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*SPRING SCHOOL ON SUPERSTRING THEORY
AND RELATED TOPICS*

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Topological Strings and Black Holes

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Topological Strings and Black Holes

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Introduction to topological strings

Compactification of type II string on a Calabi-Yau

BPS black holes from Calabi-Yau compactification

Entropy of black hole and its relation to topological string

Physical interpretation as a wave function

Example involving q-deformed 2d YM

Trieste, March 2005

Topological String Theory

(1) Start with a Calabi-Yau 3-fold.

$\mathcal{N}=2$ superconformal sigma-model

$$\Sigma_g \to CY_3$$

(2) Topologically twist the sigma-model.

(3) Couple it to the topological gravity on Σ_g.

$$F_g = \int_{\mathcal{M}_g} \Big\langle \Big| \prod_{i=1}^{3g-3} (\eta_i, G^-) \Big|^2 \Big\rangle$$

supercurrent

Beltrami differential

$$Z^{top}_{pert.} = \exp\Big[\sum_{g=0}^{\infty} \lambda^{2g-2} F_g \Big]$$

The series is typically not summable.

Topological String Partition Function = Wave Function

Consider the topological B-model

moduli space $\mathcal{M}_B$: z^i complex structure

λ topological string coupling

tangent space to $\mathcal{M}_B$ $= H^3(CY_3, \mathbb{R})$

$$\dim_{\mathbb{C}} H^3 = \underset{\uparrow \, \delta z^i}{h^{2,1}} + \underset{\uparrow \, \lambda}{1}$$

For a given background, we can define the B-model.

F_g is also a holomorphic function of $H^3(CY_3)$.

$$F_g \sim \left\langle \int_{\mathcal{M}_g} (G_{\bar{i}})^{3g-3} (G_{\bar{z}})^{3g-3} \, e^{x^I \int_{\Sigma_g} \mathcal{O}_I} \right\rangle$$

$x^I \in H^3$, $I = 0, 1, \cdots, h^{2,1}$

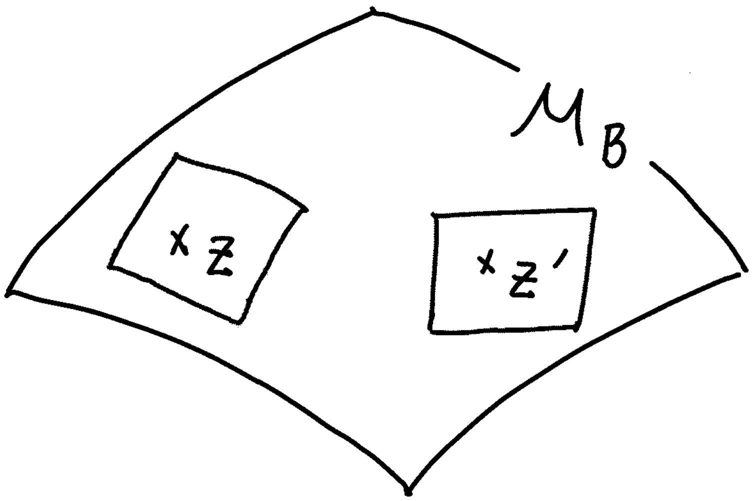
$$\psi_{top}(x^I; z^i, \bar{z}^i) = \exp\left(\sum_g F_g\right)$$

including λ

The holomorphic anomaly equations   BCOV / Witten / DVV

$$\frac{\partial}{\partial z^I} \psi_{top}(x; z, \bar{z}) = \left(\frac{\partial}{\partial x^I} + \cdots \right) \psi_{top}$$

$$\frac{\partial}{\partial \bar{z}^I} \psi_{top}(x; z, \bar{z}) = \left(\bar{C}_I{}^{JK} \frac{\partial^2}{\partial x^J \partial x^K} + \cdots \right) \psi_{top}$$



Interptetation:

(1) On each tangent space, there is a Hilbert space.

(2) The holomorphic anomaly equation is describing the paralell transport between tangent spaces at different points.

More on (1):

- $H^3(CY_3, \mathbb{R})$ has a symplectic structure.
- Topological string uses a holomorphic polarization.

$$\underbrace{H^{3,0} \oplus H^{2,1}}_{x^I} \oplus H^{2,1} \oplus H^{0,3}$$

More on (2):

- The polarization depends on $(z^i, \bar{z}^i)$

$\Rightarrow$ Wave-functions are related by Fourier topological string partition function $Z_{top} = \exp\left(\sum_g F_g\right)$

= wave function $\psi_{top}(x; z, \bar{z})$

for quantization of $H^3(CY_3, \mathbb{R})$

(H^3 = tangent space to the moduli space of CY3)

On the other hand,
the topological string partition function gives superpotential terms for CY3 compactification of type II superstring. *(BCOV)*
(AGNT)

Can we intepret the topological string partition function as a physical wave function in type II superstring?

