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Superstring Multiloop Amplitudes and Vanishing Theorems

PART II

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ICTP Lectures on Covariant Quantization of the SuperstringNathan Berkovits¹

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These ICTP Trieste lecture notes review the pure spinor approach to quantizing the superstring with manifest D=10 super-Poincaré invariance. The first section discusses covariant quantization of the superparticle and gives a new proof of equivalence with the Brink-Schwarz superparticle. The second section discusses the superstring in a flat background and shows how to construct vertex operators and compute tree amplitudes in a manifestly super-Poincaré covariant manner. And the third section discusses quantization of the superstring in curved backgrounds which can include Ramond-Ramond flux.

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1. Introduction

The two standard formalisms for describing the superstring are the Ramond-Neveu-Schwarz (RNS) and Green-Schwarz (GS) formalisms. Although the RNS formalism has a beautiful $N=1$ worldsheet supersymmetry, its lack of manifest target-space supersymmetry is responsible for several awkward features of the formalism. For example, amplitudes involving more than four external fermions are difficult to compute in a Lorentz-covariant manner because of picture-changing and bosonization complications [1]. Furthermore, it is not known how to use the RNS formalism to describe the superstring in Ramond-Ramond backgrounds.

On the other hand, target-space supersymmetry is manifest in the GS formalism, but the worldsheet symmetries are not manifest. A lack of understanding of these worldsheet symmetries has so far prevented quantization except in light-cone gauge. Although light-cone gauge is useful for determining the physical spectrum, it is clumsy for computing scattering amplitudes because of the lack of manifest Lorentz covariance and the need to introduce interaction-point operators and contact terms. For these reasons, only four-point tree and one-loop amplitudes have been explicitly computed using the GS formalism [2]. Furthermore, the necessity of choosing light-cone gauge means that quantization is only possible in those backgrounds which allow a light-cone gauge choice.

As will be discussed in these lecture notes, a new formalism for the superstring was proposed recently [3] which combines the advantages of the RNS and GS formalisms without including their disadvantages. In this new approach, the worldsheet action is quadratic in a flat background so quantization is as easy as in the RNS formalism. And since $D=10$ super-Poincaré covariance is manifest in this formalism, there is no problem with computing spacetime-supersymmetric N -point tree amplitudes or with quantizing the superstring in Ramond-Ramond backgrounds.

There are three new ingredients in this formalism as compared with the standard GS formalism. The first new ingredient is fermionic canonical momenta d_α for the θ^α variables. These canonical momenta were first introduced by Siegel [4] and allow the GS action to be written in quadratic form after including appropriate constraints. The second new ingredient is the bosonic “pure spinor” λ^α which plays the role of a ghost variable. And the third new ingredient is the nilpotent BRST operator $Q = \int \lambda^\alpha d_\alpha$ whose cohomology is used to define physical states. But before entering into more details about this new formalism, it will be useful to say a few words about where it came from.

In 1989, in an attempt to better understand the worldsheet symmetries of the GS superstring, Sorokin, Tkach, Volkov and Zheltukhin [5] replaced the worldline kappa symmetry of the Brink-Schwarz superparticle with worldline supersymmetry. The bosonic worldline superpartner for θ^α was called λ^α , and worldline supersymmetry of the action implied that λ^α satisfied the twistor-like relation

$$\lambda\gamma^m\lambda = \dot{x}^m + \frac{1}{2}\theta\gamma^m\dot{\theta}. \quad (1.1)$$

This twistor-like approach was then generalized by several authors to the classical heterotic superstring with from one to eight worldsheet supersymmetries [6][7] [8] and it was argued in [9] that quantization of the version with two worldsheet supersymmetries leads to a critical N=2 superconformal field theory. For two worldsheet supersymmetries, θ^α has two superpartners, λ^α and $\bar{\lambda}^\alpha$, which satisfy the relations

$$\lambda\gamma^m\lambda = \bar{\lambda}\gamma^m\bar{\lambda} = 0, \quad \lambda\gamma^m\bar{\lambda} = \partial x^m + \frac{1}{2}\theta\gamma^m\partial\theta. \quad (1.2)$$

In ten dimensions, a complex Weyl spinor λ^α satisfying $\lambda\gamma^m\lambda = 0$ is called a pure spinor and, as was shown by Howe [10] [11] in 1991, is useful for describing the on-shell constraints of super-Yang-Mills and supergravity.²

Unfortunately, direct quantization of the N=2 worldsheet superconformal field theory requires solving the constraints of (1.2) and breaking the manifest SO(9,1) Lorentz invariance down to U(4) [9][15]. In later papers, this U(4) formalism was related to other critical N=2 superconformal field theories called “hybrid” formalisms with manifest SO(3,1)×U(3) [16], SO(5,1)×U(2) [17], SO(1,1)×U(4) [18], or (after Wick-rotation) U(5) [19] subgroups of the Lorentz group. Together with Cumrun Vafa [20][17], it was shown that all of these formalisms are related by a field redefinition to an N=1 → N=2 embedding of the standard RNS formalism where, after twisting the worldsheet N=2, the RNS BRST current and b ghost are mapped to the fermionic N=2 superconformal generators.

Finally, in [3], it was proposed that these hybrid formalisms are equivalent to a manifestly SO(9,1) super-Poincaré covariant formalism using a BRST operator $Q = \int \lambda^\alpha d_\alpha$ constructed from the worldsheet variables $[x^m, \theta^\alpha, d_\alpha, \lambda^\alpha, w_\alpha]$ where d_α is the conjugate momentum to θ^α , w_α is the conjugate momentum to λ^α , and λ^α is a pure spinor satisfying $\lambda\gamma^m\lambda = 0$. As will be shown later, λ^α and w_α each contain 11 independent components so

² Pure spinors were originally studied by Cartan [12]. They have also been used for defining grand unified models [13] and for constructing super-Yang-Mills auxiliary fields [14].

the covariant formalism contains 32 bosons and 32 fermions. Since the hybrid formalisms all contain 12 bosons and 12 fermions (which are related by a field redefinition to the RNS variables $[x^m, \psi^m, b, c, \beta, \gamma]$), the proposal is based on the conjecture that, in addition to obeying the usual physical state conditions, states in the cohomology of $Q = \int \lambda^\alpha d_\alpha$ are independent of the extra 20 bosons and 20 fermions.

This conjecture was suggested by the U(5) version [19] of the hybrid formalism whose variables are $[x^m, \theta^a, \theta^+, d_a, d_+, \lambda^+, w_+]$ where $a = 1$ to 5. If $\lambda^\alpha = \lambda^+$ is interpreted as choosing a U(5) direction in SO(10), the extra 20 bosons can be understood as parameterizing the SO(10)/U(5) coset space. In this sense, the projective part of the pure spinor variable plays the role of an SO(10)/U(5) harmonic variable, similar to the attempts of [21] to covariantly quantize the superstring.

After the proposal was made in [3], there have been various consistency checks of its validity. These include a proof that the cohomology of $Q = \int \lambda^\alpha d_\alpha$ reproduces the superstring spectrum [22] and the construction of an explicit map from states in the cohomology of Q to physical states in the RNS formalism [23]. Also, the pure spinor description has been generalized to curved backgrounds and it has been shown that BRST invariance implies the correct low-energy equations of motion for the background fields [24][25]. Furthermore, it has recently been shown (at least at the classical level) that the pure spinor description can be obtained by directly gauge-fixing the original N=2 worldsheet supersymmetric description [7][9] of (1.2) without passing through the hybrid or RNS descriptions [26].

Although on-shell states in the pure spinor description can be related to on-shell states in the RNS description [23], there is no such relation for off-shell states. Note that the super-Poincaré algebra closes for both on-shell and off-shell states in the pure spinor description. But in the RNS descriptions, the super-Poincaré algebra closes up to picture-changing [27], which is only defined for on-shell states. Since there is no off-shell map between the descriptions, it is tricky to guess the correct rules for computing scattering amplitudes. Nevertheless, a manifestly super-Poincaré covariant prescription was given for tree amplitudes using the pure spinor description and was shown in [28][23] to coincide with the RNS prescription. However, it is still unknown how to compute manifestly super-Poincaré covariant loop amplitudes using the pure spinor description. It is possible that recent generalizations of the pure spinor approach which explicitly introduce $[b, c]$ reparameterization ghosts may be useful for defining a loop amplitude prescription [29][30].

In section 2 of these notes, covariant quantization of the superparticle using pure spinors will be reviewed and a previously unpublished proof will be given for equivalence with the Brink-Schwarz superparticle. In section 3, the pure spinor approach will be generalized to the superstring and it will be shown how to construct massless and massive vertex operators and compute tree amplitudes in a manifestly super-Poincaré covariant manner. In section 4, the open and closed superstring will be described in a curved background and it will be shown how to obtain the low-energy supersymmetric Born-Infeld and supergravity equations of motion for the background fields from the condition of BRST invariance. It will also be shown how this approach can be used to quantize the superstring in an $AdS_5 \times S^5$ background (or its plane wave limit) with Ramond-Ramond flux.³

2. Covariant Quantization of the Superparticle

Before discussing the pure spinor description, it will be useful to review the standard description of the superparticle and the superspace equations for ten-dimensional super-Yang-Mills. It will then be shown that just as D=3 Chern-Simons theory can be obtained from BRST quantization of a particle action, D=10 super-Yang-Mills theory can be obtained from BRST quantization of a superparticle action involving pure spinors.

2.1. Review of standard superparticle description

The standard Brink-Schwarz action for the ten-dimensional superparticle is [31]

$$S = \int d\tau (\Pi^m P_m + e P^m P_m) \quad (2.1)$$

where

$$\Pi^m = \dot{x}^m - \frac{1}{2} \dot{\theta}^\alpha \gamma_{\alpha\beta}^m \theta^\beta, \quad (2.2)$$

P_m is the canonical momentum for x^m , and e is the Lagrange multiplier which enforces the mass-shell condition. The gamma matrices $\gamma_{\alpha\beta}^m$ and $\gamma_m^{\alpha\beta}$ are 16×16 symmetric matrices which satisfy $\gamma_{\alpha\beta}^{(m} \gamma^{n)}{}^{\beta\gamma} = 2\eta^{mn} \delta_\alpha^\gamma$. In the Weyl representation, $\gamma_{\alpha\beta}^m$ and $\gamma_m^{\alpha\beta}$ are the off-diagonal blocks of the 32×32 Γ^m matrices. Throughout these notes, the conventions

³ Some material in this review, such as massive vertex operators and supersymmetric Born-Infeld, were not included in the ICTP lectures. Also, the lecture on quantization of the d=11 superparticle and supermembrane was not included in this review since it involves work in progress.

for factors of i and 2 will be chosen such that the supersymmetry algebra is $\{q_\alpha, q_\beta\} = \gamma_{\alpha\beta}^m \partial_m = iP_m \gamma_{\alpha\beta}^m$.

The action of (2.1) is spacetime-supersymmetric under

$$\delta\theta^\alpha = \epsilon^\alpha, \quad x^m = \frac{1}{2}\theta\gamma^m\epsilon, \quad \delta P_m = \delta e = 0,$$

and is also invariant under the local κ transformations [32]

$$\delta\theta^\alpha = P^m(\gamma_m\kappa)^\alpha, \quad \delta x^m = -\frac{1}{2}\theta\gamma^m\delta\theta, \quad \delta P_m = 0, \quad \delta e = \dot{\theta}^\beta\kappa_\beta. \quad (2.3)$$

The canonical momentum to θ^α , which will be called p_α , satisfies

$$p_\alpha = \delta L / \delta \dot{\theta}^\alpha = -\frac{1}{2}P^m(\gamma_m\theta)_\alpha,$$

so canonical quantization requires that physical states are annihilated by the fermionic Dirac constraints defined by

$$d_\alpha = p_\alpha + \frac{1}{2}P_m(\gamma^m\theta)_\alpha. \quad (2.4)$$

Since $\{p_\alpha, \theta^\beta\} = -i\delta_\alpha^\beta$, these constraints satisfy the Poisson brackets

$$\{d_\alpha, d_\beta\} = -iP_m\gamma_{\alpha\beta}^m, \quad (2.5)$$

and since $P^m P_m = 0$ is also a constraint, eight of the sixteen Dirac constraints are first-class and eight are second-class. One can easily check that the eight first-class Dirac constraints generate the κ transformations of (2.3), however, there is no simple way to covariantly separate out the second-class constraints.

Nevertheless, one can easily quantize the superparticle in a non-Lorentz covariant manner and obtain the physical spectrum. Assuming non-zero P^+ , the local fermionic κ -transformations can be used to gauge-fix $(\gamma^+\theta)_\alpha = 0$ where $\gamma^\pm = \frac{1}{\sqrt{2}}(\gamma^0 \pm \gamma^9)$. In this “semi-light-cone” gauge, the action of (2.1) simplifies to the quadratic action

$$S = \int d\tau (\dot{x}^m P_m + \frac{1}{2}P^+(\dot{\theta}\gamma^-\theta) + eP^m P_m) \quad (2.6)$$

$$= \int d\tau (\dot{x}^m P_m + \frac{1}{2}\dot{S}_a S_a + eP^m P_m), \quad (2.7)$$

where $S_a = \sqrt{P^+}(\gamma^-\theta)_a$ and $a = 1$ to 8 is an $SO(8)$ chiral spinor index.

Canonical quantization of (2.7) implies that $\{S_a, S_b\} = i\delta_{ab}$. So S_a acts like a ‘spinor’ version of $SO(8)$ Pauli matrices σ_{ab}^j which are normalized to satisfy

$$\sigma_{ac}^j \sigma_{bd}^j + \sigma_{bc}^j \sigma_{ad}^j = i\delta_{ab} \delta_{cd}$$

where j and $\dot{b}$ are $SO(8)$ vector and antichiral spinor indices. One can therefore define the quantum-mechanical wavefunction $\Psi(x)$ to carry either an $SO(8)$ vector index, $\Psi_j(x)$, or an $SO(8)$ antichiral spinor index, $\Psi_{\dot{a}}(x)$, and the anticommutation relations of S_a are reproduced by defining

$$S^a \Psi_j(x) = \sigma_j^{a\dot{b}} \Psi_{\dot{b}}(x), \quad S_a \Psi_{\dot{b}}(x) = \sigma_{a\dot{b}}^j \Psi_j(x). \quad (2.8)$$

Furthermore, the constraint $P_m P^m$ implies the linearized equations of motion $\partial_m \partial^m \Psi_j = \partial_m \partial^m \Psi_{\dot{b}} = 0$.

So the physical states of the superparticle are described by a massless $SO(8)$ vector $\Psi_j(x)$ and a massless $SO(8)$ antichiral spinor $\Psi_{\dot{a}}(x)$ which are the physical states of D=10 super-Yang-Mills theory. However, this description of super-Yang-Mills theory only manifestly preserves an $SO(8)$ subgroup of the super-Poincaré group, and one would like a more covariant method for quantizing the theory. Covariant quantization can be extremely useful if one wants to compute more than just the physical spectrum in a flat background. For example, non-covariant methods are clumsy for computing scattering amplitudes or for generalizing to curved backgrounds.

As will be shown in the following subsection, a manifestly super-Poincaré covariant description of on-shell super-Yang-Mills is possible using N=1 D=10 superspace. This covariant description will later be obtained from quantization of a superparticle action involving pure spinors.

2.2. Superspace description of super-Yang-Mills theory

Although on-shell super-Yang-Mills theory can be described by the $SO(8)$ wavefunctions $\Psi_j(x)$ and $\Psi_{\dot{a}}(x)$ of (2.8) satisfying the linearized equations of motion $\partial_m \partial^m \Psi_j = \partial_m \partial^m \Psi_{\dot{a}} = 0$, there are more covariant descriptions of the theory. Of course, there is a Poincaré-covariant description using an $SO(9, 1)$ vector field $a_m(x)$ and an $SO(9, 1)$ spinor field $\chi^\alpha(x)$ transforming in the adjoint representation of the gauge group which satisfy the equations of motion

$$\partial^m f_{mn} + ig[a^m, f_{mn}] = 0, \quad \gamma_{\alpha\beta}^m (\partial_m \chi^\beta + ig[a_m, \chi^\beta]) = 0, \quad (2.9)$$

and gauge invariance

$$\delta a_m = \partial_m s + ig[a_m, s], \quad \delta \chi^\alpha = ig[\chi^\alpha, s], \quad \delta f_{mn} = ig[f_{mn}, s], \quad (2.10)$$

where $f_{mn} = \partial_{[m} a_{n]} + ig[a_m, a_n]$ is the Yang-Mills field strength and g is the super-Yang-Mills coupling constant. However, there is also a super-Poincaré covariant description using an $SO(9, 1)$ spinor wavefunction $A_\alpha(x, \theta)$ defined in D=10 superspace. As will be explained below, on-shell super-Yang-Mills theory can be described by a spinor superfield $A_\alpha(x, \theta)$ transforming in the adjoint representation which satisfies the superspace equation of motion[33]

$$\gamma_{mnpqr}^{\alpha\beta} (D_\alpha A_\beta + ig A_\alpha A_\beta) = 0 \quad (2.11)$$

for any five-form direction $mnpqr$, with the gauge invariance

$$\delta A_\alpha = D_\alpha \Lambda + ig[A_\alpha, \Lambda] \quad (2.12)$$

where $\Lambda(x, \theta)$ is any scalar superfield and

$$D_\alpha = \frac{\partial}{\partial \theta^\alpha} + \frac{1}{2} (\gamma^m \theta)_\alpha \partial_m$$

is the supersymmetric derivative.

