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SPRING SCHOOL ON SUPERSTRING THEORY
AND RELATED TOPICS

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Classifying Supergravity Solutions and Applications

PART III

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AdS_5 Solutions in M-theory and New Sasaki-Einstein Metrics

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Introduction

Supersymmetric solutions to $D = 10, 11$ SUGRA with AdS_5 factors should be dual to $N = 1$ susy CFTs in $D = 4$.

Maldacena c. 1997 $> 3.4k$ citations.

BUT....until recently

⋆ Very few explicit examples were known

⋆ Lacked any general classification of the underlying geometry - essential for understanding the general class of susy CFTs with string or M-theory duals.

Type IIB SUGRA:

Basic example: consider flat $D3$-branes with worldvolume $\mathbb{R}^{1,3}$ transverse to $\mathbb{R}^6$ i.e. the D3-branes are embedded in $\mathbb{R}^{1,3} \times \mathbb{R}^6$.

Find that $N = 4$ SYM is dual to type IIB on

$$AdS_5 \times S^5$$
$$F_5 \sim Vol(AdS_5) + Vol(S^5)$$

Metric on $AdS_5 \times S^5$:

$$ds^2 = \rho^2 ds^2(\mathbb{R}^{1,3}) + \frac{1}{\rho^2}[d\rho^2 + \rho^2 ds^2(S^5)]$$

reveals $\mathbb{R}^6$ transverse to $D3$-branes.

Consider a Calabi-Yau cone with metric

$$ds^2 = d\rho^2 + \rho^2 ds^2(X_5)$$

where X_5 is known as a "Sasaki-Einstein" manifold.

Consider flat D3-branes at the apex of the CY cone. Leads to new AdS/CFT examples: Type IIB on

$$AdS_5 \times X_5$$
$$F_5 \sim Vol(AdS_5) + Vol(X_5)$$

is dual to the N=1 SCFT living on the D3-branes at the apex of the cone.

Until recently: only two *explicit* examples of X_5 were known S^5 (flat CY) and $T^{1,1}$ (conifold).

Examples with more fluxes?

M-Theory:

The most general type of solution:

$$AdS_5 \times_w M_6$$
$$G_4 \neq 0$$

until recently, very few examples known. e.g. Maldacena and Nunez found N=1 and N=2 examples which are dual to M5-branes wrapped on holomorphic curves in CY_3 and $K3$, respectively.

PLAN

1. Classification of all such solutions in D=11

2. Explicit examples in D=11

3. New Sasaki-Einstein metrics $\rightarrow$ type IIB examples.

$AdS_5 \times_w M_6$ solutions in $D = 11$

$$\begin{aligned} ds^2 &= e^{2\lambda(x)}[ds^2(AdS_5) + ds^2(M_6)(x)] \\ G_4 &= G_4(x) \end{aligned}$$

i.e. G is a 4-form on M_6. Ansatz preserves symmetries of AdS_5.

D=11 susy solutions equivalent to:

$$\left[\nabla_m + \frac{1}{2} i m \gamma_m \gamma_7 - \frac{1}{4} e^{-3\lambda} (\gamma \cdot G)_m\right] \epsilon = 0$$

$$\left[\gamma^m \nabla_m \lambda + \frac{1}{6} e^{-3\lambda} (\gamma \cdot G) - i m \gamma_7\right] \epsilon = 0 \ .$$

plus

$$\begin{aligned} dG &= 0 \\ d * G + (1/2) G \wedge G &= 0 \end{aligned}$$

Analyse using the G-structure tools. Find that M_6 has an $SU(2)$ structure. Specified by two vectors K^1, K^2 and three two-forms J^i orthogonal to K^i.

ANALYZE to find the most general geometries. Includes two interesting properties:

1. K^2 is a Killing vector which corresponds to the $U(1)_R$ symmetry of N=1 SCFT.

2. Most general configurations involve a one parameter family of D=4 Kähler metrics on a base space B_4 (satisfying various constraints). The K^2 vector is non-trivially fibred over it.

Explicit Solutions

Assume that M_6 is complex. Then can explicitly construct all compact regular solutions by solving ODEs.