Compactification of type II string theory on a Calabi-Yau 3-fold

$\mathcal{N}=2$ supersymmetry on $\mathbb{R}^4$

($\Rightarrow$ $\mathcal{N}=1$ by branes and/or fluxes)

◎ Supergravity multiplet

graviton , $U(1)$ gauge field (gravi-photon) A_μ^0

+ superpartners

◎ vector multiplets $i=1,\cdots,n_V$

z^i : complex scalar field

A_μ^i : gauge field

+ superpartners

z^i's parametrize the moduli space $\mathcal{M}_V$.

$\mathcal{M}_V = \begin{cases} \cdot \text{ Complexified Kähler moduli space for IIA} \\ \cdot \text{ Complex structure moduli space for IIB} \end{cases}$

Sometimes we combine $A_\mu^{i=1 \cdots n_V}$ with the gravi-photon $A_\mu^{i=0}$, and consider (n_V+1) vector multiplets : $X^{I=0,1,\cdots,n_V}$.

But, the scalar $X^{I=0}$ is not physical.

There is a scaling symmetry, which allows us to set $X^0 = 1$.

X^I may be regarded as projective coordinates of $\mathcal{M}_V$

◎ hypermultiplets $\quad i = 1, \cdots, n_H$

2 complex scalar fields in each multiplet.

- They are neutral with respect to A_μ^I.
- The moduli space $\mathcal{M}_H$ is quaternionic Kähler.
- There is no neutral coupling between vector and hypermultiplets.

For a generic (smooth) CY_3 compactification, the gauge fields A_μ^I are abelian, and the moduli space is locally $\mathcal{M}_V \times \mathcal{M}_H$.

This factorization breaks down at special points in the moduli space, e.g. a conifold --- a charged hypermultiplet becomes massless (e.g. D3 in IIB wrapping a collapsing 3 cycle)

In the following, we will consider generic compactifications.

- IIA $\quad n_V = h^{1,1}$ ← complexified Kähler moduli

$\quad n_H = h^{2,1} + 1$

each comes with 2 complex scalar (complex structure moduli + RR fields)

+1 = universal hypermultiplet : dilaton, axion (dual of $B_{\mu\nu}$) RR fields

We will be interested in the vector multiplets

$\mathcal{M}_V$: special Kähler manifold.

$z^i \quad (i = 1, \cdots, n_V)$

$X^I(z) \quad I = 0, 1, \cdots, n_V$: projective coordinates

They can be understood geometrically in IIB case.

(IIA case involves worldsheet instantons.)

IIB : $n_V = h^{2,1}$ complex str moduli space

Ω : holomorphic (3, 0) - form

$T\mathcal{M}_V \Leftrightarrow \quad \Omega \to \Omega + \delta z^i \eta_i$

$\eta_i \in H^{2,1}$

Choose a basis $\{ a_I , b^I \}$ $I = 0, 1, \cdots, h^{2,1}$

of H_3 ($\dim H_3 = 2 + 2h^{2,1}$)

s.t. $a_I \cap a_J = 0$, $b^I \cap b^J = 0$

$a_I \cap b^J = \delta_I{}^J$

Define $X^I = \int_{A_I} \Omega$ as holomorphic functions of z^i.

As we change $z^i \to z^i + \delta z^i$, there is a possibility of rescaling Ω.

$$\Rightarrow \quad \frac{X^i(z)}{X^0(z)} \qquad i = 1, 2, \cdots, h^{2,1}$$

are well-defined functions of z.

($\Rightarrow$ flat coordinates)

$$F_I = \int_{B^I} \Omega = \frac{\partial}{\partial X^I} F(X)$$

$F(X)$: prepotential

- $X^I \partial_I F = 2F$ or $F(CX) = C^2 F(X)$
- $\tau_{IJ} = \partial_I \partial_J F$: period matrix
- $C_{IJK} = \partial_I \partial_J \partial_K F$: Yukawa coupling

$$X^I C_{IJK} = X^I \partial_I \partial_J \partial_K F = 0$$

$$\Rightarrow \text{ effectively } \quad C_{ijk}(z).$$

$\mathcal{M}_V$ is a Kähler manifold, but a special kind

Kähler potential

$$K = -\log\left[i\left(\bar{X}^I F_I(X) - X^I \bar{F}_I(\bar{X})\right)\right]$$

$$= -\log\left[-2\,\mathrm{Im}\,\tau_{IJ} \cdot X^I \bar{X}^J\right]$$

If we change $X^I(z) \to C(z)\, X^I(z)$,

$$K(z,\bar{z}) \to K(z,\bar{z}) - \log C(z) - \log \bar{C}(\bar{z}).$$

Thus $G_{i\bar{j}} = \partial_i \partial_{\bar{j}} K(z,\bar{z})$ gives a metric on $\mathcal{M}_V$.