One can also define field strengths constructed from A_α by

$$B_m = \frac{1}{8} \gamma_m^{\alpha\beta} (D_\alpha A_\beta + ig A_\alpha A_\beta), \quad W^\alpha = \frac{1}{10} \gamma_m^{\alpha\beta} (D_\alpha B^m - \partial^m A_\alpha + ig[A_\alpha, B^m]), \quad (2.13)$$

$$F_{mn} = \partial_{[m} B_{n]} + ig[B_m, B_n] = \frac{1}{8} (\gamma_{mn})_\alpha{}^\beta (D_\beta W^\alpha + ig\{A_\beta, W^\alpha\})$$

which transform under the gauge transformation of (2.12) as

$$\delta B_m = \partial_m \Lambda + ig[B_m, \Lambda], \quad \delta W^\alpha = ig[W^\alpha, \Lambda], \quad \delta F^{mn} = ig[F^{mn}, \Lambda]. \quad (2.14)$$

To show that $A_\alpha(x, \theta)$ describes on-shell super-Yang-Mills theory, it will be useful to first note that in ten dimensions any symmetric bispinor $f_{\alpha\beta}$ can be decomposed in terms of a vector and a five-form as $f_{\alpha\beta} = \gamma_{\alpha\beta}^m f_m + \gamma_{\alpha\beta}^{mnpqr} f_{mnpqr}$ and any antisymmetric bispinor $f_{\alpha\beta}$ can be decomposed in terms of a three-form as $f_{\alpha\beta} = \gamma_{\alpha\beta}^{mnp} f_{mnp}$. Since $\{D_\alpha, D_\beta\} = \gamma_{\alpha\beta}^m \partial_m$, one can check that $\delta A_\alpha = D_\alpha \Lambda + ig[A_\alpha, \Lambda]$ is indeed a gauge invariance of (2.11).

Using $\Lambda(x, \theta) = h_\alpha(x)\theta^\alpha + j_{\alpha\beta}(x)\theta^\alpha\theta^\beta$, one can gauge away $(A_\alpha(x))|_{\theta=0}$ and the three-form part of $(D_\alpha A_\beta(x))|_{\theta=0}$. Furthermore, equation (2.11) implies that the five-form part of $(D_\alpha A_\beta(x))|_{\theta=0}$ vanishes. So the lowest non-vanishing component of $A_\alpha(x, \theta)$ in this gauge is the vector component $(D\gamma_m A(x))|_{\theta=0}$ which will be defined as $8a_m(x)$. Continuing this type of argument to higher order in θ^α , one finds that there exists a gauge choice such that

$$A_\alpha(x, \theta) = \frac{1}{2}(\gamma^m \theta)_\alpha a_m(x) + \frac{i}{12}(\theta \gamma^{mnp} \theta)(\gamma_{mnp})_{\alpha\beta} \chi^\beta(x) + \dots \quad (2.15)$$

where $a_m(x)$ and $\chi^\beta(x)$ are $SO(9, 1)$ vector and spinor fields satisfying (2.9) and where the component fields in ... are functions of spacetime derivatives of $a_m(x)$ and $\chi^\beta(x)$. Furthermore, this gauge choice leaves the residual gauge transformations of (2.10) where $s(x) = (\Lambda(x))|_{\theta=0}$. Also, one can check that the $\theta = 0$ components of the superfields B_m , W^α and F_{mn} of (2.13) are a_m , χ^α and f_{mn} respectively. So the equations of motion and gauge invariances of (2.11) and (2.12) correctly describe on-shell super-Yang-Mills theory.

One would now like to obtain this super-Poincaré covariant description of super-Yang-Mills theory by quantizing the superparticle. Since the super-Yang-Mills spectrum contains a massless vector, one expects the covariant superparticle constraints to generate the spacetime gauge invariances of this vector. Note that these constraints are not present in the gauge-fixed action of (2.7) since Ψ_j describes only the transverse degrees of freedom of the $SO(9, 1)$ vector. Before describing the covariant constraints which generate the gauge invariances of this vector, it will be useful to first review the worldline action for Chern-Simons theory which also has constraints related to spacetime gauge invariances.

2.3. Worldline description of Chern-Simons theory

Since the gauge invariance of a massless vector field is $\delta A_\mu = \partial_\mu \Lambda$, one might guess that the worldline action for such a field should contain the constraints P_μ . Although these constraints are too strong for describing Yang-Mills theory, they are just right for describing D=3 Chern-Simons theory where the field-strength of A_μ vanishes on-shell.

As was shown in [34], Chern-Simons theory can be described using the worldline action⁴

$$S = \int d\tau (\dot{x}^\mu P_\mu + l^\mu P_\mu) \quad (2.16)$$

⁴ Although [34] discusses only a worldsheet action for Chern-Simons string theory, the methods easily generalize to a worldline action.

where $\mu = 0$ to 2 and l^μ are Lagrange multipliers for the constraints. Since the constraints are first-class, the action can be quantized using the BRST method. After gauging $l^\mu = -\frac{1}{2}P^\mu$, the gauge-fixed action is

$$S = \int d\tau (\dot{x}^\mu P_\mu - \frac{1}{2} P^\mu P_\mu + \dot{c}^\mu b_\mu) \quad (2.17)$$

with the BRST operator

$$Q = c^\mu P_\mu \quad (2.18)$$

where (c^μ, b_μ) are fermionic Fadeev-Popov ghosts and anti-ghosts.

To show that the cohomology of the BRST operator describes Chern-Simons theory, note that the most general wavefunction constructed from a ground state annihilated by b^μ is

$$\Psi(c, x) = C(x) + c^\mu A_\mu(x) + \frac{i}{2} \epsilon_{\mu\nu\rho} c^\mu c^\nu A^{*\rho}(x) + \frac{i}{6} \epsilon_{\mu\nu\rho} c^\mu c^\nu c^\rho C^*(x) \quad (2.19)$$

where the expansion in c^μ terminates since c^μ is fermionic. One can check that

$$Q\Psi = -ic^\mu \partial_\mu C - \frac{i}{2} c^\mu c^\nu \partial_{[\mu} A_{\nu]} + \frac{1}{6} \epsilon_{\mu\nu\rho} c^\mu c^\nu c^\rho \partial_\sigma A^{*\sigma}(x). \quad (2.20)$$

So $Q\Psi = 0$ implies that $A_\mu(x)$ satisfies the equations of motion $\partial_{[\mu} A_{\nu]} = 0$ which is the linearized equation of motion of the Chern-Simons field. Furthermore, if one defines the gauge parameter $\Omega(c, x) = i\Lambda(x) - c^\mu \omega_\mu(x) + \dots$, the gauge transformation $\delta\Psi = Q\Omega$ implies $\delta A_\mu = \partial_\mu \Lambda$ which is the linearized gauge transformation of the Chern-Simons field.

If one defines physical fields in BRST quantization to carry ghost-number one, one finds that the spacetime ghosts carry ghost-number zero, the antifields carry ghost number two, and the antighosts carry ghost-number three. From the equations of motion and gauge invariances $Q\Psi = 0$ and $\delta\Psi = Q\Omega$, one learns that the gauge invariances of the antifields are related to the equations of motion of the fields, and the equations of motion of the ghosts are related to the gauge invariances of the fields. For example, from $Q\Psi = 0$ and $\delta\Psi = Q\Omega$ for the Chern-Simons wavefunction of (2.19), one learns that $A^{*\rho}$ satisfies the equation of motion $\partial_\sigma A^{*\sigma} = 0$ with the gauge invariance $\delta A^{*\sigma} = \epsilon^{\sigma\mu\nu} \partial_\mu w_\nu$, which are the linearized equations of motion and gauge invariance of the Chern-Simons antifield. And the remaining fields, $C(x)$ and $C^*(x)$, describe the spacetime ghost and antighost of Chern-Simons theory.

These equations of motion and gauge invariances can be obtained from the Batalin-Vilkovisky version [35] of the abelian Chern-Simons spacetime action

$$\mathcal{S} = \int d^3x \left(\frac{1}{2} \epsilon^{\mu\nu\rho} A_\mu \partial_\nu A_\rho + i A^{*\mu} \partial_\mu C \right), \quad (2.21)$$

where, in addition to the usual Chern-Simons action for A_μ , there is a term coupling the antifield $A^{*\mu}$ to the gauge variation of A_μ . The action of (2.21) can be written compactly in terms of the wavefunction Ψ of (2.19) as

$$\mathcal{S} = \frac{1}{2} \int d^3x \langle \Psi Q \Psi \rangle \quad (2.22)$$

where $\langle \rangle$ is normalized such that $\langle c^\mu c^\nu c^\rho \rangle = i \epsilon^{\mu\nu\rho}$.

Up to now, only abelian Chern-Simons theory has been discussed, but it is easy to generalize to the non-abelian case. For example, the Batalin-Vilkovisky version of the non-abelian Chern-Simons action is

$$\begin{aligned} \mathcal{S} = Tr \int d^3x & \left(\epsilon^{\mu\nu\rho} \left(\frac{1}{2} A_\mu \partial_\nu A_\rho + \frac{ig}{3} A_\mu A_\nu A_\rho \right) \right. \\ & \left. + i A^{*\mu} (\partial_\mu C + ig[A_\mu, C]) - g C C C^* \right), \end{aligned} \quad (2.23)$$

which can be written compactly as

$$\mathcal{S} = Tr \int d^3x \left\langle \frac{1}{2} \Psi Q \Psi + \frac{g}{3} \Psi \Psi \Psi \right\rangle \quad (2.24)$$

where g is the Chern-Simons coupling constant and the fields in Ψ of (2.19) now carry Lie algebra indices. Note that the non-linear equations of motion and gauge invariances associated with this action are

$$Q\Psi + g\Psi\Psi = 0, \quad \delta\Psi = Q\Omega + g[\Omega, \Psi]. \quad (2.25)$$

Using intuition learned from this worldline description of Chern-Simons theory, it will now be shown how to quantize the superparticle in a similar manner.

2.4. Pure spinor description of the superparticle

In the case of Chern-Simons theory, the gauge transformation $\delta A_\mu = \partial_\mu \Lambda$ was generated by the constraints P_μ . So for the superparticle, the gauge transformation $\delta A_\alpha = D_\alpha \Lambda$ suggests using the constraints d_α . However, the constraints d_α are not all first-class, so

$$Q = \lambda^\alpha d_\alpha \quad (2.26)$$

would not be a nilpotent operator for generic λ^α . But since (2.5) implies that $Q^2 = (\lambda^\alpha d_\alpha)^2 = -\frac{i}{2} \lambda^\alpha \lambda^\beta \gamma_{\alpha\beta}^m P_m$, Q is nilpotent if λ^α satisfies the pure spinor condition

$$\lambda^\alpha \gamma_{\alpha\beta}^m \lambda^\beta = 0 \quad (2.27)$$

for $m = 0$ to 9 . Note that λ^α must be complex in order to have solutions to (2.27). However, its complex conjugate $\bar{\lambda}^\alpha$ never appears in the formalism so one is free to define λ^α to be a hermitian operator. Defining $(\lambda^\alpha)^\dagger = \lambda^\alpha$ does not lead to any inconsistencies since λ^α carries ghost number and therefore does not have any c -number eigenvalues. In other words, $\lambda^\alpha (\lambda^\beta)^\dagger = \lambda^\alpha \lambda^\beta$ takes states of ghost-number g to states of ghost-number $g + 2$. So $\lambda^\alpha (\lambda^\alpha)^\dagger$ has no c -number eigenvalues and there is therefore no reason that it should be positive-definite.

The pure spinor condition of (2.27) appears strange since bosonic ghosts in the BRST formalism are normally unconstrained and come from gauge-fixing fermionic Lagrange multipliers. However, as will now be argued, the BRST operator and pure spinor constraint of (2.26) and (2.27) can be derived by starting with the Brink-Schwarz superparticle in semi-light-cone gauge, adding additional fermionic degrees of freedom and gauge invariances, and then gauge-fixing in a non-standard manner.

The action of (2.7) for the Brink-Schwarz superparticle in semi-light-cone gauge is

$$\int d\tau (\dot{x}^m P_m + \frac{1}{2} \dot{S}_a S_a + e P^m P_m) \quad (2.28)$$

where $m = 0$ to 9 , $\alpha = 1$ to 16 , and $a = 1$ to 8 . Suppose one now introduces a new set of $(p_\alpha, \theta^\alpha)$ variables which are unrelated to S_a and defines $d_\alpha = p_\alpha + \frac{1}{2} P_m (\gamma^m \theta)_\alpha$. Using $\{d_\alpha, d_\beta\} = -i P_m \gamma_{\alpha\beta}^m$ and $\{S_a, S_b\} = i \delta_{ab}$, one can check that

$$\hat{d}_\alpha = d_\alpha + (\gamma_m \gamma^+ S)_\alpha P^m (P^+)^{-\frac{1}{2}} \quad (2.29)$$

describes first-class constraints which close to $\{\hat{d}_\alpha, \hat{d}_\beta\} = -\frac{i}{2P^+} P_m P^m \gamma_{\alpha\beta}^+$. So (2.28) is equivalent to

$$S = \int d\tau (\dot{x}^m P_m + \dot{\theta}^\alpha p_\alpha + \frac{1}{2} \dot{S}_a S_a + e P^m P_m + f^\alpha \hat{d}_\alpha) \quad (2.30)$$

where f^α are fermionic Lagrange multipliers. Since $\hat{d}_\alpha$ are first-class, they could be used to gauge $\theta^\alpha = 0$ which would return (2.30) to the original action of (2.28).

Using the usual BRST method, the action of (2.30) can be gauge-fixed to

$$S = \int d\tau (\dot{x}^m P_m - \frac{1}{2} P^m P_m + \dot{\theta}^\alpha p_\alpha + \frac{1}{2} \dot{S}_a S_a + \dot{c} b + \dot{\lambda}^\alpha \hat{w}_\alpha) \quad (2.31)$$

together with the BRST operator

$$\hat{Q} = \hat{\lambda}^\alpha \hat{d}_\alpha + c P^m P_m + \frac{i}{4P^+} b (\hat{\lambda} \gamma^+ \hat{\lambda}) \quad (2.32)$$

where $\hat{\lambda}^\alpha$ is an unconstrained bosonic spinor variable which comes from gauge-fixing $f^\alpha = 0$. To relate $\hat{Q}$ with $Q = \lambda^\alpha d_\alpha$, it will first be argued that the cohomology of $\hat{Q}$ is equivalent to the cohomology of $Q' = \lambda'^\alpha \hat{d}_\alpha$ in a Hilbert space without (b, c) ghosts and where λ'^α is constrained to satisfy $\lambda' \gamma^+ \lambda' = 0$ (but is not constrained to satisfy $\lambda' \gamma^j \lambda' = 0$ or $\lambda' \gamma^- \lambda' = 0$). To show that Q' has the same cohomology as $\hat{Q}$, consider a state V annihilated by Q' up to terms proportional to $\lambda' \gamma^+ \lambda'$, i.e. $Q'V = (\lambda' \gamma^+ \lambda')W$ for some W . Then $(Q')^2 = -\frac{i}{4P^+} (\lambda' \gamma^+ \lambda') P^m P_m$ implies that $Q'W = -\frac{i}{4P^+} P^m P_m V$. Using this information, one can check that $\hat{V} = V + 4iP^+ cW$ is annihilated by $\hat{Q}$. Furthermore, if V is BRST-trivial up to terms involving $\lambda' \gamma^+ \lambda'$, i.e. $V = Q'\Omega + (\lambda' \gamma^+ \lambda')Y$ for some Y , then $V + 4iP^+ cW = \hat{Q}(\Omega - 4iP^+ cY)$, so $\hat{V}$ is also BRST-trivial. So any state in the cohomology of Q' is in the cohomology of $\hat{Q}$, and reversing the previous arguments, one can show that any state in the cohomology of $\hat{Q}$ is in the cohomology of Q' .

Finally, it will be shown that the cohomology of $Q' = \lambda'^\alpha \hat{d}_\alpha$ is equivalent to the cohomology of $Q = \lambda^\alpha d_\alpha$ where λ^α is a pure spinor and the Hilbert space is independent of S_a . Since $(\gamma^+ \lambda')_{\dot{a}}$ is a null $\text{SO}(8)$ antichiral spinor, it is preserved up to a phase by some $\text{U}(4)$ subgroup of $\text{SO}(8)$. Under this $\text{U}(4)$ subgroup, the chiral $\text{SO}(8)$ spinor $(\gamma^- \lambda')_a$ splits into a 4 and $\bar{4}$ representation which will be called $(\gamma^- \lambda')_A$ and $(\gamma^- \lambda')_{\bar{A}}$ for $A, \bar{A} = 1$ to 4. Similarly, the chiral $\text{SO}(8)$ spinors $(\gamma^+ d)_a$ and S_a split into the representations $[(\gamma^+ d)_A, (\gamma^+ d)_{\bar{A}}]$ and $[S_A, S_{\bar{A}}]$. Note that the 4 and $\bar{4}$ representations are defined with respect to the null spinor $(\gamma^+ \lambda')_{\dot{a}}$ such that $\sigma_j^{A\dot{a}} (\gamma^+ \lambda')_{\dot{a}}$ is zero for $j = 1$ to 8, and

$\sigma_j^{\bar{A}\dot{a}}(\gamma^+\lambda')_{\dot{a}}$ is non-zero. After performing a similarity transformation which shifts $S_A \rightarrow S_A + (P^+)^{-\frac{1}{2}}(\gamma^+d)_A$, one finds that Q' transforms as

$$Q' \rightarrow e^{-iS_{\bar{A}}(\gamma^+d)_A(P^+)^{-\frac{1}{2}}} Q' e^{iS_{\bar{A}}(\gamma^+d)_A(P^+)^{-\frac{1}{2}}} \quad (2.33)$$

$$= (\gamma^+\lambda')_{\dot{a}}(\gamma^-d)_{\dot{a}} + (\gamma^-\lambda')_A(\gamma^+d)_{\bar{A}} + (\gamma^-\lambda')_{\bar{A}}S_A\sqrt{P^+}.$$

So $Q' = \lambda^\alpha d_\alpha + (\gamma^-\lambda')_{\bar{A}}S_A\sqrt{P^+}$ where λ^α is a pure spinor defined by

$$[(\gamma^+\lambda)_{\dot{a}}, (\gamma^-\lambda)_A, (\gamma^-\lambda)_{\bar{A}}] = [(\gamma^+\lambda')_{\dot{a}}, (\gamma^-\lambda')_A, 0]. \quad (2.34)$$

Using the standard quartet argument, the cohomology of $Q' = Q + (\gamma^-\lambda')_{\bar{A}}S_A\sqrt{P^+}$ is equivalent to the cohomology of $Q = \lambda^\alpha d_\alpha$ in the Hilbert space independent of $(\gamma^-\lambda')_{\bar{A}}$, S_A , and its conjugate momenta $(\gamma^+w')_A$ and $S_{\bar{A}}$. So the Brink-Schwarz superparticle action has been shown to be equivalent to the action

$$S = \int d\tau (\dot{x}^m P_m - \frac{1}{2} P^m P_m + \dot{\theta}^\alpha p_\alpha + \dot{\lambda}^\alpha w_\alpha) \quad (2.35)$$

together with the BRST operator $Q = \lambda^\alpha d_\alpha$ where $\lambda\gamma^m\lambda = 0$.

