphi(s=i\omega+\varepsilon) - \varphi(s=i\omega-\varepsilon)\right] + \text{c.c.}\right\} =$$

$$\mathrm{Re}\int_{2m}^{\infty}\frac{d\omega}{\pi}\, e^{i\omega t}\left[\varphi(s=i\omega+\varepsilon) - \varphi(s=i\omega-\varepsilon)\right]$$

Look at contribution for $\omega > 2m$

$$\varphi(s=i\omega+\varepsilon) = \frac{\delta_i}{s}\left[1 - \frac{M_\phi^2}{s^2+M_\phi^2+\ell(s)}\right]_{\substack{s=i\omega+\varepsilon\\ \varepsilon\to 0^+}}, \quad \omega > 2m$$

Now $\ell(i\omega+\varepsilon) = 2G\left[\frac{1}{2}\sqrt{1-\frac{4m^2}{\omega^2(1-i\eta)}}\ \ln\left[\frac{\sqrt{1-\frac{4m^2}{\omega^2(1-i\eta)}}+1}{\sqrt{1-\frac{4m^2}{\omega^2(1-i\eta)}}-1}\right] - 1\right]$;

with $\sqrt{\ } < 1$, denom is < 0 in $\ln$: $\sqrt{1-\frac{4m^2}{\omega^2(1-i\eta)}} = -1 + \sqrt{1-\frac{4m^2}{\omega^2}} - i\delta$

$$= -\underbrace{\left(1-\sqrt{1-\frac{4m^2}{\omega^2}}\right)}_{>0} - i\delta = \rho\, e^{i\theta} \qquad \theta = -\pi$$

$\ln\frac{1}{\sqrt{\ }-1} = \ln\rho^{-1} + i\pi$ so $\ln\frac{\sqrt{\ }+1}{\sqrt{\ }-1} = \ln\frac{1+\sqrt{\ }}{1-\sqrt{\ }} + i\pi$

Thus we can write

$$\varphi(s = i\omega + \varepsilon) = \frac{\delta_i}{i\omega}\left[1 + \frac{M^2}{\omega^2 - M^2 - \Sigma_R - i\Sigma_I}\right]$$ with

$$\Sigma_R(\omega) = 2G\left[\frac{1}{2}\sqrt{1-\frac{4m^2}{\omega^2}}\ \ln\left[\frac{1+\sqrt{1-4m^2/\omega^2}}{1-\sqrt{1-4m^2/\omega^2}}\right] - 1\right]$$

$$\Sigma_I(\omega) = 2G\cdot\frac{\pi}{2}\sqrt{1-\frac{4m^2}{\omega^2}}, \quad \equiv G\pi\, r(\omega), \quad r(\omega) = \sqrt{1-\frac{4m^2}{\omega^2}}$$

For $\varepsilon \to -\varepsilon$, $i\pi \to -i\pi$, $\Sigma_I \to -\Sigma_I$ so that the cut integrand is:

$$\varphi(s = i\omega+\varepsilon) - \varphi(s = i\omega - \varepsilon) = \delta_i \frac{M^2}{i\omega}\left[\frac{1}{\omega^2 - M^2 - \Sigma_R - i\Sigma_I} - \frac{1}{\omega^2 - M^2 - \Sigma_R + i\Sigma_I}\right]$$

$$= \frac{2M^2}{\omega}\,\delta_i\left[\frac{\Sigma_I}{(\omega^2 - M^2 - \Sigma_R^2)^2 + \Sigma_I^2}\right].$$

The cut contribution then becomes:

$$\underbrace{\frac{2M^2}{\cancel{\omega}}\,\delta_i\cdot 2G\cdot\frac{\pi}{2}\cdot\frac{1}{\pi}}_{2G M^2 \delta_i}\int_{2m}^{\infty}\frac{d\omega}{\omega}\,\frac{r(\omega)\cos\omega t}{(\omega^2 - M^2 - \Sigma_R^2)^2 + G^2\pi^2 r(\omega)^2}$$

Pole contributions: the pole at 0 does not contribute, while those at $\pm\omega_0$ contribute.