(It satisfies the special geometry relation

$$R_{i\bar{j}k\bar{l}} = G_{i\bar{j}} G_{k\bar{l}} + G_{i\bar{l}} G_{k\bar{j}} - e^{-2K} G^{m\bar{n}} C_{ikm} C_{\bar{j}\bar{l}\bar{n}}$$
)

The low energy effective action for the vector multiplets is completely determined by $F(X)$.

... This is exact at the string tree level.

- In the IIB string, $F(X)$ is computable by the classical geometric method. $\left(\begin{array}{l} \text{Remember:} \\ \int_{B^I} \Omega = \frac{\partial}{\partial X^I} F \end{array} \right)$

  This is because the Kähler moduli are in $\mathcal{M}_H$. Thus, by the factorization, we can take $\mathrm{vol}(CY_3) \to \infty$.

- In the IA string, $F(X)$ receives worldsheet instanton corrections

  - Compute the Gromov-Witten invariants directly

  or

  - Use the mirror manifold to reduce the computation to IIB.

BPS states

$\mathcal{N}=2$ supersymmetry $\qquad i, j = 1, 2$

$\qquad\qquad\qquad\qquad\qquad \alpha, \beta = 1, 2$ spinor index

$$\{ Q_\alpha^i , Q_\beta^j \} = \epsilon_{\alpha\beta} \epsilon^{ij} Z \leftarrow \text{central charge}$$

$$\{ \bar{Q}_{\dot\alpha}^i , \bar{Q}_{\dot\beta}^j \} = \epsilon_{\dot\alpha\dot\beta} \epsilon^{ij} Z$$

$$\{ Q_\alpha^i , \bar{Q}_{\dot\beta}^j \} = \sigma^\mu_{\alpha\dot\beta} P_\mu \delta^{ij}$$

If P_μ is timelike (massive),

we can go to its rest frame, where

$$\{ Q_\alpha^i , \bar{Q}_{\dot\beta}^j \} = 2M \delta_{\alpha\dot\beta} \delta^{ij}$$

Define $\quad A_\alpha = \frac{1}{\sqrt{2}} (Q_\alpha^1 + \epsilon_{\alpha\beta} Q_\beta^2)$

$\qquad\qquad B_\alpha = \frac{1}{\sqrt{2}} (Q_\alpha^1 - \epsilon_{\alpha\beta} Q_\beta^2)$

$$\Rightarrow \quad \{ A_\alpha , A_\beta \} = \{ B_\alpha , B_\beta \} = \{ A_\alpha , B_\beta \} = 0$$

$$\{ A_\alpha , A^+_{\dot\beta} \} = (2M + Z) \delta_{\alpha\dot\beta}$$

$$\{ B_\alpha , B_{\dot\beta}^+ \} = (2M - Z) \delta_{\alpha\dot\beta}$$

$\mathcal{N}=2$ SUSY in 4d can be regarded as a dimensional reduction of minimal SUSY in 5d.

This view is particularly natural in IIA

since IIA on $CY_3 \times \mathbb{R}^4 \Leftarrow M$ on $CY_3 \times S^1 \times \mathbb{R}^4$

In this set-up, the central charge Z

is a momentum p_5 around the S^1

and $M^2 = m^2 + (p_5)^2$,

where m is a 5d mass
M is a 4d mass.

This also explains the bound $M^2 \geq Z^2$.

In the IIA case, $p5$ is the M theory momentum

= Kaluza-Klein charge

= graviphoton charge.

The unitarity requires $2M \geqq Z$.

- long representation $2M > Z$

SO(3) little group

$$[j] \otimes \left([1] + 4[1/2] + 5[0] \right)$$

- short representation $2M = Z$ ··· BPS states

$$[j] \otimes \left(2[1/2] + 4[0] \right)$$

We see $\mathrm{tr}\,(-1)^{2\lambda} = 0$ for both long / short

$$\mathrm{tr}\,\lambda^2 (-1)^{2\lambda} = \begin{cases} 0 & \text{for long} \\ (-1)^{2j+1} d_j & \text{for short} \end{cases}$$

In CY_3 compactification of type II string, such BPS states arise from D branes wrapping on cycles in the CY_3, Again consider D3 brane in IIB.