Although the above derivation of the pure spinor description from the Brink-Schwarz superparticle was not manifestly Lorentz covariant, the final result of (2.35) is manifestly covariant. As will be shown in the next subsection, quantization using this description provides a manifestly super-Poincaré covariant description of D=10 super-Yang-Mills theory.

2.5. Covariant quantization of the D=10 superparticle

The most general super-Poincaré covariant wavefunction that can be constructed from $(x^m, \theta^\alpha, \lambda^\alpha)$ is

$$\Psi(x, \theta, \lambda) = C(x, \theta) + \lambda^\alpha A_\alpha(x, \theta) + (\lambda\gamma^{mnpqr}\lambda) A_{mnpqr}^*(x, \theta) + \lambda^\alpha \lambda^\beta \lambda^\gamma C_{\alpha\beta\gamma}^*(x, \theta) + \dots \quad (2.36)$$

where ... includes superfields with more than three powers of λ^α . Note that the names for the superfields appearing in (2.36) have been chosen to coincide with the names for the Chern-Simons fields in (2.19). As in Chern-Simons, the ghost-number zero superfield C contains the spacetime ghost, the ghost-number one superfield A_α contains the super-Yang-Mills fields, the ghost-number two superfield A_{mnpqr}^* contains the super-Yang-Mills

antifields, and the ghost-number three superfield $C_{\alpha\beta\gamma}^*$ contains the spacetime antighost. All superfields in ... with ghost-number greater than three will have trivial cohomology.

For example, $Q\Psi = -i\lambda^\alpha D_\alpha C - i\lambda^\alpha \lambda^\beta D_\alpha A_\beta + \dots$, so $Q\Psi = 0$ implies that $A_\alpha(x, \theta)$ satisfies the equation of motion $\lambda^\alpha \lambda^\beta D_\alpha A_\beta = 0$. But since $\lambda^\alpha \lambda^\beta$ is proportional to $(\lambda \gamma^{mnpqr} \lambda) \gamma_{mnpqr}^{\alpha\beta}$, this implies that $D\gamma^{mnpqr} A = 0$, which is the linearized version of the super-Yang-Mills equation of motion of (2.11). Furthermore, if one defines the gauge parameter $\Omega = i\Lambda + \lambda^\alpha \omega_\alpha + \dots$, the gauge transformation $\delta\Psi = Q\Omega$ implies $\delta A_\alpha = D_\alpha \Lambda$ which is the linearized super-Yang-Mills gauge transformation of (2.12).

So as described in (2.15), $A_\alpha(x, \theta)$ contains the on-shell super-Yang-Mills gluon and gluino, $a_m(x)$ and $\chi^\alpha(x)$, which satisfy the linearized equations of motion and gauge invariances

$$\partial^m \partial_{[m} a_{n]} = \gamma_{\alpha\beta}^m \partial_m \chi^\beta = 0, \quad \delta a_m = \partial_m s.$$

And since gauge invariances of antifields correspond to equations of motion of fields, one expects to have antifields $a^{*m}(x)$ and $\chi_\alpha^*(x)$ in the cohomology of Q which satisfy the linearized equations of motion and gauge invariances

$$\partial_m a^{*m} = 0, \quad \delta a^{*m} = \partial_n (\partial^n s^m - \partial^m s^n), \quad \delta \chi_\alpha^* = \gamma_{\alpha\beta}^m \partial_m \kappa^\beta \quad (2.37)$$

where s^m and κ^β are gauge parameters. Indeed, these antifields a^{*m} and χ_α^* appear in components of the ghost-number +2 superfield A_{mnpqr}^* of (2.36). Using $Q\Psi = 0$ and $\delta\Psi = Q\Omega$, A_{mnpqr}^* satisfies the linearized equation of motion $\lambda^\alpha (\lambda \gamma^{mnpqr} \lambda) D_\alpha A_{mnpqr}^* = 0$ with the linearized gauge invariance $\delta A_{mnpqr}^* = \gamma_{mnpqr}^{\alpha\beta} D_\alpha \omega_\beta$. Expanding ω_α and A_{mnpqr}^* in components, one learns that A_{mnpqr}^* can be gauged to the form

$$A_{mnpqr}^* = (\theta \gamma_{[mnp} \theta) (\theta \gamma_{qr}]^\alpha \chi_\alpha^*(x) + (\theta \gamma_{[mnp} \theta) (\theta \gamma_{qr}]_s \theta) a^{*s}(x) + \dots \quad (2.38)$$

where χ_α^* and a^{*s} satisfy the equations of motion and residual gauge invariances of (2.37), and ... involves terms higher order in θ^α which depend on derivatives of χ_α^* and a^{*s} .

In addition to these fields and antifields, one also expects to find the Yang-Mills ghost $c(x)$ and antighost $c^*(x)$ in the cohomology of Q . The ghost $c(x)$ is found in the $\theta = 0$ component of the ghost-number zero superfield, $C(x, \theta) = c(x) + \dots$, and the antighost $c^*(x)$ is found in the $(\theta)^5$ component of the ghost-number +3 superfield, $C_{\alpha\beta\gamma}^*(x, \theta) = \dots + c^*(x) (\gamma^m \theta)_\alpha (\gamma^n \theta)_\beta (\gamma^p \theta)_\gamma (\theta \gamma_{mnp} \theta) + \dots$. It was proven in [36] that the above states are the only states in the cohomology of Q and therefore, although Ψ of (2.36) contains

superfields of arbitrarily high ghost number, only superfields with ghost-number between zero and three contain states in the cohomology of Q .

The linearized equations of motion and gauge invariances $Q\Psi = 0$ and $\delta\Psi = Q\Omega$ are easily generalized to the non-linear equations of motion and gauge invariances

$$Q\Psi + g\Psi\Psi = 0, \quad \delta\Psi = Q\Omega + g[\Psi, \Omega] \quad (2.39)$$

where Ψ and Ω transform in the adjoint representation of the gauge group. For the superfield $A_\alpha(x, \theta)$, (2.39) implies the super-Yang-Mills equations of motion and gauge transformations of (2.11) and (2.12). Furthermore, the equation of motion and gauge transformation of (2.39) can be obtained from the spacetime action⁵

$$\mathcal{S} = \text{Tr} \int d^{10}x \left\langle \frac{1}{2} \Psi Q\Psi + \frac{g}{3} \Psi\Psi\Psi \right\rangle \quad (2.40)$$

using the normalization definition that

$$\langle (\lambda\gamma^m\theta)(\lambda\gamma^n\theta)(\lambda\gamma^p\theta)(\theta\gamma_{mnp}\theta) \rangle = 1. \quad (2.41)$$

Although (2.41) may seem strange, it resembles the normalization of (2.22) in that $\langle \Psi \rangle = c^*(x)$ where $c^*(x)$ is the spacetime antighost. After expressing (2.40) in terms of component fields and integrating out auxiliary fields, it should be possible to show that (2.40) reduces to the standard Batalin-Vilovsky action for super-Yang-Mills,

$$\begin{aligned} \mathcal{S} = \text{Tr} \int d^{10}x & \left(\frac{1}{4} f_{mn} f^{mn} + \chi^\alpha \gamma_{\alpha\beta}^m (\partial_m \chi^\beta + ig[a_m, \chi^\beta]) \right. \\ & \left. + ia^{*m} (\partial_m c + ig[a_m, c]) - g\chi_\alpha^* \{\chi^\alpha, c\} - gccc^* \right). \end{aligned} \quad (2.42)$$

Because the action of (2.40) only involves integration over five θ 's, it is not manifestly spacetime supersymmetric. This is not surprising since it is not known how to construct a manifestly supersymmetric action for D=10 super-Yang-Mills. Nevertheless, the equations of motion coming from this action have the same physical content as the manifestly spacetime supersymmetric equations of motion $Q\Psi + g\Psi\Psi = 0$. This is because all components in $Q\Psi + g\Psi\Psi = 0$ with more than five θ 's are auxiliary equations of motion. So removing these equations of motion only changes auxiliary fields to gauge fields but does not affect the physical content of the theory. By defining the normalization of (2.41) to involve $\lambda^\alpha(\sigma)$ and $\theta^\alpha(\sigma)$ at the midpoint $\sigma = \frac{\pi}{2}$ as in [37], it should be possible to generalize the action of (2.40) to a cubic open superstring field theory action.

⁵ This spacetime action was first proposed to me by John Schwarz and Edward Witten.

2.6. Pure spinor description for $d \neq 10$

It is interesting to ask if the pure spinor description can also be used to covariantly quantize the superparticle when $d \neq 10$. Note that unlike the GS superstring action, the Brink-Schwarz superparticle action is invariant under κ -symmetry in any spacetime dimension. If one defines a pure spinor in d dimensions⁶ by $\lambda\gamma^m\lambda = 0$ for $m = 0$ to $d-1$, a pure spinor contains $(3N-4)/4$ independent components where N is the number of components in an unconstrained $SO(d-1, 1)$ spinor. This counting can be derived using a construction similar to the counting in $d = 10$ where $(\gamma^+\lambda)$ is a null $SO(d-2)$ spinor with $(N-2)/2$ components and $(\gamma^-\lambda)$ is half of an $SO(d-2)$ spinor with $N/4$ components. So λ^α has 2 components when $d = 4$, 5 components when $5 \leq d \leq 6$, 11 components when $7 \leq d \leq 10$, and 23 components when $d = 11$.

For $d = 11$, it was shown in [38] that the pure spinor description correctly describes a superparticle whose physical spectrum is linearized d=11 supergravity with 32 supersymmetries. As discussed in [38], physical states for the $d = 11$ superparticle carry ghost-number three and the state $\Psi = \lambda^\alpha\lambda^\beta\lambda^\gamma B_{\alpha\beta\gamma}(x, \theta)$ describes the $d = 11$ supergravity multiplet where $B_{\alpha\beta\gamma}$ is the spinor component of the three-form superfield [39]. And for $7 \leq d < 10$, one can easily check that the pure spinor description correctly describes a superparticle whose physical spectrum is a dimensional reduction of super-Yang-Mills with 16 supersymmetries. However, for $d \leq 6$, the situation is more subtle. Note that a $d = 6$ spinor is described by λ_a^J where $J = 1$ to 2 is an $SU(2)$ spinor index and $a = 1$ to 4 is an $SU^*(4)$ index. The constraint $\lambda\gamma^m\lambda = 0$ implies $\lambda_a^J\lambda_b^K\epsilon_{JK} = 0$, which implies that $\lambda_a^J = c^J h_a$ for some c^J and h_a . And for $d = 4$, $\lambda\gamma^m\lambda = 0$ implies that either $\lambda_a = 0$ or $\lambda_{\dot{a}} = 0$ where $(a, \dot{a}) = 1$ to 2 are the standard $SU(2)$ Weyl indices.

Using techniques similar to the $d = 10$ case, one finds that for $5 \leq d \leq 6$ or $d = 4$, the cohomology of $Q = \lambda^\alpha d_\alpha$ describes off-shell super-Yang-Mills with 8 or 4 supersymmetries. As in $d = 10$, $Q\Psi = 0$ implies that $\lambda^\alpha\lambda^\beta D_\alpha A_\beta = 0$, which implies that $D_{(\alpha}A_{\beta)} = \gamma_{\alpha\beta}^m B_m$ for some vector gauge field B_m . However, unlike $d = 10$, the theory is off-shell since $D_{(\alpha}A_{\beta)} = \gamma_{\alpha\beta}^m B_m$ does not impose equations of motion when $d \leq 6$. This might seem surprising since the Brink-Schwarz superparticle contains the $P_m P^m = 0$ mass-shell constraint for any d . But note that for $d \leq 6$, there are also subtleties in the light-cone

⁶ In arbitrary spacetime dimension, this is not the pure spinor definition used by Cartan. For example, in $d = 11$, Cartan would define a pure spinor to satisfy both $\lambda\gamma^m\lambda = 0$ and $\lambda\gamma^{mn}\lambda = 0$ [11].

quantization of the Brink-Schwarz superparticle. When $d = 6$, the light-cone S_a variable contains 4 components, which naively suggests $2^{4/2} = 4$ states in the physical spectrum instead of the 8 states of $d = 6$ super-Yang-Mills. And when $d = 4$, S_a contains 2 components, which naively suggests $2^{2/2} = 2$ physical states instead of the 4 states of $d = 4$ super-Yang-Mills. Since light-cone quantization of the superparticle is not straightforward in $d \leq 6$, it is not so surprising that there are subtleties in the pure spinor description in these dimensions.

3. Covariant Quantization of the Superstring

In this section, the pure spinor description of the superparticle will be generalized to the superstring. Although there have been several previous approaches to covariantly quantizing the superparticle, this is the first approach which successfully generalizes to covariant quantization of the superstring. But before discussing the pure spinor approach, it will be useful to discuss an alternative approach of Siegel [4] which contains some of the same features as the pure spinor approach.

3.1. Review of GS formalism using the approach of Siegel

In conformal gauge, the classical covariant GS action for the heterotic superstring is [40]

$$S_{het} = \int d^2z \left[\frac{1}{2} \Pi^m \bar{\Pi}_m + \frac{1}{4} \Pi_m \theta^\alpha \gamma_{\alpha\beta}^m \bar{\partial} \theta^\beta - \frac{1}{4} \bar{\Pi}_m \theta^\alpha \gamma_{\alpha\beta}^m \partial \theta^\beta \right] + S_R \quad (3.1)$$

where x^m and θ^α are the worldsheet variables ($m = 0$ to 9 , $\alpha = 1$ to 16), S_R describes the right-moving degrees of freedom for the $E_8 \times E_8$ or $SO(32)$ lattice, and $\Pi^m = \partial x^m + \frac{1}{2} \theta^\alpha \gamma_{\alpha\beta}^m \partial \theta^\beta$ and $\bar{\Pi}^m = \bar{\partial} x^m + \frac{1}{2} \theta^\alpha \gamma_{\alpha\beta}^m \bar{\partial} \theta^\beta$ are supersymmetric combinations of the momentum. In what follows, the right-moving degrees of freedom play no role and will be ignored. Also, all of the following remarks are easily generalized to the Type I and Type II superstrings.

Since the action of (3.1) is in conformal gauge, it needs to be supplemented with the Virasoro constraint $T = -\frac{1}{2} \Pi^m \Pi_m = 0$. Also, since the canonical momentum to θ^α does not appear in the action, one has the Dirac constraint $p_\alpha = \delta \Lambda / \delta \partial_0 \theta^\alpha = \frac{1}{2} (\Pi_m - \frac{1}{4} \theta \gamma_m \partial_1 \theta) (\gamma^m \theta)_\alpha$ where p_α is the canonical momentum to θ^α . If one defines

$$d_\alpha = p_\alpha - \frac{1}{2} (\Pi_m - \frac{1}{4} \theta \gamma_m \partial_1 \theta) (\gamma^m \theta)_\alpha, \quad (3.2)$$

one can use the canonical commutation relations to find $\{d_\alpha, d_\beta\} = i\gamma_{\alpha\beta}^m \Pi_m$, which implies (since $\Pi^m \Pi_m = 0$ is a constraint) that the sixteen Dirac constraints d_α have eight first-class components and eight second-class components. Since the anti-commutator of the second-class constraints is non-trivial (i.e. the anti-commutator is an operator Π^+ rather than a constant), standard Dirac quantization cannot be used since it would involve inverting an operator. So except in light-cone gauge (where the commutator becomes a constant), the covariant Green-Schwarz formalism cannot be easily quantized.

