The noncommutative Chern-Simons action and the Seiberg-Witten map

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The Seiberg-Witten map

- Open strings in a constant Neveu-Schwartz background
  - Pauli-Villars $\longrightarrow$ Commutative gauge theory

$$A_\mu \to A_\mu + D_\mu \Lambda = A_\mu + \partial_\mu \Lambda + A_\mu \Lambda - \Lambda\, A_\mu$$

  - Point-Splitting $\longrightarrow$ Noncommutative gauge theory

$$\hat{A}_\mu \to \hat{A}_\mu + \hat{D}_\mu \hat{\Lambda} = \hat{A}_\mu + \partial_\mu \hat{\Lambda} + \hat{A}_\mu * \hat{\Lambda} - \hat{\Lambda} * \hat{A}_\mu$$

- Is there some map relating them? yes! (and it is more general)

$$\hat{A}_\mu \quad \longleftrightarrow \quad A_\mu$$

$$\theta^{\mu\nu} \neq 0 \quad \longleftrightarrow \quad \theta'^{\mu\nu} = 0 \; (\text{or}\; \theta'^{\mu\nu} \neq 0)$$

- Seiberg-Witten map relates two noncommutative gauge theories

$$\theta^{\mu\nu} \quad \longleftrightarrow \quad \theta'^{\mu\nu}$$

The Seiberg-Witten map

- Explicit form of the map $\theta^{\rho\sigma} \longleftrightarrow \theta^{\rho\sigma} + \delta\theta^{\rho\sigma}$

$$\hat{A}_\mu \longleftrightarrow A_\mu + \frac{1}{4}\delta\theta^{\rho\sigma}\left\{A_\rho, \partial_\sigma A_\mu + F_{\sigma\mu}\right\}$$

$$\hat{F}_{\mu\nu} \longleftrightarrow F_{\mu\nu} + \frac{1}{4}\delta\theta^{\rho\sigma}\left(-2\left\{F_{\mu\rho}, F_{\nu\sigma}\right\} + \left\{A_\rho, D_\sigma F_{\mu\nu} + \partial_\sigma F_{\mu\nu}\right\}\right)$$

- For a general action, the dynamics is not invariant

$$S^\theta[\hat{A}_\mu] \longleftrightarrow S^{\theta+\delta\theta}[A_\mu] + \delta S[A_\mu]$$

- For example, for the Yang-Mills action

$$\delta S_{YM} = \frac{1}{2}\delta\theta^{\rho\sigma}\int_{\mathcal{M}}\left(F_{\mu\nu}F^{\mu}_{\ \rho}F^{\nu}_{\ \sigma} - \frac{1}{4}F_{\mu\nu}\left\{A_\rho, D_\sigma F^{\mu\nu} + \partial_\sigma F^{\mu\nu}\right\}\right) \neq 0$$

The Seiberg-Witten map

- Explicit form of the map $\theta^{\rho\sigma} \longleftrightarrow \theta^{\rho\sigma} + \delta\theta^{\rho\sigma}$

$$\hat{A}_\mu \longleftrightarrow A_\mu + \frac{1}{4}\delta\theta^{\rho\sigma}\{A_\rho, \partial_\sigma A_\mu + F_{\sigma\mu}\}$$

$$\hat{F}_{\mu\nu} \longleftrightarrow F_{\mu\nu} + \frac{1}{4}\delta\theta^{\rho\sigma}\left(-2\{F_{\mu\rho}, F_{\nu\sigma}\} + \{A_\rho, D_\sigma F_{\mu\nu} + \partial_\sigma F_{\mu\nu}\}\right)$$

- For a general action, the dynamics is not invariant

$$S^\theta[\hat{A}_\mu] \longleftrightarrow S^{\theta+\delta\theta}[A_\mu] + \delta S[A_\mu]$$

- But, for the Chern-Simons action we have

$$\delta S_{CS} = \int_{\mathcal{M}} \partial_\mu f^\mu$$

Then, <u>in a manifold without boundary</u>, the CS action is invariant

The Noncommutative CS action

- The noncommutative Chern-Simons action is

$$S^{\theta}_{CS}[\hat{A}_\mu] = \frac{\kappa}{4\pi}\mathrm{Tr}\int_{\mathcal{M}} \epsilon^{\mu\nu\rho}\left(\hat{A}_\mu * \partial_\nu \hat{A}_\rho + \frac{2}{3}\hat{A}_\mu * \hat{A}_\nu * \hat{A}_\rho\right)$$

- What should we expect?

$$\begin{array}{ccccl}
\hat{F}_{\mu\nu} = 0 & \longleftrightarrow & F_{\mu\nu} = 0 & & \\
\updownarrow & & \updownarrow & & \text{EOM} \\
S^{\theta}_{CS}[\hat{A}] & \Longleftrightarrow & S^{\theta'}_{CS}[A] & & \\
\updownarrow & & \updownarrow & & A_0 \text{ integration} \\
S^{\theta}_{WZW}[\hat{g}] & \longleftrightarrow & S^{\theta'}_{WZW}[g] & & \\
 & & & & \text{Bosonization} \\
 & S^{\theta}[\psi] \equiv S^{\theta'}[\psi] & & &
\end{array}$$