γ : 3 cycle in CY_3

Wrap D3 brane on γ $\rightarrow$ particle in $\mathbb{R}^4$

BPS bound

$$M \geq \left| e^{+\frac{1}{2}K} \int_\gamma \Omega \right| .$$

Note :

$g_{i\bar{j}}$: metric on CY_3

$$\det g = c \, \Omega \bar{\Omega}$$

$$R_{i\bar{j}} = \partial_i \partial_{\bar{j}} \log \det g = 0$$

$$\int_{CY_3} k \wedge k \wedge k = c \int_{CY_3} \Omega \wedge \bar{\Omega} = c \, e^{-K}$$

Thus $c \sim e^{K(z,\bar{z})}$ as far as z-dep. concerned.

$$\gamma : \text{BPS} \iff \begin{array}{l} k|_\gamma = 0 \ : \text{Lagrangian} \\ c^{1/2} \Omega|_\gamma = \text{volume form on } \gamma \end{array}$$

$$M \sim \int_\gamma (\text{volume form})$$

Thus $M \geqq \left| e^{K/2} \int_\gamma \Omega \right|$: minimum volume for a given homology.

If γ wraps a_I q_I times
b^I p^I times

$$\int_\gamma \Omega = q_I \int_{a_I} \Omega + p^I \int_{b^I} \Omega$$

$$= q_I X^I + p^I F_I(X)$$

$$= W_{p,q}(X)$$

$$M_{BPS} = \left| e^{K/2} W_{p,q}(X) \right|$$

q_I , p^I are electric / magnetic charges.

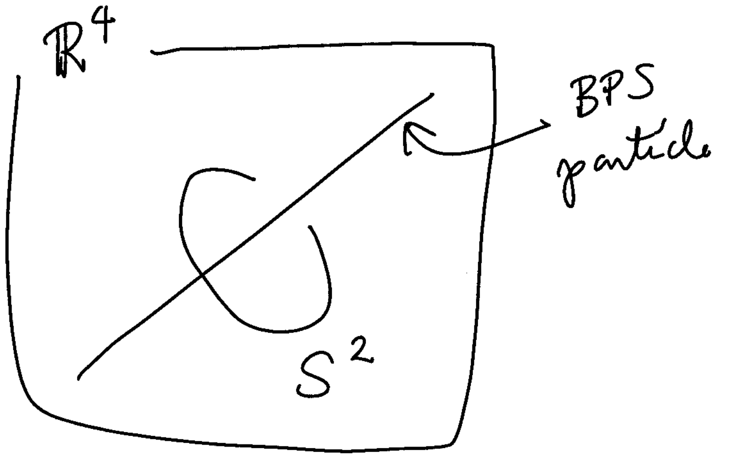
with respect to the gauge field $A_\mu{}^I$

The D3 brane is charged with respect to

the 4-form potential C_4 : $F_5 = dC_4$

The 4d gauge field $A_\mu{}^I$ is defined

as $F_2{}^I = \int_{a_I} F_5$



The magnetic charge is defined by

$$\int_{S^2} F_2{}^I = \int_{S^2 \times a_I} F_5$$

a_I intersects once with b^I.

Thus $\int_{S^2 \times a_I} F_5 = p^I$

By using the self-duality $F_5 = {}^*F_5$, one finds that the electric charge of the D3 brane is $g\, I$.

Black hole attractor and entropy

Low energy effective action

$$\mathcal{L} = -R + G_{i\bar{j}}\, \partial z^i \partial \bar{z}^{\bar{j}}$$
$$+ \mathcal{N}_{IJ} F_{\mu\nu}^{I} F_{\mu\nu}^{J} + \theta_{IJ} F_{\mu\nu}^{I} {}^{*}F_{\mu\nu}^{I}$$

$$\mathcal{N}_{IJ} = \mathrm{Im}\left(\bar{\tau}_{IJ} + 2i \frac{\mathrm{Im}(\tau_{IK}) z^K \, \mathrm{Im}(\tau_{JL}) z^L}{\mathrm{Im}(\tau_{KL}) z^K z^L} \right)$$
$$\theta_{IJ} = \mathrm{Re}\left(\cdots\cdots\cdots\cdots \right)$$

We want to find a classical solution that corresponds to the D3 brane on $\gamma \subset CY_3$.

Static, stationary metric :

$$ds^2 = -e^{2g(r)} dt^2 + e^{2f(r)} (dr^2 + r^2 d\Omega^2)$$

$$\text{SUSY} \rightarrow f = -g$$

Given (q, p), the gauge field $F_{\mu\nu}^{I}$ are determined by $f(r)$.