In 1986, Siegel suggested an alternative approach in which the canonical momentum to θ^α is an independent variable using the free-field action [4]

$$S = \int d^2z \left[\frac{1}{2} \partial x^m \bar{\partial} x_m + p_\alpha \bar{\partial} \theta^\alpha \right]. \quad (3.3)$$

In this approach, Siegel attempted to replace the problematic constraints of the covariant GS action with some suitable set of first-class constraints constructed out of the supersymmetric objects Π^m , d_α and $\partial\theta^\alpha$ where

$$d_\alpha = p_\alpha - \frac{1}{2} (\partial x^m + \frac{1}{4} \theta \gamma^m \partial \theta) (\gamma_m \theta)_\alpha \quad (3.4)$$

is defined as in (3.2) and is no longer constrained to vanish. The first-class constraints should include the Virasoro constraint $A = -\frac{1}{2} \Pi^m \Pi_m - d_\alpha \partial \theta^\alpha = -\frac{1}{2} \partial x^m \partial x_m - p_\alpha \partial \theta^\alpha$ and the κ -symmetry generator $B^\alpha = \Pi^m (\gamma_m d)^\alpha$. To get to light-cone gauge, one also needs constraints such as $C^{mnp} = d_\alpha (\gamma^{mnp})^{\alpha\beta} d_\beta$ which is supposed to replace the second-class constraints in d_α . Although this approach was successfully used for quantizing the superparticle [41], a set of constraints which closes at the quantum level and which reproduces the correct physical superstring spectrum was never found.

Nevertheless, the approach of Siegel has the advantage that all worldsheet fields are free which makes it trivial to compute the OPE's that

$$x^m(y) x^n(z) \rightarrow -2\eta^{mn} \log|y-z|, \quad p_\alpha(y) \theta^\beta(z) \rightarrow \delta_\alpha^\beta (y-z)^{-1}, \quad (3.5)$$

$$d_\alpha(y) d_\beta(z) \rightarrow -\frac{1}{(y-z)} \gamma_{\alpha\beta}^m \Pi_m(z), \quad d_\alpha(y) \Pi^m(z) \rightarrow \frac{1}{(y-z)} \gamma_{\alpha\beta}^m \partial \theta^\beta(z). \quad (3.6)$$

This gives some useful clues about the appropriate ghost degrees of freedom. Since $(\theta^\alpha, p_\alpha)$ contributes -32 to the conformal anomaly, the total matter contribution is -22 which is expected to be cancelled by a ghost contribution of $+22$. Furthermore, the spin contribution to the $SO(9,1)$ Lorentz currents in Siegel's approach is $M_{mn} = \frac{1}{2} p \gamma_{mn} \theta$, as compared with the spin contribution to the $SO(9,1)$ Lorentz currents in the RNS formalism which is $\psi_m \psi_n$. These two Lorentz currents satisfy similar OPE's except for the numerator in the double pole of M_{mn} with M_{mn} , which is $+4$ in Siegel's approach and $+1$ in the RNS formalism. This suggests that the worldsheet ghosts should have Lorentz currents which contribute -3 to the double pole.

3.2. Superstring quantization using pure spinors

In fact, there exists an $SO(9, 1)$ irreducible representation contributing $c = 22$ and with a -3 coefficient in the double pole of its Lorentz current [3]. This representation consists of a bosonic pure spinor λ^α satisfying the condition that

$$\lambda^\alpha \gamma_{\alpha\beta}^m \lambda^\beta = 0 \quad (3.7)$$

for $m = 0$ to 9 . To show that this representation has the desired properties, it is useful to temporarily break manifest Lorentz invariance by explicitly solving the pure spinor constraint of (3.7).

A parameterization of λ^α which preserves a $U(5)$ subgroup of (Wick-rotated) $SO(10)$ is [3][23]

$$\lambda^+ = e^s, \quad \lambda_{ab} = u_{ab}, \quad \lambda^a = -\frac{1}{8} e^{-s} \epsilon^{abcde} u_{bc} u_{de} \quad (3.8)$$

where $a = 1$ to 5 , $u_{ab} = -u_{ba}$ are ten independent variables, and the $SO(10)$ spinor λ^α has been written in terms of its irreducible $U(5)$ components which transform as $(1_{\frac{5}{2}}, \overline{10}_{\frac{1}{2}}, 5_{-\frac{3}{2}})$ representations of $SU(5)_{U(1)}$. A simple way to obtain these $U(5)$ representations is to write an $SO(10)$ spinor using $[\pm \pm \pm \pm \pm]$ notation where Weyl/anti-Weyl spinors have an odd/even number of $+$ signs. The $1_{\frac{5}{2}}$ component of λ^α is the component with five $+$ signs, the $\overline{10}_{\frac{1}{2}}$ component has three $+$ signs, and the $5_{-\frac{3}{2}}$ component has one $+$ sign. The λ^α parameterization of (3.8) is possible whenever $\lambda^+ \neq 0$.

Using the above parameterization of λ^α , one can define the action S_λ for the worldsheet ghosts as

$$S_\lambda = \int d^2 z [\bar{\partial} t \partial s - \frac{1}{2} v^{ab} \bar{\partial} u_{ab}] \quad (3.9)$$

where t and v^{ab} are the conjugate momenta to s and u_{ab} satisfying the OPE's

$$t(y) s(z) \rightarrow \log(y - z), \quad v^{ab}(y) u_{cd}(z) \rightarrow \delta_c^{[a} \delta_d^{b]} (y - z)^{-1}. \quad (3.10)$$

Note that the factor of $\frac{1}{2}$ in the $v^{ab} \bar{\partial} u_{ab}$ term has been introduced to cancel the factor of 2 from $u_{ab} = -u_{ba}$. Also note that s and t are chiral bosons, so their contribution to (3.9) needs to be supplemented by a chirality constraint.

One can construct $SO(10)$ Lorentz currents N^{mn} out of these free variables as

$$N = \frac{1}{\sqrt{5}} \left(\frac{1}{4} u_{ab} v^{ab} + \frac{5}{2} \partial t - \frac{5}{2} \partial s \right), \quad N_a^b = u_{ac} v^{bc} - \frac{1}{5} \delta_a^b u_{cd} v^{cd}, \quad (3.11)$$

$$N^{ab} = e^s v^{ab}, \quad N_{ab} = e^{-s} (2\partial u_{ab} - u_{ab}\partial t - 2u_{ab}\partial s + u_{ac}u_{bd}v^{cd} - \frac{1}{2}u_{ab}u_{cd}v^{cd})$$

where N^{mn} has been written in terms of its $U(5)$ components $(N, N_a^b, N^{ab}, N_{ab})$ which transform as $(1_0, 24_0, 10_2, \overline{10}_{-2})$ representations of $SU(5)_{U(1)}$. The Lorentz currents of (3.11) can be checked to satisfy the OPE's

$$N^{mn}(y)\lambda^\alpha(z) \rightarrow \frac{1}{2}(\gamma^{mn})^\alpha{}_\beta \frac{\lambda^\beta(z)}{y-z}, \quad (3.12)$$

$$N^{kl}(y)N^{mn}(z) \rightarrow \frac{\eta^{m[l}N^{k]n}(z) - \eta^{n[l}N^{k]m}(z)}{y-z} - 3\frac{\eta^{kn}\eta^{lm} - \eta^{km}\eta^{ln}}{(y-z)^2}. \quad (3.13)$$

So although S_λ of (3.9) is not manifestly Lorentz covariant, any OPE's of λ^α and N^{mn} which are computed using this action are manifestly covariant.

In terms of the free fields, the stress tensor is

$$T_\lambda = \frac{1}{2}v^{ab}\partial u_{ab} + \partial t\partial s + \partial^2 s \quad (3.14)$$

where the $\partial^2 s$ term is included so that the Lorentz currents of (3.11) are primary fields. This stress tensor has central charge +22 and can be written in manifestly Lorentz invariant notation as[42]

$$T_\lambda = \frac{1}{10}N_{mn}N^{mn} - \frac{1}{8}J^2 - \partial J \quad (3.15)$$

where J is defined in terms of the free fields by

$$J = \frac{1}{2}u_{ab}v^{ab} + \partial t + 3\partial s. \quad (3.16)$$

Note that J has no singularities with N^{mn} and satisfies the OPE's

$$J(y)J(z) \rightarrow -4(y-z)^{-2}, \quad J(y)\lambda^\alpha(z) \rightarrow (y-z)^{-1}\lambda^\alpha(z).$$

The operator $\oint J$ can be identified with the ghost-number operator so that λ^α carries ghost number +1.

3.3. Physical vertex operators

Physical states in the pure spinor formalism for the open superstring are defined as ghost-number one states in the cohomology of $Q = \int \lambda^\alpha d_\alpha$ where λ^α is constrained to satisfy $\lambda \gamma^m \lambda = 0$. The constraint $\lambda \gamma^m \lambda = 0$ implies that the canonical momentum for λ^α , which will be called w_α , only appears in combinations which are invariant under the gauge transformation

$$\delta w_\alpha = (\gamma^m \lambda)_\alpha \Lambda_m \quad (3.17)$$

for arbitrary Λ_m . This implies that w_α only appears in the Lorentz-covariant combinations $N_{mn} = \frac{1}{2} : w \gamma_{mn} \lambda :$ and $J =: w_\alpha \lambda^\alpha :$ where the normal-ordered expressions can be explicitly defined using the parameterization of (3.11) and (3.16). When $(mass)^2 = n/2$, open superstring vertex operators are constructed from arbitrary combinations of $[x^m, \theta^\alpha, d_\alpha, \lambda^\alpha, N^{mn}, J]$ which carry ghost number one and conformal weight n at zero momentum. Note that $[d_\alpha, N_{mn}, J]$ carry conformal weight one and λ^α carries ghost number one.

For example, the most general vertex operator at $(mass)^2 = 0$ is

$$U = \lambda^\alpha A_\alpha(x, \theta) \quad (3.18)$$

where $A_\alpha(x, \theta)$ is an unconstrained spinor superfield. As was shown in subsection (2.5), $QU = 0$ and $\delta U = Q\Omega$ implies $\gamma_{mnpqr}^{\alpha\beta} D_\alpha A_\beta = 0$ and $\delta A_\alpha = D_\alpha \Omega$, which are the super-Maxwell equations of motion and gauge invariances written in terms of a spinor superfield.

At the next mass level, the physical states of the open superstring form a massive spin-2 multiplet containing 128 bosons and 128 fermions. Although it was not previously known how to covariantly describe this multiplet in D=10 superspace, such a superspace description was found with Osvaldo Chandía using the pure spinor approach [43]. When $(mass)^2 = \frac{1}{2}$, the most general vertex operator is

$$U = \partial \lambda^\alpha A_\alpha(x, \theta) + : \partial \theta^\beta \lambda^\alpha B_{\alpha\beta}(x, \theta) : + : d_\beta \lambda^\alpha C^\beta_\alpha(x, \theta) : \quad (3.19)$$

$$+ : \Pi^m \lambda^\alpha H_{m\alpha}(x, \theta) : + : J \lambda^\alpha E_\alpha(x, \theta) : + : N^{mn} \lambda^\alpha F_{\alpha mn}(x, \theta) :$$

where $: O^A \lambda^\alpha \Phi_{\alpha A}(x, \theta)(z) : = \oint \frac{dy}{y-z} O^A(y) \lambda^\alpha(z) \Phi_{\alpha A}(z)$ and $\Phi_{\alpha A}(x, \theta)$ are the various superfields appearing in (3.19). Using the OPE's of (3.6), it was shown in [43] that $QU = 0$ implies the equations

$$(\gamma_{mnpqr})^{\alpha\beta} [D_\alpha B_{\beta\gamma} - \gamma_{\alpha\gamma}^s H_{s\beta}] = 0, \quad (3.20)$$

$$\begin{aligned}
& (\gamma_{mnpqr})^{\alpha\beta} [D_\alpha H_{s\beta} - \gamma_{s\alpha\gamma} C^\gamma_\beta] = 0, \\
& (\gamma_{mnpqr})^{\alpha\beta} [D_\alpha C^\gamma_\beta + \delta^\gamma_\alpha E_\beta + \frac{1}{2}(\gamma^{st})^\gamma_\alpha F_{\beta st}] = 0, \\
& (\gamma_{mnpqr})^{\alpha\beta} [D_\alpha A_\beta + B_{\alpha\beta} + 2\gamma^s_{\beta\gamma} \partial_s C^\gamma_\alpha - D_\beta E_\alpha + \frac{1}{2}(\gamma^{st} D)_\beta F_{\alpha st}] \\
& \quad = 4\gamma^{\alpha\beta}_{mnpqr} \gamma_{\alpha\beta}^{vwxyz} \eta_{st} K^t_{vwxy}, \\
& (\gamma_{mnp})^{\alpha\beta} [D_\alpha A_\beta + B_{\alpha\beta} + 2\gamma^s_{\beta\gamma} \partial_s C^\gamma_\alpha - D_\beta E_\alpha + \frac{1}{2}(\gamma^{st} D)_\beta F_{\alpha st}] \\
& \quad = 32\gamma^{\alpha\beta}_{mnp} \gamma_{\alpha\beta}^{wxyz} K^s_{wxyz}, \\
& \gamma^{\alpha\beta}_{mnpqr} D_\alpha E_\beta = \gamma^{\alpha\beta}_{mnpqr} (\gamma^{vwxy} \gamma_s)_{\alpha\beta} K^s_{vwxy}, \\
& \gamma^{\alpha\beta}_{mnpqr} D_\alpha F^{st}_\beta = -\gamma^{\alpha\beta}_{mnpqr} (\gamma^{vwxy} \gamma^{[s})_{\alpha\beta} K^{t]}_{vwxy},
\end{aligned}$$

where K^s_{vwxy} is an arbitrary superfield. And the gauge invariance $\delta U = Q\Omega$ implies the gauge transformations

$$\begin{aligned}
\delta A_\alpha &= \Omega_{1\alpha} + 2\gamma^m_{\alpha\beta} \partial_m \Omega_2^\beta - D_\alpha \Omega_4 - \frac{1}{2}(\gamma^{mn})^\beta_\alpha D_\beta \Omega_{5mn}, \\
\delta B_{\alpha\beta} &= -D_\alpha \Omega_{1\beta} + \gamma^m_{\alpha\beta} \Omega_{3m}, \\
\delta C^\beta_\alpha &= -D_\alpha \Omega_2^\beta - \delta^\beta_\alpha \Omega_4 - \frac{1}{2}(\gamma^{mn})^\beta_\alpha \Omega_{5mn}, \\
\delta H_{m\alpha} &= D_\alpha \Omega_{3m} - \gamma_{m\alpha\beta} \Omega_2^\beta, \\
\delta E_\alpha &= D_\alpha \Omega_4, \\
\delta F_{\alpha mn} &= D_\alpha \Omega_{5mn},
\end{aligned} \tag{3.21}$$

where

$$\begin{aligned}
\Omega &= : \partial\theta^\alpha \Omega_{1\alpha}(x, \theta) : + : d_\alpha \Omega_2^\alpha(x, \theta) : + : \Pi^m \Omega_{3m}(x, \theta) : \\
&+ : J \Omega_4(x, \theta) : + : N^{mn} \Omega_{5mn}(x, \theta) :,
\end{aligned} \tag{3.22}$$

and $: O^A \Omega_A(x, \theta) : = \oint \frac{dy}{y-z} O^A(y) \Omega_A(z)$. Using d=10 superspace techniques, it was argued in [43] that the equations of motion and gauge transformations of (3.20) and (3.21) imply that the superfields $\Phi_{\alpha A}(x, \theta)$ in (3.19) correctly describe a massive spin-two multiplet with $(mass)^2 = \frac{1}{2}$.

To compute scattering amplitudes, one also needs vertex operators in integrated form, $\int dz V$, where V is usually obtained from the unintegrated vertex operator U by anti-commuting with the b ghost. But since there is no natural candidate for the b ghost in

this formalism, one needs to use an alternative method for obtaining V which is from the relation $[Q, V] = \partial U$ [44]. Using this alternative method, one finds for the open superstring massless vertex operator that [3]

$$V = \partial\theta^\alpha A_\alpha(x, \theta) + \Pi^m B_m(x, \theta) + d_\alpha W^\alpha(x, \theta) + \frac{1}{2} N_{mn} F^{mn}(x, \theta). \quad (3.23)$$

To show that $QV = \partial U$, note that

$$\begin{aligned} QV &= \partial(\lambda^\alpha A_\alpha) + \lambda^\alpha \partial\theta^\beta (-D_\alpha A_\beta - D_\beta A_\alpha + \gamma_{\alpha\beta}^m B_m) \\ &+ \lambda^\alpha \Pi^m (D_\alpha B_m - \partial_m A_\alpha - \gamma_{m\alpha\beta} W^\beta) + \lambda^\alpha d_\beta (-D_\alpha W^\beta + \frac{1}{4} (\gamma^{mn})_\alpha{}^\beta F_{mn}) + \frac{1}{2} \lambda^\alpha N_{mn} D_\alpha F^{mn}. \end{aligned} \quad (3.24)$$

So $QV = \partial U$ if the superfields satisfy

$$-D_\alpha A_\beta - D_\beta A_\alpha + \gamma_{\alpha\beta}^m B_m = 0, \quad (3.25)$$

$$D_\alpha B_m - \partial_m A_\alpha - \gamma_{m\alpha\beta} W^\beta = 0,$$

$$-D_\alpha W^\beta + \frac{1}{4} (\gamma^{mn})_\alpha{}^\beta F_{mn} = 0,$$

$$\lambda^\alpha \lambda^\beta (\gamma_{mn})_\beta{}^\gamma D_\alpha F^{mn} = 0,$$

which imply the super-Maxwell equations of subsection (2.2). Note that the fourth equation of (3.25) is implied by the third equation since $\lambda^\alpha \lambda^\beta D_\alpha D_\beta W^\gamma = \frac{1}{2} (\lambda \gamma^m \lambda) \partial_m W^\gamma = 0$. It is useful to note that in components,

$$V = a_m(x) \partial x^m + \frac{1}{2} \partial_{[m} a_{n]}(x) M^{mn} + \xi^\alpha(x) q_\alpha + O(\theta^2), \quad (3.26)$$

where $M^{mn} = \frac{1}{2} p \gamma^{mn} \theta + N^{mn}$ is the spin contribution to the Lorentz current and $q_\alpha = p_\alpha + \frac{1}{2} (\partial x^m + \frac{1}{12} \theta \gamma^m \partial \theta) (\gamma_m \theta)_\alpha$ is the spacetime-supersymmetry current. So (3.26) closely resembles the RNS vertex operator [27] for the gluon and gluino. If one drops the $\frac{1}{2} N_{mn} F^{mn}$ term, the vertex operator of (3.23) was suggested by Siegel [4] based on superspace arguments.