Explicit calculation

- Writing the CS action as

$$S^{\theta}_{CS}[\hat{A}_0, \hat{A}_i] \;=\; \frac{\kappa}{4\pi}\mathrm{Tr}\int_{\mathcal{M}} \epsilon^{ij}\Big(\hat{A}_0\hat{F}_{ij} - \hat{A}_i * \dot{\hat{A}}_j \;+ B^{(1)}_{ij} + B^{(2)}_{ij}\Big)$$

  where the boundary terms are

$$B^{(1)}_{ij} = \partial_i(\hat{A}_j * \hat{A}_0) + [\partial_i\hat{A}_j + \tfrac{2}{3}\hat{A}_i * \hat{A}_j \,,\, \hat{A}_0] - \tfrac{2}{3}\left[\hat{A}_i * \hat{A}_0 \,,\, \hat{A}_j\right]$$

$$B^{(2)}_{ij} = (\hat{A}_0 * \hat{F}_{ij} - \hat{A}_0\hat{F}_{ij})$$

- $\hat{A}_0$ enforces the constraint $\hat{F}_{ij} = 0$ then $\hat{A}_i = \hat{g}^{-1}\partial_i\hat{g}$

$$-\frac{\kappa}{4\pi}\int_{\partial\mathcal{M}} \vec{t}_i(\hat{g}^{-1} * \partial_i\hat{g}) * (\hat{g}^{-1} * \partial_t\hat{g}) + \int_{\mathcal{M}} \epsilon^{ij}(\hat{g}^{-1} * \partial_i\hat{g}) * (\hat{g}^{-1} * \partial_t\hat{g}) * (\hat{g}^{-1} * \partial_j\hat{g})$$

- $\hat{A}_0$ also enforces on the boundary an infinite set of nonlinear constraints involving $\hat{A}_i$ and all its derivatives

Explicit calculation

- Writing the CS action as

$$S^{\theta}_{CS}[\hat{A}_0,\hat{A}_i] \;=\; \frac{\kappa}{4\pi}\mathrm{Tr}\int_{\mathcal{M}}\epsilon^{ij}\Big(\hat{A}_0\hat{F}_{ij}-\hat{A}_i\dot{\hat{A}}_j\;+B^{(1)}_{ij}+B^{(2)}_{ij}\Big)$$

where the boundary terms are

$$B^{(1)}_{ij}=\partial_i(\hat{A}_j*\hat{A}_0)+[\partial_i\hat{A}_j+\tfrac{2}{3}\hat{A}_i*\hat{A}_j\,,\hat{A}_0]-\tfrac{2}{3}\left[\hat{A}_i*\hat{A}_0\,,\hat{A}_j\right]$$

$$B^{(2)}_{ij}=(\hat{A}_0*\hat{F}_{ij}-\hat{A}_0\hat{F}_{ij})-(\hat{A}_i*\dot{\hat{A}}_j-\hat{A}_j\dot{\hat{A}}_i)$$

- Applying the Seiberg-Witten map we get

$$\delta S^{\theta}_{CS} \;=\; \frac{\kappa}{4\pi}\mathrm{Tr}\int_{\mathcal{M}}\epsilon^{ij}\Big(\delta\hat{A}_0\hat{F}_{ij}+\hat{A}_0\delta\hat{F}_{ij}-2\delta\hat{A}_i\dot{\hat{A}}_j\;+\delta B^{(1)}_{ij}+\delta B^{(2)}_{ij}\Big)$$

Explicit calculation

- Writing the CS action as

$$S^{\theta}_{CS}[\hat{A}_0, \hat{A}_i] = \frac{\kappa}{4\pi}\mathrm{Tr}\int_{\mathcal{M}} \epsilon^{ij}\Big(\hat{A}_0\hat{F}_{ij} - \hat{A}_i\dot{\hat{A}}_j + B^{(1)}_{ij} + B^{(2)}_{ij}\Big)$$

where the boundary terms are

$$B^{(1)}_{ij} = \partial_i(\hat{A}_j * \hat{A}_0) + [\partial_i\hat{A}_j + \tfrac{2}{3}\hat{A}_i * \hat{A}_j\, , \hat{A}_0] - \tfrac{2}{3}\left[\hat{A}_i * \hat{A}_0\, , \hat{A}_j\right]$$

$$B^{(2)}_{ij} = (\hat{A}_0 * \hat{F}_{ij} - \hat{A}_0\hat{F}_{ij}) - (\hat{A}_i * \dot{\hat{A}}_j - \hat{A}_j\dot{\hat{A}}_i)$$

- Applying the Seiberg-Witten map we get

$$\delta S^{\theta}_{CS} = \frac{\kappa}{4\pi}\mathrm{Tr}\int_{\mathcal{M}} \epsilon^{ij}\Big(\delta\theta^{kl}\partial_i\Big(2A_j\partial_k A_l A_0 + A_j A_k \dot{A}_l\Big) + \delta B^{(1)}_{ij} + \delta B^{(2)}_{ij}\Big)$$

- Then in a boudaryless manifold, the CS action is invariant under the Seiberg-Witten map

Closing remarks

- In a boundaryless manifold, the noncommutative Chern-Simons action is invariant under the Seiberg-Witten map, this meaning (N.E.G and G.A. Silva hep-th/0310113)

$$S_{CS}^{\theta} \longleftrightarrow S_{CS}^{\theta'}$$

- In a manifold with boundary, we need to impose a infinite set of nonlinear relations involving all the derivatives of the gauge field as boundary conditions (A.R. Lugo hep-th/0111064).
- The "boundary theory" is not a NC chiral Wess-Zumino-Witten model. In fact it is not a boundary theory at all because it involves infinite derivatives of the fields in the directions of the bulk.
- The stated equivalence holds at the classical level. The quantum result would require the study of the of the behavior of the measure $\mathcal{D}A_\mu$ under the map (Kaminsky, Okawa, Ooguri hep-th/0101133, Kaminsky hep-th/0310011).