For convenience, parametrize the metric as

$$ds^2 = - e^{2U(\rho)} e^{2\rho} dt^2 + e^{-2U(\rho)} d\rho^2 + e^{-2U(\rho)} d\Omega^2 + ds^2_{CY_3}.$$

where $r = e^\rho$, $U(\rho) = g(\rho) - \rho$

We also assume that CY_3 moduli z^i are functions of ρ. The hypermultiplet moduli are constant.

BPS equations

$$\begin{cases} \dfrac{dU}{d\rho} = -1 + e^U M_{BPS}(z, \bar{z}) \\ \dfrac{dz^i}{d\rho} = 2 e^U G^{i\bar{j}} \partial_{\bar{j}} M_{BPS}(z, \bar{z}) \end{cases}$$

We can write the BPS equation concisely:

Choose any holomorphic section $X_0^I(z)$.

Write
$$X^I = 2ie^{-U}\left(e^{K(X_0,\bar{X}_0)}\frac{\bar{W}(\bar{X}_0)}{W(X_0)}\right)^{1/2} X_0^I$$

This combines the metric component e^{-U}

with the complex moduli z^i.

The BPS equation becomes

$$\frac{dX^I}{d\rho} = X^I + \left(\frac{i}{\operatorname{Im}\tau}\right)^{IJ}\bar{\partial}_J \bar{W}(\bar{X}).$$

We can also write it as

$$\begin{cases} \operatorname{Re}\left(X^I - \dfrac{dX^I}{d\rho}\right) = p^I \\ \operatorname{Re}\left(F_I - \dfrac{d}{d\rho}F_I\right) = q_I \end{cases}$$

with a general solution:

$$\begin{cases} \operatorname{Re} X^I = p^I + c^I e^\rho \\ \operatorname{Re} F_I = q_I + d_I e^\rho \end{cases}$$

c. d: integration constants

Whatever the boundary condition at

$$r = \infty \quad (\rho = \infty) ,$$

we have

$$\mathrm{Re}\, X^I \to p^I$$

$$\mathrm{Re}\, F_I \to q_I$$

This determines z^i and U uniquely.

"Black hole attractor"

The near horizon geometry is $AdS_2 \times S^2$

(since U is constant)

and the complex moduli are fixed.

This is closely related to the flux compactification

- $CY_3 \times \mathbb{R}^4$

3-form flux F_3 $\Rightarrow$ GVW potential

$$W = \int_{CY_3} F_3 \wedge \Omega$$

This fixes the complex str uniquely

$\because$)

$$D_i W = e^{-K} \partial_i e^K W$$

$$= \int_{CY_3} F_3 \wedge \eta_i = 0$$

$$\eta_i \in H^{2,1}$$

$$\Rightarrow \quad F_3 \in H^{3,0} \oplus H^{0,3}$$

i.e. $\quad F_3 = \mathrm{Re}\,(c\,\Omega)$

for some const c.

$\Rightarrow$ AdS_4 as the rest.

What we are considering is

$CY_3 \times S^2$ compactification with F_5

of the form volume form on S^2

$$F_5 = F_3 \wedge \omega + \text{dual}$$

$$F_3 = p^I \alpha_I + q_I \beta^I$$

where $\alpha_I, \beta^I \in H^3$ are dual

to $A_I, B^I \in H_3$.

The superpotential is

$$W = \int_{CY_3 \times S^2} F_5 \wedge \Omega = \int_{CY_3} F_3 \wedge \Omega$$

which is the same as before

$\Rightarrow$ $DW = 0$ implies

$$\mathrm{Re}(C\Omega) = F_3$$

$$\Rightarrow \quad \mathrm{Re}(CX^I) = p^I , \quad \mathrm{Re}(CF_I) = q_I$$

The condition $DW=0$ is equivalent to $\partial_i M_{BPS} = 0$. $\left(M_{BPS}^2 = e^K W \bar{W} \right)$

So the attractor condition minimizes the BPS mass.

The entropy of this black hole is given by $\frac{1}{4}$ of the area of S^2 that is

$$S = \frac{1}{4} \pi M_{BPS}^2$$

This is in the semi-classical limit.

We can generalize this to all order in string perturbation theory.