For the Type II superstring, the unintegrated massless vertex operator is $U = \lambda^\alpha \hat{\lambda}^{\hat{\beta}} A_{\alpha\hat{\beta}}(x, \theta, \hat{\theta})$ where $\hat{\lambda}^{\hat{\alpha}}$ and $\hat{\theta}^{\hat{\alpha}}$ are right-moving worldsheet fields and the chirality of the $\hat{\alpha}$ index depends if the superstring is IIA or IIB. The physical state condition $QU = \hat{Q}U = 0$ and gauge invariance $\delta U = Q\hat{\Omega} + \hat{Q}\Omega$ where $\hat{Q}\hat{\Omega} = Q\Omega = 0$ implies that

$$\gamma_{mnpqr}^{\alpha\beta} D_\alpha A_{\beta\hat{\gamma}} = \gamma_{mnpqr}^{\hat{\alpha}\hat{\gamma}} \hat{D}_{\hat{\alpha}} A_{\beta\hat{\gamma}} = 0, \quad \delta A_{\alpha\hat{\beta}} = D_\alpha \hat{\Omega}_{\hat{\beta}} + \hat{D}_{\hat{\beta}} \Omega_\alpha, \quad (3.27)$$

$$\gamma_{mnpqr}^{\alpha\beta} D_\alpha \Omega_\beta = \gamma_{mnpqr}^{\hat{\alpha}\hat{\gamma}} \hat{D}_{\hat{\alpha}} \hat{\Omega}_{\hat{\gamma}} = 0$$

for any five-form direction $mnpqr$, which are the linearized equations of motion and gauge invariances of the Type IIA or Type IIB supergravity multiplet. The integrated form of the closed superstring massless vertex operator is the left-right product of the open superstring vertex operator of (3.23) and is given by

$$V_{SG} = \int d^2z \quad (3.28)$$

$$\begin{aligned} & [\partial\theta^\alpha \bar{\partial}\hat{\theta}^{\hat{\beta}} A_{\alpha\hat{\beta}}(x, \theta, \hat{\theta}) + \partial\hat{\theta}^\alpha \bar{\Pi}^m A_{\alpha m}(x, \theta, \hat{\theta}) + \Pi^m \bar{\partial}\hat{\theta}^{\hat{\alpha}} A_{m\hat{\alpha}}(x, \theta, \hat{\theta}) + \Pi^m \bar{\Pi}^n A_{mn}(x, \theta, \hat{\theta}) \\ & + d_\alpha (\bar{\partial}\hat{\theta}^{\hat{\beta}} E_{\hat{\beta}}^\alpha(x, \theta, \hat{\theta}) + \bar{\Pi}^m E_m^\alpha(x, \theta, \hat{\theta})) + \hat{d}_{\hat{\alpha}} (\partial\theta^\beta E_{\hat{\beta}}^{\hat{\alpha}}(x, \theta, \hat{\theta}) + \Pi^m E_m^{\hat{\alpha}}(x, \theta, \hat{\theta})) \\ & + \frac{1}{2} N_{mn} (\bar{\partial}\hat{\theta}^{\hat{\beta}} \Omega_{\hat{\beta}}^{mn}(x, \theta, \hat{\theta}) + \bar{\Pi}^p \Omega_p^{mn}(x, \theta, \hat{\theta})) + \frac{1}{2} \hat{N}_{mn} (\partial\theta^\beta \hat{\Omega}_{\hat{\beta}}^{mn}(x, \theta, \hat{\theta}) + \Pi^p \hat{\Omega}_p^{mn}(x, \theta, \hat{\theta})) \\ & + d_\alpha \hat{d}_{\hat{\beta}} P^{\alpha\hat{\beta}}(x, \theta, \hat{\theta}) + N_{mn} \hat{d}_{\hat{\alpha}} C^{mn\hat{\alpha}}(x, \theta, \hat{\theta}) + d_\alpha \hat{N}_{mn} \hat{C}^{\alpha mn}(x, \theta, \hat{\theta}) + N_{mn} \hat{N}_{pq} S^{mnpq}(x, \theta, \hat{\theta})]. \end{aligned}$$

3.4. Tree-level scattering amplitudes

As usual, the N -point tree-level open superstring scattering amplitude will be defined as the correlation function of 3 unintegrated vertex operators U_r and $N - 3$ integrated vertex operators $\int dz V_r$ as

$$A = \int dz_4 \dots \int dz_N \langle U_1(z_1) U_2(z_2) U_3(z_3) \prod_{r=4}^N V_r(z_r) \rangle. \quad (3.29)$$

For massless external states, the vertex operators are given in (3.18) and (3.23).

The first step to evaluate the correlation function is to eliminate all worldsheet fields of non-zero dimension (i.e. ∂x^m , $\partial\theta^\alpha$, p_α , J and N^{mn}) by using their OPE's with other worldsheet fields and the fact that they vanish at $z \rightarrow \infty$. One then integrates over the x^m zero modes to get a Koba-Nielsen type formula,

$$A = \int dz_4 \dots dz_N \langle \lambda^\alpha \lambda^\beta \lambda^\gamma f_{\alpha\beta\gamma}(z_r, k_r, \eta_r, \theta) \rangle \quad (3.30)$$

where $\lambda^\alpha \lambda^\beta \lambda^\gamma$ comes from the three unintegrated vertex operators and $f_{\alpha\beta\gamma}$ is some function of the z_r 's, the momenta k_r , the polarizations η_r , and the remaining θ zero modes.

One would like to define the correlation function $\langle \lambda^\alpha \lambda^\beta \lambda^\gamma f_{\alpha\beta\gamma} \rangle$ such that A is supersymmetric and gauge invariant. An obvious way to make A supersymmetric is to require

that the correlation function vanishes unless all sixteen θ zero modes are present, but this gives the wrong answer by dimensional analysis. The correct answer comes from realizing that $Y = \lambda^\alpha \lambda^\beta \lambda^\gamma f_{\alpha\beta\gamma}$ satisfies the constraint $QY = 0$ when the external states are on-shell. Furthermore, gauge invariance implies that $\langle Y \rangle$ should vanish whenever $Y = Q\Omega$.

As discussed in subsection (2.5), there is precisely one state in the cohomology of Q at zero momentum and ghost-number three which is $(\lambda\gamma^m\theta)(\lambda\gamma^n\theta)(\lambda\gamma^p\theta)(\theta\gamma_{mnp}\theta)$. So if

$$f_{\alpha\beta\gamma}(\theta) = A_{\alpha\beta\gamma} + \theta^\delta B_{\alpha\beta\gamma\delta} + \dots + (\gamma^m\theta)_\alpha (\gamma^n\theta)_\beta (\gamma^p\theta)_\gamma (\theta\gamma_{mnp}\theta) F + \dots, \quad (3.31)$$

it is natural to define

$$\langle \lambda^\alpha \lambda^\beta \lambda^\gamma f_{\alpha\beta\gamma}(z_r, k_r, \eta_r, \theta) \rangle = F(z_r, k_r, \eta_r). \quad (3.32)$$

This definition is supersymmetric when all external states are on-shell since

$$(\lambda\gamma^m\theta)(\lambda\gamma^n\theta)(\lambda\gamma^p\theta)(\theta\gamma_{mnp}\theta)$$

cannot be written as the supersymmetric variation of a quantity which is annihilated by Q . And the definition is gauge invariant since

$$(\lambda\gamma^m\theta)(\lambda\gamma^n\theta)(\lambda\gamma^p\theta)(\theta\gamma_{mnp}\theta) \neq Q\Omega$$

for any Ω . Note that (3.32) can be interpreted as integration over an on-shell harmonic superspace involving five θ 's since $\langle \lambda^\alpha \lambda^\beta \lambda^\gamma f_{\alpha\beta\gamma} \rangle$ is proportional to

$$\left(\frac{\partial}{\partial\theta}\gamma_m\right)^\alpha \left(\frac{\partial}{\partial\theta}\gamma_n\right)^\beta \left(\frac{\partial}{\partial\theta}\gamma_p\right)^\gamma \left(\frac{\partial}{\partial\theta}\gamma^{mnp}\frac{\partial}{\partial\theta}\right) f_{\alpha\beta\gamma}|_{\theta=0} = \int (d^5\theta)^{\alpha\beta\gamma} f_{\alpha\beta\gamma}. \quad (3.33)$$

For three-point scattering, $A = \langle \lambda^\alpha A_\alpha^1(z_1) \lambda^\beta A_\beta^2(z_2) \lambda^\gamma A_\gamma^3(z_3) \rangle$, it is easy to check that the prescription of (3.32) reproduces the usual super-Yang-Mills cubic vertex. In the gauge of (2.15), each A_α contributes one, two or three θ 's. If the five θ 's are distributed as (1, 1, 3), one gets the $a_m^1 a_n^2 \partial^{[m} a^{3n]}$ vertex for three gluons, whereas if they are distributed as (2, 2, 1), one gets the $(\xi^1 \gamma^m \xi^2) a_m^3$ vertex for two gluinos and one gluon. Together with Brenno Vallilo, it was proven that the above prescription agrees with the standard RNS prescription of [27] for N-point massless tree amplitudes involving up to four fermions [28]. The relation of (3.26) to the RNS massless vertex operator was used in this proof, and the restriction on the number of fermions comes from the need for different pictures in the RNS prescription. Furthermore, using the map from on-shell states in the pure spinor BRST cohomology to on-shell states in the RNS formalism, it was argued in [23] that tree amplitudes involving massive states must also agree with the RNS prescription.

4. Quantization of the Superstring in a Curved Background

Although it is not known how to covariantly quantize the GS superstring, one can construct the classical GS superstring action in a curved background. It has been shown that when the background fields satisfy their on-shell equations to lowest order in α' , the classical worldsheet action is invariant under κ -symmetry. However, because of quantization problems, it is not known how to compute α' corrections to the background equations of motion using the GS formalism.

As will be reviewed here, one can use the pure spinor description to construct an analogous action for the superstring in a curved background. In this case, classical BRST invariance will imply the on-shell equations for the background to lowest order in α' . Since quantization is straightforward using the pure spinor description, one can now compute α' corrections to the background equations by requiring quantum BRST invariance of the action. Note that in the pure spinor description, the equations coming from classical BRST invariance are expected to imply that the action is conformally invariant to one-loop order. Since the one-loop beta function vanishes, it is sensible to ask if there are finite corrections to the background equations coming from one-loop BRST invariance. Similarly, n -loop BRST invariance is expected to imply $(n+1)$ -loop conformal invariance of the action, so this method can in principle be extended to all orders in α' .

4.1. Relation between κ -symmetry and classical BRST invariance

The fact that classical BRST invariance in the pure spinor description is related to κ -symmetry in the GS description can be understood by computing the Poisson brackets of $Q = \int \lambda^\alpha d_\alpha$ with the worldsheet fields. One finds that

$$\delta_Q x^m = \lambda \gamma^m \theta, \quad \delta_Q \theta^\alpha = \lambda^\alpha, \quad \delta_Q d_\alpha = -\Pi^m (\gamma_m \lambda)^\alpha, \quad \delta_Q w_\alpha = d_\alpha, \quad (4.1)$$

which resemble the κ -symmetry transformations

$$\delta x^m = \xi \gamma^m \theta, \quad \delta \theta^\alpha = \xi^\alpha, \quad (4.2)$$

where $\xi^\alpha = \Pi^m (\gamma_m \kappa)^\alpha$. As shown by Oda and Tonin [45], this relation is useful for constructing BRST-invariant actions from κ -invariant GS actions.

If the GS action S_{GS} satisfies $\delta S_{GS} = 0$ under (4.2) up to the Virasoro constraint $\Pi_m \Pi^m = 0$ when $\xi^\alpha = \Pi^m (\gamma_m \kappa)^\alpha$, then when ξ^α is arbitrary, $\delta S_{GS} = \int d^2 z \Pi_m (\xi \gamma^m \Omega)$ for some Ω^α . Since S_{GS} is independent of d_α and w_α , this implies from (4.1) that the BRST

transformation of S_{GS} is $\delta_Q S_{GS} = \int d^2 z \Pi^m (\lambda \gamma^m \Omega)$. One can therefore define a classically BRST-invariant action as

$$\begin{aligned} S_{BRST} &= S_{GS} + \int d^2 z \delta_Q (w_\alpha \Omega^\alpha) \\ &= S_{GS} + \int d^2 z [d_\alpha \Omega^\alpha + w_\alpha \delta_Q \Omega^\alpha]. \end{aligned} \quad (4.3)$$

Although $Q^2 = 0$ naively implies that $\int d^2 z \delta_Q (w_\alpha \Omega^\alpha)$ is BRST invariant by itself, one can check from (4.1) that $\delta_Q \delta_Q w_\alpha = -\Pi^m (\gamma_m \lambda)_\alpha$. Note that such a transformation for w_α is not inconsistent with $Q^2 = 0$ since $\delta w_\alpha = -\Pi^m (\gamma_m \lambda)_\alpha$ is a gauge transformation of the type discussed in (3.17). So

$$\delta_Q \delta_Q (w_\alpha \Omega^\alpha) = (\delta_Q \delta_Q w_\alpha) \Omega^\alpha = -\Pi_m (\lambda \gamma^m \Omega),$$

which implies that S_{BRST} of (4.3) is BRST-invariant.

It can be easily checked that this construction of S_{BRST} agrees with the superparticle and superstring actions constructed using pure spinors. For example, for the heterotic superstring in an on-shell super-Yang-Mills background,

$$S_{GS} = S_{het} + \int d^2 z [\partial \theta^\alpha A_\alpha^I + \Pi^m B_m^I] \bar{J}^I \quad (4.4)$$

where S_{het} is defined in (3.1), $\bar{J}^I$ are the right-moving $E_8 \times E_8$ or $SO(32)$ currents, I is a Lie algebra index, and A_α and B_m satisfy (2.11) and (2.13). One can use (4.2) together with $\delta \bar{J}^I = -ig[A_\alpha, \bar{J}]^I$ to compute that $\Omega^\alpha = \bar{\partial} \theta^\alpha + W^{\alpha I} \bar{J}^I$ where W^α is defined in (2.13). So

$$\begin{aligned} S_{BRST} &= S_{GS} + \int d^2 z \delta_Q (w_\alpha \bar{\partial} \theta^\alpha + w_\alpha W^{\alpha I} \bar{J}^I) \\ &= S_{GS} + \int d^2 z [d_\alpha \bar{\partial} \theta^\alpha + w_\alpha \bar{\partial} \lambda^\alpha + (d_\alpha W^{\alpha I} + w_\alpha \lambda^\beta (\nabla_\beta W^\alpha)^I) \bar{J}^I] \\ &= \int d^2 z [\frac{1}{2} \partial x^m \bar{\partial} x_m + p_\alpha \bar{\partial} \theta^\alpha + w_\alpha \bar{\partial} \lambda^\alpha + (\partial \theta^\alpha A_\alpha^I + \Pi^m B_m^I + d_\alpha W^{\alpha I} + \frac{1}{2} N_{mn} F^{mn I}) \bar{J}^I], \end{aligned} \quad (4.5)$$

which is the pure spinor version of the heterotic superstring action in a super-Yang-Mills background.

4.2. Open superstring and supersymmetric Born-Infeld equations

Over fifteen years ago, it was shown that one-loop conformal invariance of the bosonic open string in an electromagnetic background implies that the background satisfies the Born-Infeld equations, and higher-loop conformal invariance implies higher-derivative corrections to these equations [46]. However, because of problems with describing fermionic backgrounds, this result was generalized only to the bosonic sector of supersymmetric Born-Infeld theory using the Ramond-Neveu-Schwarz formalism of the open superstring [47]. Although fermionic backgrounds can be classically described using the Green-Schwarz formalism of the superstring, quantization problems have prevented computation of the equations implied by one-loop or higher-loop conformal invariance. Nevertheless, it has been argued that κ -symmetry of the classical Green-Schwarz superstring action in an abelian background implies the abelian supersymmetric Born-Infeld equations for the background [48] [49].