$\star$ Topology: $S^2 \to M_6 \to B_4$

$\star$ Metric for M_6: completely explicit given metric on B_4 which can be in one of two classes;

(a) B_4 is Kähler-Einstein with positive scalar curvature (Kähler and $R_{ij} = \lambda g_{ij}$ with $\lambda > 0$). These have been classified by Tian and Yau:
explicit: $S^2 \times S^2$, CP^2
implicit: del Pezzo P_k $k = 3, \ldots 8$ (CP^2 blown up at k points).

(b) B_4 is a product.
All explicit: $S^2 \times S^2$, $S^2 \times H^2$, $S^2 \times T^2$

A special case of the S^2 bundle over $S^2 \times H^2$ case gives the $N = 1$ Maldacena Nunez solution.

Nice.

What is dual conformal field theory? Something to do with M5-branes.

Consider D=11 solution with $S^2 \times T^2$ base:

Dimensional reduction on one of the S^1s of $T^2 \to AdS_5 \times_w M_5$ solution of type IIA with various fluxes $\neq 0$.

T-dualise on the other S^1 of the $T^2 \to$ type IIB solution

$$AdS_5 \times X_5$$
$$F_5 \sim Vol(AdS_5) + Vol(X_5)$$

$\Rightarrow X_5$ must be Sasaki-Einstein, at least locally. In fact gives an infinite number of new explicit Sasaki-Einstein metrics on $S^2 \times S^3$!

Sasaki-Einstein

A SE X_5 is equivalent to the cone

$$ds^2 = dr^2 + r^2 ds^2(X_5)$$

being CY_3.

There is a canonical vector which is Killing:

$$(\partial_{\psi'})^j = r(\partial_r)^i J_i{}^j$$

This corresponds to the "U(1)" R-symmetry of the D=4 SCFT.

Locally, metric can be written

$$ds^2(X_5) = (d\psi' + \sigma) + ds^2(B_4)$$

where B_4 is Kähler-Einstein and $d\sigma = 2J_4$

Three possibilities:

1. Regular SE:

Have a $U(1)$ R-symmetry and it is free.
B_4 is globally defined and hence can classify using Tian and Yau:
Explicit:
$B_4 = CP^2 \rightarrow S^5$
$B_4 = S^2 \times S^2 \rightarrow T^{1,1}$
Implicit: $B_4 = P_k$ del Pezzo $k = 3, \ldots 8$.

2. Quasi regular SE:

$U(1)$ R-symmetry with finite isotropy groups.
B_4 is an orbifold.

3. Irregular SE:

Have a non-compact $\mathbb{R}$ R-symmetry.
B_4 is not a manifold.

The metrics obtained from $D = 11$ provide the first explicit example in the quasi-regular class, and the very first examples in the irregular class!

The SE metrics $Y^{p,q}$:

$$\begin{aligned} ds^2 &= \frac{1-cy}{6}(d\theta^2+\sin^2\theta d\phi^2) \\ &+ \frac{1}{w(y)q(y)}dy^2+\frac{q(y)}{9}[d\psi-\cos\theta d\phi]^2 \\ &+ w(y)\left[d\alpha+f(y)(d\psi-\cos\theta d\phi)\right]^2 \end{aligned}$$

with

$$\begin{aligned} w(y) &= \frac{2(b-y^2)}{1-cy} \\ q(y) &= \frac{b-3y^2+2cy^3}{a-y^2} \\ f(y) &= \frac{bc-2y+y^2c}{6(a-y^2)} \end{aligned}$$

For regularity we demand

$$b=\frac{1}{2}-\frac{p^2-3q^2}{4p^3}\sqrt{4p^2-3q^2}$$

with $0<q<p$.