(Lopes Cardoso, de Wit and Mohaupt)

$\Rightarrow$ Topological String Theory

More on topological strings

$F_g(z)$: g-loop partition functs

- A-model : depends (almost) holomorphically on the Kähler moduli
- B-model : depends (almost) holomorphically on the complex structure moduli

Type II in RNS formalism :

$$\mathcal{N}=2 \text{ SCFT } (\Sigma \to CY_3)$$

$$\otimes \quad (X^\mu, \psi^\mu : \mu = 0,1,2,3) \otimes (\text{ghosts})$$

focus on this part

$$\mathcal{N}=2 \text{ SCA} : \begin{array}{lll} T_L, & G_L^{\pm}, & J_L \\ T_R, & G_R^{\pm}, & J_R \end{array}$$

$\hat{c} = 3$: complex dim of CY_3

Topological twisting

- $T_L \longrightarrow T_L \pm \frac{1}{2} \partial J_L$
- add a coupling $\int_\Sigma d^2z \ \pm \frac{1}{2} \omega_{\bar{z}} J_L$

  $\omega_{\bar{z}}$: spin connection

- T_L : dim $(2, 0)$

  G_L^+ : dim $(1, 0)$ $\quad G_L^-$: dim $(2, 0)$

  (or G_L^- : dim $(2, 0)$ $\quad G_L^-$: dim $(1, 0)$)

  J_L : dim $(1, 0)$

  $\hat{c} = 3 \longrightarrow \hat{c} = 0$.

4 types of twisting : A, A*, B, B*

Let us choose our notation s.t.

$G_L^+ : (1, 0)$, $\quad G_R^+ : (0, 1)$

$G_L^- : (2, 0)$, $\quad G_R^- : (0, 2)$

$Q_{BRST} = \oint G_L^+ + \oint G_R^+$: topological BRST

$\{ Q_{BRST} , G_L^- \} = T_L$, $\quad \{ Q_{BRST} , G_R^- \} = T_R$

G_L^-, G_R^- behave as ghosts of bosonic string

moreover $\langle (G_L^-)^{3g-3} (G_R^-)^{3g-3} \rangle \neq 0$

if the original $\hat{c} = 3$.

Thus we can define

$$F_g = \int_{\mathcal{M}_g} \langle \prod_{i=1}^{3g-3} (\eta_i, G_L^-)(\bar{\eta}_i, G_R^-) \rangle$$

moduli space of Σ_g

η_i : Beltrami-differential

$$(\eta_i, G^-) = \int_\Sigma \eta_i G_L^-$$

In the B-model, it depends only on the complex structure moduli.

$\mathcal{L}$: line bundle over $\mathcal{M}_V$

with e^K as a metric

F_g : section of $\mathcal{L}^{2-2g}$

The fact that F_g is a section of $\mathcal{L}^{1-g}$ and not a function is a consequence of the topological twisting.

$\because$) $J_L = i\,\partial\phi$

$$\frac{1}{2}\int_{\Sigma} \bar{\omega}\, J_L + \omega\, J_R = -\frac{1}{2}\int R\,\phi$$

we can choose $R = -\sum_{i=1}^{2g-2} \delta(z-z_i)$

Thus the twisting amounts to

$(2g-2)$ insertions of $e^{\frac{i}{2}\phi}$

On the other hand

$$\Omega_{ijk}\,\psi_L^i\,\psi_L^j\,\psi_L^k = e^{-i\phi_L}$$

rescaling of $\Omega \longleftrightarrow$ shift of ϕ $\lrcorner$

If F_g were holomorphic section of $\mathcal{L}^{2-2g}$, we could express it as a weight $(2-2g)$ function of X's :

functions of z^i

$$F_g(X) = (X^0)^{2-2g} F_g\left(\frac{X^i}{X^0}\right)$$

But, there is a holomorphic anomaly :

$$\bar{\partial}_{\bar{i}} F_g = \frac{1}{2} \bar{C}_{\bar{i}\bar{j}\bar{k}}\, e^{2K} G^{j\bar{j}} G^{k\bar{k}} \left(D_j D_k F_{g-1} + \sum_{g'=1}^{g-1} D_j F_{g'} D_k F_{g-g'} \right).$$

(BCOV '93)

What this is telling us is that $\Psi = \exp\left(\sum_g F_g \lambda^{2g-2}\right)$ should be regarded as a wave function on $H_3(CY_3, \mathbb{R})$

Fix background $(z^i, \bar{z}^i) \in \mathcal{M}_V$.

Deform by x^i.

$$F_g = \int_{\mathcal{M}_g} \langle (G_L^-)^{3g-3} (G_R^-)^{3g-3} \times \exp\left[\int_{\Sigma_g} x^i G_L^- G_R^- \mathcal{O}_i \right] \rangle$$

$\mathcal{O}_i$: chiral primary $\leftrightarrow$ $x^i \in H^{2,1}$

This shifts $\frac{z^i}{z^0} \rightarrow \frac{z^i}{z^0} + x^i$

keeping $\bar{z}^I$: fixed.

$$F \equiv \sum_{g=0}^{\infty} \lambda^{2g-2} F_g (x^i ; z, \bar{z})$$

We can combine λ, x^i into extended moduli x^I.