Using the pure spinor description of the superstring, physical states are defined using the left and right-moving BRST charges

$$Q = \int d\sigma(\lambda^\alpha d_\alpha) \quad \text{and} \quad \hat{Q} = \int d\sigma(\hat{\lambda}^\alpha \hat{d}_\alpha) \quad (4.6)$$

where d_α and $\hat{d}_\alpha$ are left and right-moving worldsheet variables for the N=2 D=10 supersymmetric derivatives and λ^α and $\hat{\lambda}^\alpha$ are left and right-moving pure spinor variables satisfying

$$\lambda \gamma_{\alpha\beta}^m \lambda^\beta = \hat{\lambda}^\alpha \gamma_{\alpha\beta}^m \hat{\lambda}^\beta = 0 \quad (4.7)$$

for $m = 0$ to 9. As was shown with Vladimir Pershin, classical BRST invariance of the open superstring in a background implies that the background fields satisfy the full non-linear supersymmetric Born-Infeld equations of motion. This was verified by computing the boundary conditions of the open superstring worldsheet variables in the presence of the background and showing that the left and right-moving BRST currents satisfy

$$\lambda^\alpha d_\alpha = \hat{\lambda}^\alpha \hat{d}_\alpha \quad (4.8)$$

on the boundary if and only if the background fields satisfy the supersymmetric Born-Infeld equations of motion. Since $\lambda^\alpha d_\alpha$ is left-moving and $\hat{\lambda}^\alpha \hat{d}_\alpha$ is right-moving, $\frac{\partial}{\partial \tau}(Q + \hat{Q}) = \int d\sigma \frac{\partial}{\partial \sigma}(\lambda^\alpha d_\alpha - \hat{\lambda}^\alpha \hat{d}_\alpha)$. So (4.8) implies that classical BRST invariance is preserved in the presence of the open superstring background. Although similar results can be obtained

using κ -symmetry in the classical Green-Schwarz formalism, this pure spinor approach has the advantage of allowing the computation of higher-derivative corrections through the requirement of quantum BRST invariance.

The first step in computing the equations implied by classical BRST invariance is to determine the appropriate boundary conditions for the open superstring worldsheet variables in the presence of the background. Recall that for the bosonic string in an electromagnetic background, the Neumann boundary conditions $\frac{\partial}{\partial\sigma}x^m = 0$ are modified to

$$\frac{\partial}{\partial\sigma}x^m = F^{mn}\dot{x}^n \quad (4.9)$$

where F^{mn} is the electromagnetic field strength. For the bosonic string, these modified boundary conditions do not affect classical BRST invariance since (4.9) together with $F^{mn} = -F^{nm}$ implies that the left-moving stress-tensor $T = \frac{1}{2}\partial x^m\partial x_m$ remains equal to the right-moving stress-tensor $\hat{T} = \frac{1}{2}\bar{\partial}x^m\bar{\partial}x_m$ on the boundary where $\partial = \frac{\partial}{\partial\tau} + \frac{\partial}{\partial\sigma}$ and $\bar{\partial} = \frac{\partial}{\partial\tau} - \frac{\partial}{\partial\sigma}$. So by defining the left and right-moving reparameterization ghosts to satisfy $c = \hat{c}$ and $b = \hat{b}$ on the boundary, one is guaranteed that the left and right-moving BRST currents coincide on the boundary in the presence of the background.

However, for the superstring using the pure spinor formalism, the boundary conditions on the worldsheet variables in the presence of a background do not automatically imply that the left and right-moving BRST currents coincide on the boundary. As will be reviewed here, $\lambda^\alpha d_\alpha = \hat{\lambda}^\alpha \hat{d}_\alpha$ on the boundary if and only if the background superfields satisfy the supersymmetric Born-Infeld equations of motion.

In a background, the open superstring action using the pure spinor description is $S = S_0 + V$ where

$$S_0 = -\frac{1}{\alpha'} \int d\tau d\sigma \left\{ \frac{1}{2} \partial x^m \bar{\partial} x_m + p_\alpha \bar{\partial} \theta^\alpha + \hat{p}_\alpha \partial \hat{\theta}^\alpha + w_\alpha \bar{\partial} \lambda^\alpha + \hat{w}_\alpha \partial \hat{\lambda}^\alpha \right\} \quad (4.10)$$

is the action in a flat background and V is the super-Maxwell integrated vertex operator defined in (3.23). Before computing the boundary conditions on the worldsheet variables in the presence of V , it is convenient to add a surface term S_b to the action such that $S = S_0 + S_b$ is manifestly invariant under the N=1 D=10 supersymmetry transformations

$$\delta\theta_+^\alpha = \epsilon^\alpha, \quad \delta x^m = \frac{1}{2}\theta_+ \gamma^m \epsilon, \quad \delta\theta_-^\alpha = 0, \quad (4.11)$$

where $\theta_\pm^\alpha = \frac{1}{\sqrt{2}}(\theta^\alpha \pm \hat{\theta}^\alpha)$. Note that although S_0 is invariant under (4.11) using the flat boundary conditions $\theta_-^\alpha = \partial_\sigma x^m = 0$, it is not invariant under (4.11) for more general

boundary conditions. However, it was shown in [24] that by choosing S_b appropriately, one can make $S = S_0 + S_b$ invariant under (4.11) for arbitrary boundary conditions. Furthermore, it is convenient to modify the vertex operator V to

$$V = \dot{\theta}_+^\alpha A_\alpha(x, \theta_+) + \Pi_+^m B_m(x, \theta_+) + d_+^\alpha W^\alpha(x, \theta_+) + \frac{1}{2} N_+^{mn} F_{mn}(x, \theta^+) \quad (4.12)$$

where the $+/-$ index denotes the sum/difference of left and right-moving worldsheet variables. With this modification of V , the background superfields transform covariantly under the N=1 D=10 supersymmetry transformations of (4.11).

As was shown in [24], cancellation of the surface term equations of motion implies that the flat boundary conditions

$$\theta_-^\alpha = \Pi_-^m = d_-^\alpha = \lambda_-^\alpha = w_-^\alpha = 0 \quad (4.13)$$

are modified in the presence of V to

$$\theta_-^\alpha = -W^\alpha(x, \theta_+), \quad (4.14)$$

$$\begin{aligned} \Pi_-^m &= \dot{\theta}_+^\alpha (\partial^m A_\alpha - D_\alpha B^m + \gamma_{\alpha\beta}^m W^\beta + \frac{1}{6} \gamma_{\alpha\beta}^n \gamma_{n\gamma\delta} W^\beta W^\gamma \partial^m W^\delta) \\ &\quad + \Pi_+^n (\partial^m B_n - \partial_n B^m) + d_+^\alpha \partial^m W^\alpha + \frac{1}{2} N_+^{nk} \partial^m F_{nk}, \\ \sqrt{2} d_-^\alpha &= \dot{\theta}_+^\beta (D_\alpha A_\beta + D_\beta A_\alpha - \gamma_{\alpha\beta}^m B_m + \frac{1}{6} \gamma_{\alpha\gamma}^n \gamma_{n\delta\lambda} W^\gamma W^\delta D_\beta W^\lambda \\ &\quad + \frac{1}{6} \gamma_{\beta\gamma}^n \gamma_{n\delta\lambda} W^\gamma W^\delta D_\alpha W^\lambda) \\ &\quad + \Pi_+^m (\partial_m A_\alpha - D_\alpha B_m + \gamma_{m\alpha\beta} W^\beta + \frac{1}{6} \gamma_{\alpha\beta}^n \gamma_{n\gamma\delta} W^\beta W^\gamma \partial_m W^\delta) \\ &\quad + d_+^\gamma D_\alpha W^\gamma - \frac{1}{2} N_+^{mn} D_\alpha F_{mn}, \\ \lambda_-^\alpha &= -\frac{1}{4} \lambda_+^b (\gamma_{mn})_\beta{}^\alpha F^{mn}, \quad w_-^\alpha = \frac{1}{4} F^{mn} (\gamma_{mn})_\alpha{}^\beta w_+^\beta. \end{aligned} \quad (4.15)$$

Using the boundary conditions of (4.14) and (4.15), the difference between the left and right-moving BRST currents on the boundary is

$$\begin{aligned} 2(\lambda^\alpha d_\alpha - \hat{\lambda}^\alpha \hat{d}_\alpha) &= \\ \lambda_+^\alpha \dot{\theta}_+^\beta &\left[D_\alpha A_\beta + D_\beta A_\alpha - \gamma_{\alpha\beta}^m B_m + \frac{1}{6} \gamma_{\alpha\gamma}^m \gamma_{m\delta\lambda} W^\gamma W^\delta D_\beta W^\lambda + \frac{1}{6} \gamma_{\beta\gamma}^m \gamma_{m\delta\lambda} W^\gamma W^\delta D_\alpha W^\lambda \right. \end{aligned}$$

$$\begin{aligned}
& + \frac{1}{8}(\gamma F)_\alpha{}^\kappa \gamma_{\kappa\lambda}^m W^\lambda (\partial_m A_\beta - D_\beta B_m + \gamma_{m\beta\gamma} W^\gamma + \frac{1}{6} \gamma_{\beta\sigma}^n \gamma_{n\gamma\delta} W^\sigma W^\gamma \partial_m W^\delta) \Big] \\
& + \lambda_+^\alpha \Pi_+^m \left[\partial_m A_\alpha - D_\alpha B_m + \gamma_{m\alpha\beta} W^\beta + \frac{1}{6} \gamma_{\alpha\beta}^n \gamma_{n\gamma\delta} W^\beta W^\gamma \partial_m W^\delta \right. \\
& \quad \left. - \frac{1}{8} (\gamma F)_\alpha{}^\beta \gamma_{\beta\gamma}^n W^\gamma (\partial_n B_m - \partial_m B_n) \right] \\
& + \lambda_+^\alpha d_\gamma^+ \left[D_\alpha W^\gamma - \frac{1}{4} (\gamma F)_\alpha{}^\gamma + \frac{1}{8} (\gamma F)_\alpha{}^\beta \gamma_{\beta\lambda}^n W^\lambda \partial_n W^\gamma \right] \\
& - \frac{1}{2} \lambda_+^\alpha N_+^{mn} \left[D_\alpha F_{mn} + \frac{1}{8} (\gamma F)_\alpha{}^\beta \gamma_{\beta\lambda}^k W^\lambda \partial_k F_{mn} \right], \tag{4.16}
\end{aligned}$$

where $(\gamma F)_\beta{}^\alpha = F_{mn}(\gamma^{mn})_\beta{}^\alpha$.

Requiring this to be zero implies the equations:

$$\begin{aligned}
& D_\alpha A_\beta + D_\beta A_\alpha - \gamma_{\alpha\beta}^m B_m + \frac{1}{6} \gamma_{\alpha\gamma}^m \gamma_{\delta\lambda} W^\gamma W^\delta D_\beta W^\lambda + \frac{1}{6} \gamma_{\beta\gamma}^m \gamma_{m\delta\lambda} W^\gamma W^\delta D_\alpha W^\lambda \\
& + \frac{1}{64} (\gamma F)_\alpha{}^\gamma (\gamma F)_\beta{}^\delta \gamma_{\gamma\lambda}^n \gamma_{\delta\sigma}^m W^\lambda W^\sigma (\partial_m B_n - \partial_n B_m) = 0, \tag{4.17}
\end{aligned}$$

$$\begin{aligned}
& \partial_m A_\alpha - D_\alpha B_m + \gamma_{m\alpha\beta} W^\beta + \frac{1}{6} \gamma_{\alpha\beta}^n \gamma_{n\gamma\delta} W^\beta W^\gamma \partial_m W^\delta \\
& - \frac{1}{8} (\gamma F)_\alpha{}^\beta \gamma_{\beta\lambda}^n W^\lambda (\partial_n B_m - \partial_m B_n) = 0, \tag{4.18}
\end{aligned}$$

$$D_\alpha W^\gamma - \frac{1}{4} (\gamma F)_\alpha{}^\gamma + \frac{1}{8} (\gamma F)_\alpha{}^\beta \gamma_{\beta\lambda}^n W^\lambda \partial_n W^\gamma = 0, \tag{4.19}$$

$$\lambda_+^\alpha \lambda_+^\beta (\gamma^{mn})_\beta{}^\gamma \left[D_\alpha F_{mn} + \frac{1}{8} (\gamma F)_\alpha{}^\beta \gamma_{\beta\lambda}^k W^\lambda \partial_k F_{mn} \right] = 0. \tag{4.20}$$

As in the super-Maxwell equations of (3.25), the contraction of (4.17) with $\gamma_{mnpqr}^{\alpha\beta}$ implies the equations of motion for A_α , the contraction of (4.17) with $\gamma_m^{\alpha\beta}$ defines B_m , the contraction of (4.18) with $\gamma^{m\alpha\gamma}$ defines W^γ , the contraction of (4.19) with $(\gamma_{rs})_\beta{}^\alpha$ defines F_{rs} , and the remaining contractions of (4.18) and (4.19) are implied by these equations through Bianchi identities. Note that because of the non-linear terms in (4.17)-(4.19), W^γ and F_{mn} are now complicated functions of the spinor and vector field strengths constructed from the gauge fields A_α and B_m .

Finally, equation (4.20) vanishes as a consequence of (4.19) and the pure spinor property

$$\lambda_+ \gamma^m \lambda_+ + \frac{1}{16} (\gamma F)_\gamma{}^\alpha (\gamma F)_\delta{}^\beta \gamma_{\alpha\beta}^m \lambda_+^\gamma \lambda_+^\delta = \lambda_+ \gamma^m \lambda_+ + \lambda_- \gamma^m \lambda_- = \lambda \gamma^m \lambda + \hat{\lambda} \gamma^m \hat{\lambda} = 0. \tag{4.21}$$

To show that (4.20) vanishes, it is useful to write (4.19) and (4.20) as $\hat{D}_\alpha W^\gamma = \frac{1}{4}(\gamma F)_\alpha{}^\gamma$ and $\lambda_+^\alpha \lambda_+^\beta \hat{D}_\alpha \hat{D}_\beta W^\gamma = 0$ where

$$\hat{D}_\alpha = D_\alpha + \frac{1}{2} D_\alpha W^\gamma \left(\delta_\beta^\gamma - \frac{1}{2} \gamma_{\beta\lambda}^n W^\lambda \partial_n W^\gamma \right)^{-1} (\gamma^r W)_\beta \partial_r. \quad (4.22)$$

One can check that

$$\{\hat{D}_\alpha, \hat{D}_\beta\} = (\gamma_{\alpha\beta}^m + \frac{1}{16} (\gamma F)_\alpha{}^\gamma (\gamma F)_\beta{}^\delta \gamma_{\gamma\delta}^m) \hat{\partial}_m \quad (4.23)$$

where

$$\hat{\partial}_m = \partial_m + \frac{1}{2} \partial_m W^\gamma \left(\delta_\beta^\gamma - \frac{1}{2} \gamma_{\beta\lambda}^n W^\lambda \partial_n W^\gamma \right)^{-1} (\gamma^r W)_\beta \partial_r, \quad (4.24)$$

so (4.21) implies that $\lambda_+^\alpha \lambda_+^\beta \hat{D}_\alpha \hat{D}_\beta W^\gamma = 0$.

To prove that equations (4.17)- (4.19) are the abelian supersymmetric Born-Infeld equations, it was shown in [24] that they are invariant under N=2 D=10 supersymmetry where the second supersymmetry acts non-linearly on the superfields. Except for factors of i coming from different conventions for the supersymmetry algebra, equations (4.17)-(4.19) are easily shown to coincide with the superspace Born-Infeld equations (33)-(35) of reference [49] which were independently derived using the superembedding method [48].

4.3. Closed superstring and Type II supergravity equations

In a curved background, the classical GS superstring action can be written as

$$S = \frac{1}{4\pi\alpha'} \int d^2z (G_{MN}(Z) + B_{MN}(Z)) \partial Z^M \bar{\partial} Z^N \quad (4.25)$$

where $M = [m, \mu, \hat{\mu}]$ are curved N=2 D=10 superspace indices, $Z^M = [x^m, \theta^\mu, \hat{\theta}^{\hat{\mu}}]$, μ and $\hat{\mu}$ denote SO(9,1) spinors of opposite chirality for the Type IIA superstring and of the same chirality for the Type IIB superstring, and G_{MN} and B_{MN} describe the background superfields. When the background fields satisfy the Type II supergravity equations of motion, the action of (4.25) is invariant under κ -symmetry. However, because of quantization problems, it is not known how to use the action of (4.25) to compute α' corrections to the supergravity equations. This is an important question since it is not yet understood how the superspace structure of Type II supergravity equations is modified by these α' corrections.

As will be reviewed in this subsection, an analogous action can be constructed using the pure spinor description of the Type II superstring in a curved background. As was

shown with Paul Howe in [25], classical BRST invariance of this action implies the Type II supergravity equations and quantum BRST invariance is expected to imply α' corrections to these equations. Except for the Fradkin-Tseytlin term which couples the dilaton to worldsheet curvature, the Type II sigma model action in a curved background can be constructed by adding the massless integrated closed superstring vertex operator of (3.28) to the flat action of (4.10), and then covariantizing with respect to N=2 D=10 super-reparameterization invariance. Alternatively, one can consider the most general action constructed from the closed superstring worldsheet variables which is classically invariant under worldsheet conformal transformations.