$\star$ Family includes $T^{1,1}$ and S^5.
($Y^{p,0} \equiv T^{1,1}/Z_p$ and $Y^{p,p} \equiv S^5/Z_2 \times Z_p$)

$\star$ SE Killing vector is $\partial_{\psi'} = \partial_\psi - (1/6)\partial_\alpha$

$\star$ Two classes of b:

(i) $4p^2 - 3q^2 = n^2$ then SE vector $\partial_{\psi'}$ has $U(1)$ orbits $\Rightarrow$ Quasi-Regular SE

(ii) $4p^2 - 3q^2 \neq n^2$ then SE vector $\partial_{\psi'}$ has $\mathbb{R}$ orbits $\Rightarrow$ Irregular SE

$\star$ Isometry group $\sim SU(2) \times U(1) \times U(1)$

$\star$ Topology: $S^2 \times S^3$ just as for $T^{1,1}$

Dual to new SCFTs

⋆ Symmetries: $SU(2) \times U(1) \times U(1) \times U(1)_B$

⋆ Central charges:

$$\frac{a(Y^{p,q})}{a(S^5)} = Vol(S^5)/Vol(Y^{p,q})$$
$$= \frac{3p^2[3q^2 - 2p^2 + p(4p^2 - 3q^2)^{1/2}]}{q^2[2p + (4p^2 - 3q^2)^{1/2}]}$$

Quasi-regular case: $U(1)$ R-symmetry and rational a.

Irregular case: $\mathbb{R}$ R-symmetry and quadratic irrational a.

⋆ Baryons arise from D3-branes wrapped on supersymmetric 3-cycles [Martelli, Sparks; Herzog, Ejaz, Klebanov]. R-charges of baryons:

$$R \propto \frac{Vol(\Sigma_i)}{Vol(Y^{p,q})} = ...$$

Identifying the dual field theory

There is a toric description of the Calabi-Yau cone since the $Y^{p,q}$ have a a $U(1)^3$ action [Martelli, Sparks]. The Calabi-Yau cone is an ALE space fibred over S^2.

The toric description gives information about how to resolve singularity of cone. Also leads to an identification of the dual quantum field theory. Find a quiver gauge theory, with superpotential terms [Benvenuti, Franco, Hanany, Martelli, Sparks] that generalises that for the conifold.

Can use the procedure of a-maximisation [Intriligator, Wecht] to find a as well as the the R-charges of the baryon.
Idea: the exact R-symmetry is the linear combination of all possible $U(1)$s which maximises

$$a_{trial}(R) = \frac{3}{32}(3TrR^3 - TrR)$$

Find exact agreement with calculations found from the geometry!

Now have an infinite number of AdS/CFT examples where both the geometry and the field theory are known.

Nice.

Conclusions

⋆ The conifold (the cone over $T^{1,1}$) can be resolved (blow up an S^2) and deformed (blow up an S^3) while preserving the CY condition and geometries are known exactly. What are the analogous geometries smoothing the singularity of the cones over $Y^{p,q}$? Co-homogeneity two.

⋆ β-deformations: exactly marginal deformations of the field theories based on $Y^{p,q}$ [Benvenuti, Hanany] and exact AdS_5 geometries found [Lunin, Maldacena] !

⋆ Wrap $D5$-branes on S^2's: breaks conformal invariance and leads to duality cascades. Singular solutions that describe the UV have been found [Herzog, Ejaz, Klebanov] . Can the singularity be smoothed, generalising the Klebanov/Strassler solution?

⋆ Can generalise to give new SE manifolds in all odd dimensions.
Physics: Gives new $AdS_4 \times X_7(SE)$ susy solutions of M-theory. X_7 are co-homogeneity one and generalise the known homogeneous examples $Q^{1,1,1}$ and $M^{3,2}$. Recall:

$$\begin{aligned} S^1 &\to Q^{1,1,1} \to S^2 \times S^2 \times S^2 \\ S^1 &\to M^{3,2} \to S^2 \times CP^2 \end{aligned}$$

What is the field theory? Also what is the field theory for $AdS_5 \times_w M_6$ solutions.

$AdS_5 \times_w M_6$ in $D = 11$. Explicit solutions were obtained when M_6 is assumed to be complex. There may well be more. Interesting that Maldacena Nunez solution with $N = 2$ susy is in the non-complex class.

⋆ Classify all type IIB AdS_5 solutions - in progress.

⋆ Could also classify AdS_n for other n.

⋆ Geometries describing renormalisation group flows between different field theories?