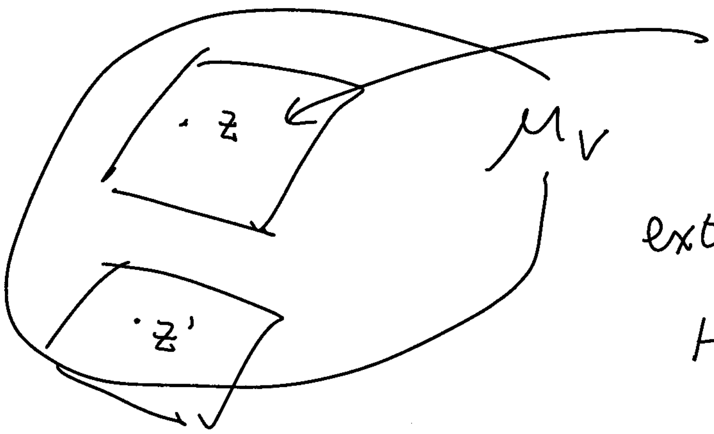
$F(x ; z, \bar{z})$ thus defined satisfies :

$$\psi_{top} = \exp\left(F(x; z, \bar{z}) \right)$$

$$\frac{\partial}{\partial \bar{z}^I} \psi_{top} = \left(\frac{1}{16\pi i} \bar{C}_I^{JK} \frac{\partial^2}{\partial x^J \partial x^K} + \frac{1}{4i} \bar{C}_{IJ}{}^J \right) \psi_{top}$$

$$\frac{\partial}{\partial z^I} \psi_{top} = \left(\frac{\partial}{\partial x^I} - \frac{i}{2} C_{IJ}{}^K x^J \frac{\partial}{\partial x^K} - i\pi C_{IJK} x^J x^K \right) \psi_{top}$$

This can be interpreted as follows. (Witten)



$T_z \mathcal{M}_V \simeq H^{2,1}(CY_3) \oplus H^{1,2}(CY_3)$

extend it to

$H^3(CY_3, \mathbb{R})$

$\Updownarrow$

chiral operator

At each z,

we can quatize H^3.

$\Rightarrow$ parallel transport between different z?

Holomorphic polarization :

$$H^3(CY_3, \mathbb{R}) = H^{3,0} \oplus H^{2,1} \oplus H^{1,2} \oplus H^{0,3}$$

w.r.t. complex structure at z.

wave function is a holomorphic function of $H^{3,0} \oplus H^{2,1}$

Going to different point $z' \in \mathcal{M}_V$

$\Rightarrow$ Canonical transformation

(~ Fourier transformation)

$\Rightarrow$ holomorphic anomaly equation as a connection formula.

Physically :

$F(x ; z, \bar{z})$ gives the prepotential for vector multiplets coupled to Weyl multiplet.

Exact Black Hole Entropy

Consider type IIB superstring on $CY_3 \times \mathbb{R}^{3,1}$

RR 4-form $\Rightarrow$ gauge field A_μ^I in 4d.

$I = 0, 1, \cdots, h^{2,1}$

D branes wrapping on 3 cycles in CY_3

q_I times on A_I

p^I times on B^I

$A_I, B^I \in H_3(CY_3)$

$A_I \cap A_J = B^I \cap B^J = 0$

$A_I \cap B^J = \delta_I{}^J$

= BPS black hole in four dimensions

with electric charges q_I , magnetic charges p^I

Conjecture (Strominger, Vafa + H.O.)

$$Z_{BH} \equiv \sum_q \Omega(p, q)\, e^{-q\phi}$$

$$= |\psi_{top}(X)|^2$$

where $X^I = p^I + \frac{i}{\pi}\phi^I$

Black hole partition function:

$$Z_{BH} = \sum_{q} \Omega(p, q)\, e^{-q \cdot \phi}$$

p : magnetic charges of the black hole

ϕ : chemical potentials for electric charges

Perturbative topological string partition function:

$$\psi_{top}^{(pert.)} = \exp\left[\sum_g F_g(X)\right]$$

$\lambda = 4\pi i / X^0$: string coupling constant

X^i / X^0 : complex structure of CY_3

Black Hole Charges <==> Calabi-Yau Moduli.

$$X^I = p^I + \frac{i}{\pi} \phi^I \qquad (I = 0, 1, \cdots, h^{2,1})$$

$$Z_{BH}(p, \phi) = |\psi_{top}(X)|^2$$

Why?

The perturbative topological string amplitudes give low energy effective action terms.

BCOV/1994
AGNT

It turns out that these are the terms that are relevant in computing perturbative string corrections to the Bekenstein-Hawking entropy formula.