Using the worldsheet variables of the previous subsection, the Type II sigma model action is defined as

$$\begin{aligned}
S = \frac{1}{2\pi\alpha'} \int d^2z & \left[\frac{1}{2} (G_{MN}(Z) + B_{MN}(Z)) \partial Z^M \bar{\partial} Z^N + P^{\alpha\hat{\beta}}(Z) d_\alpha \hat{d}_{\hat{\beta}} \right. \\
& + E_M^\alpha(Z) d_\alpha \bar{\partial} Z^M + E_M^{\hat{\alpha}}(Z) \hat{d}_{\hat{\alpha}} \partial Z^M + \Omega_{M\alpha}{}^\beta(Z) \lambda^\alpha w_\beta \bar{\partial} Z^M + \hat{\Omega}_{M\hat{\alpha}}{}^{\hat{\beta}}(Z) \hat{\lambda}^{\hat{\alpha}} \hat{w}_{\hat{\beta}} \partial Z^M \\
& \left. + C_\alpha^{\beta\hat{\gamma}}(Z) \lambda^\alpha w_\beta \hat{d}_{\hat{\gamma}} + \hat{C}_{\hat{\alpha}}^{\hat{\beta}\gamma}(Z) \hat{\lambda}^{\hat{\alpha}} \hat{w}_{\hat{\beta}} d_\gamma + S_{\alpha\hat{\gamma}}^{\beta\hat{\delta}}(Z) \lambda^\alpha w_\beta \hat{\lambda}^{\hat{\gamma}} \hat{w}_{\hat{\delta}} + \frac{1}{2} \alpha' \Phi(Z) r \right] + S_\lambda + S_{\hat{\lambda}}
\end{aligned} \tag{4.26}$$

where $M = (m, \mu, \hat{\mu})$ are curved superspace indices, $Z^M = (x^m, \theta^\mu, \hat{\theta}^{\hat{\mu}})$, $A = (a, \alpha, \hat{\alpha})$ are tangent superspace indices, S_λ and $S_{\hat{\lambda}}$ are the flat actions for the pure spinor variables, r is the worldsheet curvature, and $[G_{MN} = \eta_{cd} E_M^c E_N^d, B_{MN}, E_M^\alpha, E_M^{\hat{\alpha}}, \Omega_{M\alpha}{}^\beta, \hat{\Omega}_{M\hat{\alpha}}{}^{\hat{\beta}}, P^{\alpha\hat{\beta}}, C_\alpha^{\beta\hat{\gamma}}, \hat{C}_{\hat{\alpha}}^{\hat{\beta}\gamma}, S_{\alpha\hat{\gamma}}^{\beta\hat{\delta}}, \Phi]$ are the background superfields. Note that d_α and $\hat{d}_{\hat{\alpha}}$ can be treated as independent variables in (4.26) since p_α and $\hat{p}_{\hat{\alpha}}$ do not appear explicitly.

If the Fradkin-Tseytlin term $\int d^2z \Phi(Z) r$ is omitted, (4.26) is the most general action with classical worldsheet conformal invariance and zero (left,right)-moving ghost number which can be constructed from the Type II worldsheet variables. Note that d_α carries conformal weight $(1, 0)$, $\hat{d}_{\hat{\alpha}}$ carries conformal weight $(0, 1)$, λ^α carries ghost number $(1, 0)$ and conformal weight $(0, 0)$, $\hat{\lambda}^{\hat{\alpha}}$ carries ghost number $(0, 1)$ and conformal weight $(0, 0)$, w_α carries ghost number $(-1, 0)$ and conformal weight $(1, 0)$, and $\hat{w}_{\hat{\alpha}}$ carries ghost number $(0, -1)$ and conformal weight $(0, 1)$. Since w_α and $\hat{w}_{\hat{\alpha}}$ can only appear in combinations which commute with the pure spinor constraints, the background superfields must satisfy

$$(\gamma^{bcde})_\beta^\alpha \Omega_{M\alpha}{}^\beta = (\gamma^{bcde})_\beta^\alpha \hat{\Omega}_{M\hat{\alpha}}{}^{\hat{\beta}} = 0, \tag{4.27}$$

$$(\gamma^{bcde})_\beta^\alpha C_\alpha^{\beta\hat{\gamma}} = (\gamma^{bcde})_\beta^{\hat{\alpha}} \hat{C}_{\hat{\alpha}}^{\hat{\beta}\gamma} = (\gamma^{bcde})_\beta^\alpha S_{\alpha\hat{\gamma}}^{\beta\hat{\delta}} = (\gamma^{bcde})_{\hat{\delta}}^{\hat{\gamma}} S_{\alpha\hat{\gamma}}^{\beta\hat{\delta}} = 0,$$

and the different components of the spin connections will be defined as

$$\Omega_{M\alpha}{}^\beta = \Omega_M^{(s)} \delta_\alpha^\beta + \frac{1}{2} \Omega_M^{cd} (\gamma_{cd})_\alpha{}^\beta, \quad \hat{\Omega}_{M\hat{\alpha}}{}^{\hat{\beta}} = \hat{\Omega}_M^{(s)} \delta_{\hat{\alpha}}^{\hat{\beta}} + \frac{1}{2} \hat{\Omega}_M^{cd} (\gamma_{cd})_{\hat{\alpha}}{}^{\hat{\beta}}. \quad (4.28)$$

Although the background superfields appearing in (4.26) look unconventional, they all have physical interpretations. The superfields E_M^A , B_{MN} and Φ are the supervielbein, two-form potential and dilaton superfields, $P^{\alpha\hat{\beta}}$ is the superfield whose lowest components are the Type II Ramond-Ramond field strengths, and the superfields $C_\alpha^{\beta\hat{\gamma}} = C^\gamma \delta_\alpha^\beta + \frac{1}{2} C^{\gamma ab} (\gamma_{ab})_\alpha{}^\beta$ and $\hat{C}_{\hat{\alpha}}^{\hat{\beta}\hat{\gamma}} = \hat{C}^\gamma \delta_{\hat{\alpha}}^{\hat{\beta}} + \frac{1}{2} \hat{C}^{\gamma ab} (\gamma_{ab})_{\hat{\alpha}}{}^{\hat{\beta}}$ are related to the N=2 D=10 dilatino and gravitino field strengths. Unlike the GS sigma model of (4.25) where the spinor supervielbein is absent, the action of (4.26) contains E_M^α and $E_m^{\hat{\alpha}}$. This means that the action is invariant under two sets of local Lorentz and scale transformations which act independently on the unhatted and hatted spinor indices. One therefore has two independent sets of spin connections and scale connections, $(\Omega_M^{(s)}, \Omega_M^{ab})$ and $(\hat{\Omega}_M^{(s)}, \hat{\Omega}_M^{ab})$, which appear explicitly in the Type II sigma model action. Under the two types of local Lorentz and scale transformations,

$$\delta E_M^\alpha = \Sigma_\beta^\alpha E_M^\beta, \quad \delta E_M^{\hat{\alpha}} = \hat{\Sigma}_{\hat{\beta}}^{\hat{\alpha}} E_M^{\hat{\beta}}, \quad \delta d_\alpha = -\Sigma_\alpha^\beta d_\beta, \quad \delta \hat{d}_{\hat{\alpha}} = -\hat{\Sigma}_{\hat{\alpha}}^{\hat{\beta}} \hat{d}_{\hat{\beta}}, \quad (4.29)$$

$$\delta \Omega_{M\alpha}{}^\beta = \partial_M \Sigma_\alpha^\beta + \Sigma_\alpha^\gamma \Omega_{M\gamma}{}^\beta - \Sigma_\gamma^\beta \Omega_{M\alpha}{}^\gamma, \quad \delta \hat{\Omega}_{M\hat{\alpha}}{}^{\hat{\beta}} = \partial_M \hat{\Sigma}_{\hat{\alpha}}^{\hat{\beta}} + \hat{\Sigma}_{\hat{\alpha}}^{\hat{\gamma}} \hat{\Omega}_{M\hat{\gamma}}{}^{\hat{\beta}} - \hat{\Sigma}_{\hat{\gamma}}^{\hat{\beta}} \hat{\Omega}_{M\hat{\alpha}}{}^{\hat{\gamma}},$$

$$\delta \lambda^\alpha = \Sigma_\gamma^\alpha \lambda^\gamma, \quad \delta w_\alpha = -\Sigma_\alpha^\gamma w_\gamma, \quad \delta \hat{\lambda}^{\hat{\alpha}} = \hat{\Sigma}_{\hat{\gamma}}^{\hat{\alpha}} \hat{\lambda}^{\hat{\gamma}}, \quad \delta \hat{w}_{\hat{\alpha}} = -\hat{\Sigma}_{\hat{\alpha}}^{\hat{\gamma}} \hat{w}_{\hat{\gamma}},$$

where $\Sigma_\alpha^\beta = \Sigma^{(s)} \delta_\alpha^\beta + \frac{1}{2} \Sigma^{bc} (\gamma_{bc})_\alpha{}^\beta$, $\hat{\Sigma}_{\hat{\alpha}}^{\hat{\beta}} = \hat{\Sigma}^{(s)} \delta_{\hat{\alpha}}^{\hat{\beta}} + \frac{1}{2} \hat{\Sigma}^{bc} (\gamma_{bc})_{\hat{\alpha}}{}^{\hat{\beta}}$, Σ^{bc} and $\hat{\Sigma}^{bc}$ parameterize independent local Lorentz transformations on the unhatted and hatted spinor indices, $\Sigma^{(s)}$ and $\hat{\Sigma}^{(s)}$ parameterize independent local scale transformations on the unhatted and hatted spinor indices, and the background superfields $[P^{\alpha\hat{\alpha}}, C_\alpha^{\beta\hat{\gamma}}, \hat{C}_{\hat{\alpha}}^{\hat{\beta}\hat{\gamma}}, S_{\alpha\hat{\gamma}}^{\beta\hat{\delta}}]$ transform according to their spinor indices.

Finally, the background superfields $S_{\alpha\hat{\gamma}}^{\beta\hat{\delta}}$ appearing in (4.26) are related to curvatures constructed from the spin and scale connections. Note that a similar relation occurs in the Type II RNS sigma model action which contains the terms

$$\frac{1}{4\pi\alpha'} \int d^2 z (\Omega_m^{ab}(x) \psi_a \psi_b \bar{\partial} x^m + \hat{\Omega}_m^{ab}(x) \bar{\psi}_a \bar{\psi}_b \partial x^m + S_{abcd}(x) \psi^a \psi^b \bar{\psi}^c \bar{\psi}^d) \quad (4.30)$$

where $\psi^a = e_m^a(x) \psi^m$, $\bar{\psi}^a = e_m^a(x) \bar{\psi}^m$, and $e_m^a(x)$ is the target-space vielbein.

It is important to note that the Fradkin-Tseytlin term $\int d^2 z \Phi(Z) r$ is absent from the GS action of (4.25) since it breaks κ -symmetry. However, as was argued in [25], this term

is necessary in the pure spinor description in order to preserve quantum BRST invariance and conformal invariance. The presence of this term can also be justified by the coupling constant dependence $e^{(2g-2)\phi}$ of genus g scattering amplitudes.

As was shown in [25], classical BRST invariance of (4.26) implies that the background superfields satisfy the Type II supergravity equations. For the action of (4.26) to be BRST invariant, it is necessary that the BRST currents are nilpotent and holomorphic, i.e. that $\{Q, Q\} = \{\hat{Q}, \hat{Q}\} = \{Q, \hat{Q}\} = 0$ and that $\bar{\partial}(\lambda^\alpha d_\alpha) = \partial(\hat{\lambda}^{\hat{\alpha}} \hat{d}_{\hat{\alpha}}) = 0$.

To analyze the conditions implied by nilpotency, it is convenient to use the canonical momenta $P_M = \partial L / \partial (\partial_0 Z^M)$ to write

$$d_\alpha = E_\alpha^M [P_M + \frac{1}{2} B_{MN} (\partial Z^N - \bar{\partial} Z^N) - \Omega_{M\beta}{}^\gamma \lambda^\beta w_\gamma - \hat{\Omega}_{M\hat{\beta}}{}^{\hat{\gamma}} \hat{\lambda}^{\hat{\beta}} \hat{w}_{\hat{\gamma}}], \quad (4.31)$$

$$\hat{d}_{\hat{\alpha}} = E_{\hat{\alpha}}^M [P_M + \frac{1}{2} B_{MN} (\partial Z^N - \bar{\partial} Z^N) - \Omega_{M\beta}{}^\gamma \lambda^\beta w_\gamma - \hat{\Omega}_{M\hat{\beta}}{}^{\hat{\gamma}} \hat{\lambda}^{\hat{\beta}} \hat{w}_{\hat{\gamma}}].$$

Using the canonical commutation relations

$$[P_M, Z^N] = -i\delta_M^N, \quad [w_\alpha, \lambda^\beta] = -i\delta_\alpha^\beta, \quad [\hat{w}_{\hat{\alpha}}, \hat{\lambda}^{\hat{\beta}}] = -i\delta_{\hat{\alpha}}^{\hat{\beta}},$$

one finds that

$$\{Q, Q\} = \oint \lambda^\alpha \lambda^\beta [T_{\alpha\beta}{}^C D_C + \frac{1}{2} (\partial Z^N - \bar{\partial} Z^N) H_{\alpha\beta N} - R_{\alpha\beta\gamma}{}^\delta \lambda^\gamma w_\delta - \hat{R}_{\alpha\beta\hat{\gamma}}{}^{\hat{\delta}} \hat{\lambda}^{\hat{\gamma}} \hat{w}_{\hat{\delta}}],$$

$$\{\hat{Q}, \hat{Q}\} = \oint \hat{\lambda}^{\hat{\alpha}} \hat{\lambda}^{\hat{\beta}} [T_{\hat{\alpha}\hat{\beta}}{}^C D_C + \frac{1}{2} (\partial Z^N - \bar{\partial} Z^N) H_{\hat{\alpha}\hat{\beta} N} - R_{\hat{\alpha}\hat{\beta}\gamma}{}^\delta \lambda^\gamma w_\delta - \hat{R}_{\hat{\alpha}\hat{\beta}\hat{\gamma}}{}^{\hat{\delta}} \hat{\lambda}^{\hat{\gamma}} \hat{w}_{\hat{\delta}}],$$

$$\{Q, \hat{Q}\} = \oint \lambda^\alpha \hat{\lambda}^{\hat{\beta}} [T_{\alpha\hat{\beta}}{}^C D_C + \frac{1}{2} (\partial Z^N - \bar{\partial} Z^N) H_{\alpha\hat{\beta} N} - R_{\alpha\hat{\beta}\gamma}{}^\delta \lambda^\gamma w_\delta - \hat{R}_{\alpha\hat{\beta}\hat{\gamma}}{}^{\hat{\delta}} \hat{\lambda}^{\hat{\gamma}} \hat{w}_{\hat{\delta}}],$$

where $D_C = E_C^M (P_M - \Omega_{M\alpha}{}^\beta \lambda^\alpha w_\beta - \hat{\Omega}_{M\hat{\alpha}}{}^{\hat{\beta}} \hat{\lambda}^{\hat{\alpha}} \hat{w}_{\hat{\beta}})$, $T_{AB}{}^\alpha$ and $R_{AB\beta}{}^\gamma$ are defined using the $\Omega_{M\beta}{}^\gamma$ spin connection, and $T_{AB}{}^{\hat{\alpha}}$ and $\hat{R}_{AB\hat{\beta}}{}^{\hat{\gamma}}$ are defined using the $\hat{\Omega}_{M\hat{\beta}}{}^{\hat{\gamma}}$ spin connection.

So nilpotency of Q and $\hat{Q}$ implies that

$$\lambda^\alpha \lambda^\beta T_{\alpha\beta}{}^C = \lambda^\alpha \lambda^\beta H_{\alpha\beta B} = \lambda^\alpha \lambda^\beta \hat{R}_{\alpha\beta\hat{\gamma}}{}^{\hat{\delta}} = \lambda^\alpha \lambda^\beta \lambda^\gamma R_{\alpha\beta\gamma}{}^\delta = 0, \quad (4.32)$$

$$\hat{\lambda}^{\hat{\alpha}} \hat{\lambda}^{\hat{\beta}} T_{\hat{\alpha}\hat{\beta}}{}^C = \hat{\lambda}^{\hat{\alpha}} \hat{\lambda}^{\hat{\beta}} H_{\hat{\alpha}\hat{\beta} B} = \hat{\lambda}^{\hat{\alpha}} \hat{\lambda}^{\hat{\beta}} \hat{R}_{\hat{\alpha}\hat{\beta}\gamma}{}^\delta = \hat{\lambda}^{\hat{\alpha}} \hat{\lambda}^{\hat{\beta}} \hat{\lambda}^{\hat{\gamma}} \hat{R}_{\hat{\alpha}\hat{\beta}\hat{\gamma}}{}^{\hat{\delta}} = 0,$$

$$\lambda^\alpha \hat{\lambda}^{\hat{\beta}} T_{\alpha\hat{\beta}}{}^C = \lambda^\alpha \hat{\lambda}^{\hat{\beta}} H_{\alpha\hat{\beta} B} = \lambda^\alpha \lambda^\beta R_{\alpha\hat{\gamma}\beta}{}^\delta = \hat{\lambda}^{\hat{\alpha}} \hat{\lambda}^{\hat{\beta}} \hat{R}_{\gamma\hat{\alpha}\hat{\beta}}{}^{\hat{\delta}} = 0,$$

for any pure spinors λ^α and $\hat{\lambda}^{\hat{\alpha}}$. One can easily check that the nilpotency constraints on $R_{ABC}{}^D$ in (4.32) are implied through Bianchi identities by the nilpotency constraints on

$T_{AB}{}^C$. Since λ^α and $\hat{\lambda}^{\hat{\alpha}}$ are independent pure spinors, the remaining constraints imply that

$$\begin{aligned} (\gamma_{mnpqr})^{\alpha\beta} T_{\alpha\beta}{}^C &= (\gamma_{mnpqr})^{\hat{\alpha}\hat{\beta}} T_{\hat{\alpha}\hat{\beta}}{}^C = T_{\alpha\hat{\beta}}{}^C = 0, \\ (\gamma_{mnpqr})^{\alpha\beta} H_{\alpha\beta C} &= (\gamma_{mnpqr})^{\hat{\alpha}\hat{\beta}} H_{\hat{\alpha}\hat{\beta} C} = H_{\alpha\hat{\beta} C} = 0 \end{aligned} \quad (4.33)$$

for any self-dual five-form direction $mnpqr$.

As was shown in [25], the constraints of (4.33) can be interpreted as Type II pure spinor integrability conditions and imply all the essential Type II supergravity constraints. Furthermore, it was shown in [25] that the remaining conventional Type II supergravity constraints are implied by the holomorphicity conditions that $\bar{\partial}(\lambda^\alpha d_\alpha) = \partial(\hat{\lambda}^{\hat{\alpha}} \hat{d}_{\hat{\alpha}}) = 0$.

4.4. Superstring in $AdS_5 \times S^5$ background and Penrose limit

In this subsection, a quantizable action will be constructed for the superstring in an $AdS_5 \times S^5$ background with Ramond-Ramond flux [3][50] and its Penrose limit [51]. Since the action is quantizable, one can in principle compute vertex operators and scattering amplitudes in this background which would be very useful for testing the Maldacena conjecture. However, because of the complicated form of the action, only the simplest vertex operators [52] [50] and scattering amplitudes [53] have so far been computed. Nevertheless, it has been proven that the action in an $AdS_5 \times S^5$ background is conformally invariant up to one-loop order [54] [55], and that the action for the Penrose limit plane wave background is exactly conformally invariant [51].

The action in these backgrounds can be obtained by either plugging in the appropriate background fields into the Type IIB sigma model action of (4.26) or by requiring that the sigma model has the desired target-space isometries and is BRST invariant. Except for the contribution of the pure spinor ghosts, the $AdS_5 \times S^5$ action is a direct generalization of the $AdS_3 \times S^3$ and $AdS_2 \times S^2$ actions which were constructed with the collaboration of Cumrun Vafa and Edward Witten in [56], and with the collaboration of Michael Bershadsky, Tamas Hauer, Slava Zhukov and Barton Zwiebach in [54].

In either the $AdS_5 \times S^5$ background with R-R flux or its corresponding plane wave limit, the worldsheet action using the pure spinor description is

$$\mathcal{S} = \mathcal{S}_{GS} + \int d^2z (d_\alpha \bar{L}^\alpha + \hat{d}_{\hat{\alpha}} L^{\hat{\alpha}} - \frac{1}{2} d_\alpha \hat{d}_{\hat{\beta}} F^{\alpha\hat{\beta}}) + \mathcal{S}_{ghost} \quad (4.34)$$

where $F^{\alpha\hat{\beta}} = \frac{1}{120} F^{m_1 \dots m_5} (\gamma_{m_1 \dots m_5})^{\alpha\hat{\beta}}$ is the constant five-form self-dual Ramond-Ramond flux. For the $AdS_5 \times S^5$ background, $F^{\alpha\hat{\beta}}$ is an invertible 16×16 matrix, whereas for its Penrose limit, $F^{\alpha\hat{\beta}}$ is not invertible and has rank 8.

The first term S_{GS} in (4.34) is the standard covariant GS action

$$S_{GS} = \int d^2z \left[\frac{1}{2} \eta_{mn} L^m \bar{L}^n + \int dy \epsilon^{IJK} (\gamma_{m\alpha\beta} L_I^m L_J^\alpha L_K^\beta + \gamma_{m\hat{\alpha}\hat{\beta}} L_I^m L_J^{\hat{\alpha}} L_K^{\hat{\beta}}) \right] \quad (4.35)$$

where L^M and $\bar{L}^M$ are defined using the Metsaev-Tseytlin currents [57][58]

$$G^{-1} \partial G = P_m L^m + Q_\alpha L^\alpha + Q_{\hat{\alpha}} L^{\hat{\alpha}} + \frac{1}{2} J_{mn} L^{mn}, \quad (4.36)$$

$$G^{-1} \bar{\partial} G = P_m \bar{L}^m + Q_\alpha \bar{L}^\alpha + Q_{\hat{\alpha}} \bar{L}^{\hat{\alpha}} + \frac{1}{2} J_{mn} \bar{L}^{mn},$$

$G(x^m, \theta^\alpha, \hat{\theta}^{\hat{\alpha}}) = \exp(x^m P_m + \theta^\alpha Q_\alpha + \hat{\theta}^{\hat{\alpha}} Q_{\hat{\alpha}})$ takes values in a coset supergroup, $[x^m, \theta^\alpha, \hat{\theta}^{\hat{\alpha}}]$ are $N = 2, D = 10$ superspace variables with $m = 0$ to 9 and $[\alpha, \hat{\alpha}] = 1$ to 16, the generators $[P_m, Q_\alpha, Q_{\hat{\alpha}}, J_{mn}]$ form a super-Lie algebra with the commutation relations

$$[P^m, P^n] = \frac{1}{2} R^{mnpq} J_{pq}, \quad \{Q_\alpha, Q_\beta\} = 2\gamma_{\alpha\beta}^m P_m, \quad \{Q_{\hat{\alpha}}, Q_{\hat{\beta}}\} = 2\gamma_{\hat{\alpha}\hat{\beta}}^m P_m, \quad (4.37)$$

$$[Q_\alpha, P^m] = \gamma_{\alpha\beta}^m F^{\beta\hat{\gamma}} Q_{\hat{\gamma}}, \quad [Q_{\hat{\alpha}}, P^m] = -\gamma_{\hat{\alpha}\hat{\beta}}^m F^{\gamma\hat{\beta}} Q_\gamma, \quad \{Q_\alpha, Q_{\hat{\gamma}}\} = \frac{1}{2} J_{[mn]} \gamma_{\alpha\beta}^m F^{\beta\hat{\delta}} \gamma_{\hat{\delta}\hat{\gamma}}^n,$$

J_{mn} generate the usual Lorentz algebra, R^{mnpq} is the constant spacetime curvature tensor which is related to $F^{\alpha\hat{\beta}}$ by the identity

$$R^{mnpq} (\gamma_{pq})_\alpha^\beta = \gamma_{\alpha\gamma}^m F^{\gamma\hat{\delta}} \gamma_{\hat{\delta}\hat{\kappa}}^n F^{\beta\hat{\kappa}} - \gamma_{\alpha\gamma}^n F^{\gamma\hat{\delta}} \gamma_{\hat{\delta}\hat{\kappa}}^m F^{\beta\hat{\kappa}}, \quad (4.38)$$

and $\int dy \epsilon^{IJK} (\gamma_{m\alpha\beta} L_I^m L_J^\alpha L_K^\beta + \gamma_{m\hat{\alpha}\hat{\beta}} L_I^m L_J^{\hat{\alpha}} L_K^{\hat{\beta}})$ is the Wess-Zumino term which is constructed such that S_{GS} is invariant under κ -symmetry.

Under $G \rightarrow \Omega G H$ for global Ω and local H , the currents $G^{-1} \partial G$ are invariant up to a tangent-space Lorentz rotation using the standard coset construction where $[P_m, Q_\alpha, Q_{\hat{\alpha}}, J_{mn}]$ are the generators in Ω and J_{mn} are the generators in H . Since the action is constructed from Lorentz-invariant combinations of currents, it is therefore invariant under the global target-space isometries generated by $[P_m, Q_\alpha, Q_{\hat{\alpha}}, J_{mn}]$. Note that because the R-R field-strength is self-dual, only 20 of the 45 Lorentz generators J_{mn} appear in (4.37). So only 20 of the L^{mn} currents are nonzero in (4.36). For the $AdS_5 \times S^5$ background, these are the $SO(4,1) \times SO(5)$ currents L^{ab} and $L^{a'b'}$ for $a, b = 0$ to 4 and

$a', b' = 5$ to 9 . And for the plane wave background, these are the currents L^{jk} , $L^{j'k'}$, L^{+j} and $L^{+j'}$ for $j, k = 1$ to 4 and $j', k' = 5$ to 8 .

The terms $d_\alpha \bar{L}^\alpha$ and $\hat{d}_{\hat{\alpha}} L^{\hat{\alpha}}$ in (4.34) break kappa symmetry but allow quantization since they imply non-vanishing propagators for θ^α and $\hat{\theta}^{\hat{\alpha}}$. And the term $-\frac{1}{2}d_\alpha \hat{d}_{\hat{\beta}} F^{\alpha\hat{\beta}}$ comes from the R-R vertex operator and implies that certain components of d_α and $\hat{d}_{\hat{\beta}}$ are auxiliary fields. Finally, $\mathcal{S}_{ghost}$ describes the action for the worldsheet ghosts which is non-trivial since the pure spinors transform under Lorentz transformations and therefore couple through their Lorentz currents to the spacetime connection and curvature. This ghost action is

$$\mathcal{S}_{ghost} = \int d^2z [\mathcal{L}_{ghost}^{flat} + \frac{1}{2}N_{mn}\bar{L}^{mn} + \frac{1}{2}\hat{N}_{mn}L^{mn} + \frac{1}{4}N_{mn}\hat{N}_{pq}R^{mnpq}] \quad (4.39)$$

where $\mathcal{L}_{ghost}^{flat}$ is the free Lagrangian in a flat background for the left and right-moving worldsheet ghosts $(\lambda^\alpha, w_\alpha)$ and $(\hat{\lambda}^{\hat{\alpha}}, \hat{w}_{\hat{\alpha}})$, $N_{mn} = \frac{1}{2}\lambda\gamma_{mn}w$ and $\hat{N}_{mn} = \frac{1}{2}\hat{\lambda}\gamma_{mn}\hat{w}$ are their left and right-moving Lorentz currents, and R^{mnpq} is the target-space curvature tensor defined in (4.38). Note that $\mathcal{S}_{ghost}$ is invariant under local tangent-space Lorentz rotations, which is necessary for the action to be well-defined on the coset superspace described by $G(x, \theta, \hat{\theta})$.

To check that the action is classically BRST invariant, i.e. that $\bar{\partial}(\lambda^\alpha d_\alpha) = \partial(\hat{\lambda}^{\hat{\alpha}} \hat{d}_{\hat{\alpha}}) = 0$, it is useful to first compute the equations of motion for d_α and $\hat{d}_{\hat{\alpha}}$. Suppose one varies $Z^M = [x^m, \theta^\alpha, \hat{\theta}^{\hat{\alpha}}]$ such that $E_M^\alpha \delta Z^M = \rho^\alpha$, $E_M^{\hat{\alpha}} \delta Z^M = \bar{\rho}^{\hat{\alpha}}$, and $E_M^m \delta Z^M = 0$ where $L^\alpha = E_M^\alpha \partial Z^M$, $L^{\hat{\alpha}} = E_M^{\hat{\alpha}} \partial Z^M$, $L^m = E_M^m \partial Z^M$, and $[L^\alpha, L^{\hat{\alpha}}, L^m]$ are defined in (4.36). Then the covariant GS action $\mathcal{S}_{GS}$ transforms as

$$\delta \mathcal{S}_{GS} = 2\rho^\alpha L^m \gamma_{m\alpha\beta} \bar{L}^\beta + 2\bar{\rho}^{\hat{\alpha}} \bar{L}^m \gamma_{m\hat{\alpha}\hat{\beta}} L^{\hat{\beta}}. \quad (4.40)$$

The transformation of (4.40) is related to kappa symmetry since when $\rho^\alpha = \kappa_\beta L^m \gamma_m^{\alpha\beta}$ and $\bar{\rho}^{\hat{\alpha}} = \bar{\kappa}_{\hat{\beta}} L^m \gamma_m^{\hat{\alpha}\hat{\beta}}$, $\delta \mathcal{S}_{GS}$ is proportional to the Virasoro constraints $\eta_{mn} L^m L^n$ and $\eta_{mn} \bar{L}^m \bar{L}^n$.

Furthermore, the commutation relations of (4.37) imply that

$$\delta L^\alpha = \partial \rho^\alpha + \frac{1}{4}(\gamma^{mn})_\beta^\alpha L_{mn} \rho^\beta + F^{\alpha\hat{\beta}} \gamma_{\hat{\beta}\hat{\gamma}}^m L_m \bar{\rho}^{\hat{\gamma}}, \quad (4.41)$$

$$\delta L^{\hat{\alpha}} = \partial \bar{\rho}^{\hat{\alpha}} + \frac{1}{4}(\gamma^{mn})_{\hat{\beta}}^{\hat{\alpha}} L_{mn} \bar{\rho}^{\hat{\beta}} - F^{\beta\hat{\alpha}} \gamma_{\beta\gamma}^m L_m \rho^\gamma,$$

$$\delta L^{mn} = (\gamma^{[m} F \gamma^{n]})_{\beta\hat{\gamma}} \rho^\beta L^{\hat{\gamma}} + (\gamma^{[m} F \gamma^{n]})_{\beta\hat{\gamma}} L^\beta \bar{\rho}^{\hat{\gamma}}$$

where $(\gamma^{[m}F\gamma^{n]})_{\alpha\hat{\delta}} = \frac{1}{2}(\gamma_{\alpha\beta}^m F^{\beta\hat{\gamma}}\gamma_{\hat{\gamma}\hat{\delta}}^n - \gamma_{\alpha\beta}^n F^{\beta\hat{\gamma}}\gamma_{\hat{\gamma}\hat{\delta}}^m)$.

So by varying ρ^α and $\bar{\rho}^{\hat{\alpha}}$, one obtains the equations of motion

$$\begin{aligned}\bar{\partial}d_\alpha &= 2\gamma_{\alpha\beta}^m L_m \bar{L}^\beta + \frac{1}{4}d_\beta(\gamma_{mn})_\alpha^\beta \bar{L}^{mn} - \hat{d}_\beta F^{\gamma\hat{\beta}} \gamma_{\gamma\alpha}^m L_m + \frac{1}{2}(\gamma^{[m}F\gamma^{n]})_{\alpha\hat{\gamma}}(N_{mn}\bar{L}^{\hat{\gamma}} + \hat{N}_{mn}L^{\hat{\gamma}}), \\ \partial\hat{d}_{\hat{\alpha}} &= 2\gamma_{\hat{\alpha}\hat{\beta}}^m \bar{L}_m L^{\hat{\beta}} + \frac{1}{4}\hat{d}_\beta(\gamma_{mn})_{\hat{\alpha}}^{\hat{\beta}} L^{mn} + d_\beta F^{\beta\hat{\gamma}} \gamma_{\hat{\gamma}\hat{\alpha}}^m \bar{L}_m - \frac{1}{2}(\gamma^{[m}F\gamma^{n]})_{\gamma\hat{\alpha}}(N_{mn}\bar{L}^\gamma + \hat{N}_{mn}L^\gamma).\end{aligned}\tag{4.42}$$

Plugging into (4.42) the equations of motion $\bar{L}^\alpha = \frac{1}{2}F^{\alpha\hat{\beta}}\hat{d}_{\hat{\beta}}$ and $L^{\hat{\alpha}} = -\frac{1}{2}F^{\beta\hat{\alpha}}d_\beta$ which come from varying d_α and $\hat{d}_{\hat{\alpha}}$, one finds

$$\begin{aligned}\bar{\nabla}d_\alpha &= \frac{1}{2}(\gamma^{[m}F\gamma^{n]})_{\alpha\hat{\gamma}}(N_{mn}\bar{L}^{\hat{\gamma}} - \frac{1}{2}\hat{N}_{mn}F^{\delta\hat{\gamma}}d_\delta), \\ \nabla\hat{d}_{\hat{\alpha}} &= -\frac{1}{2}(\gamma^{[m}F\gamma^{n]})_{\gamma\hat{\alpha}}(\frac{1}{2}N_{mn}F^{\gamma\delta}\hat{d}_\delta + \hat{N}_{mn}L^\gamma),\end{aligned}\tag{4.43}$$

where the spin connections in the covariantized derivatives ∇ and $\bar{\nabla}$ are L^{mn} and $\bar{L}^{mn}$.

Furthermore, the equations of motion of λ^α and $\hat{\lambda}^{\hat{\alpha}}$ coming from (4.39) are

$$\begin{aligned}\bar{\nabla}\lambda^\alpha &= \frac{1}{8}R^{mnpq}(\gamma_{mn})_\beta^\alpha \lambda^\beta \hat{N}_{pq}, \\ \nabla\hat{\lambda}^{\hat{\alpha}} &= \frac{1}{8}R^{mnpq}(\gamma_{pq})_{\hat{\beta}}^{\hat{\alpha}} \hat{\lambda}^{\hat{\beta}} N_{mn}.\end{aligned}\tag{4.44}$$

So (4.43) and (4.44), together with the identity of (4.38), imply that

$$\begin{aligned}\bar{\partial}(\lambda^\alpha d_\alpha) &= \frac{1}{2}\lambda^\alpha(\gamma^{[m}F\gamma^{n]})_{\alpha\hat{\gamma}}N_{mn}\bar{L}^{\hat{\gamma}}, \\ \partial(\hat{\lambda}^{\hat{\alpha}}\hat{d}_{\hat{\alpha}}) &= -\frac{1}{2}\hat{\lambda}^{\hat{\alpha}}(\gamma^{[m}F\gamma^{n]})_{\gamma\hat{\alpha}}\hat{N}_{mn}L^\gamma.\end{aligned}\tag{4.45}$$

Since $N_{mn} = \frac{1}{2}(\lambda\gamma_{mn}w)$ and $\lambda^\alpha\lambda^\beta$ is proportional to $(\lambda\gamma^{pqrst}\lambda)(\gamma_{pqrst})^{\alpha\beta}$, the right-hand side of (4.45) is proportional to $\gamma_{mn}\gamma_{pqrst}\gamma^{[m}F\gamma^{n]}$. But since $\gamma_m\gamma_{pqrst}\gamma^m = 0$, one finds that

$$\gamma_{mn}\gamma_{pqrst}\gamma^{[m}F\gamma^{n]} = 2\gamma_{pqrst}\gamma^n F\gamma_n = 2\gamma_{pqrst}\gamma^n \gamma_{uvwx} \gamma_n F^{uvwx} = 0.\tag{4.46}$$

So $\bar{\partial}(\lambda^\alpha d_\alpha) = \partial(\hat{\lambda}^{\hat{\alpha}}\hat{d}_{\hat{\alpha}}) = 0$ as desired.

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