Lopez-Cardoso, de Wit + Mohaupt/1998-99

Define $$\mathcal{F} = \log |\psi_{top}^{(pert)}(X)|^2$$

(X^I : $\mathcal{N}=2$ chiral superfield)

Black Hole Attractor Eguations:

$$X^I = p^I + \frac{i}{\pi}\phi^I, \quad q_I = \frac{\partial}{\partial\phi^I}\mathcal{F}(p,\phi)$$

Black Hole Entropy (all order in string perturbation):

$$S_{BH}(p,\phi) = q_I\phi^I + \mathcal{F}(p,\phi)$$

This is the Legendre transformation:

$$q \longleftrightarrow \phi$$

Entropy and Wave Function

The OSV conjecture can be inverted as:

$$\Omega(p, q) = \int d\phi \, |\psi_{p,q}(\phi)|^2$$

where

$$\psi_{p,q}(\phi) \equiv e^{-\frac{1}{2} q_I \phi^I - \frac{1}{2} p^I \tilde{\phi}_I} \, \psi_{top}(\phi)$$

$$\left(\tilde{\phi}_I = -\frac{2i}{\pi} \frac{\partial}{\partial \phi^I} \right)$$

$e^{-\frac{1}{2} q_I \phi^I - \frac{1}{2} p^I \tilde{\phi}_I}$: flux changing operator

$\psi_{top}(\phi)$: topological string partition function

If the AdS2/CFT1 makes sense, one can interpret Ω as

(1) $\Omega(p, q) = \mathrm{Tr}_{H_{pq}} (-1)^F \lambda^2 e^{-\beta H}$

in CFT_1

AND

(2) type IIB string partition function on AdS2 x S2 x CY3, euclideanized & periodically identified.

near horizon geometry

Near Horizon Geometry of the Black Hole

$$AdS_2 \times S^2 \times CY_3$$

RR 4-form $\Rightarrow$ 4d gauge field $A_\mu{}^I$

on $AdS_2 \times S^2$

electric charge q_I , magnetic charge p^I

$\Rightarrow$ 5-form flux F_5

Flux Compactification on $S^2 \times CY_3$

GVW superpotential

$$W = \int_{S^2 \times CY_3} F_5 \wedge \Omega$$

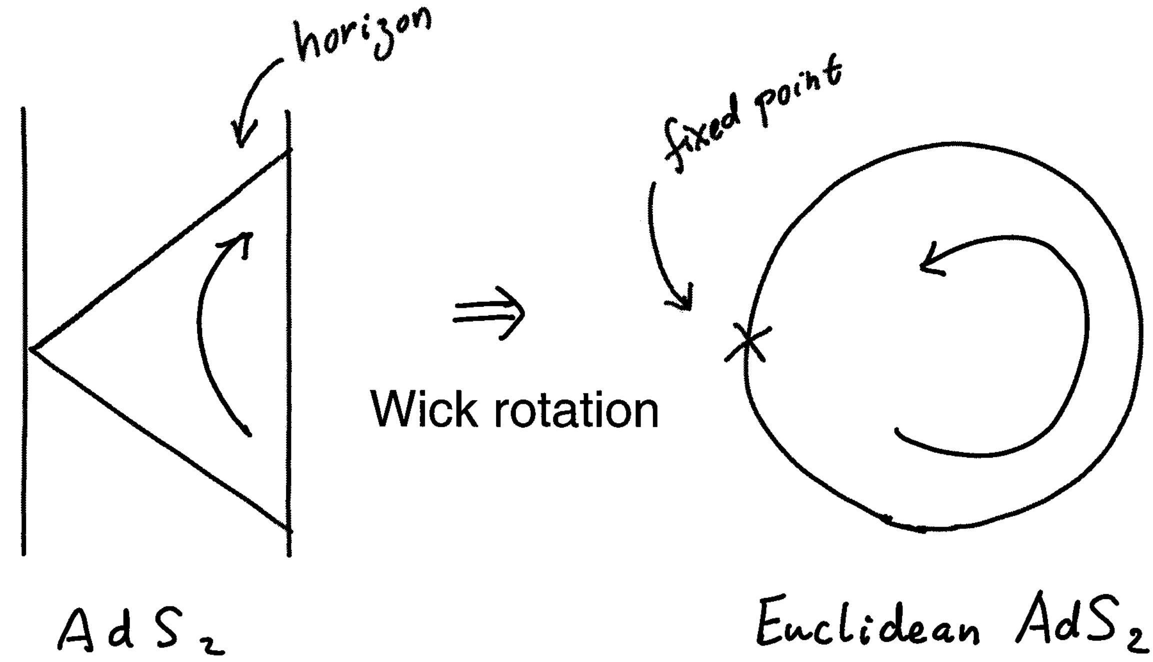
holomorphic 3 form on CY_3

$$= q_I X^I - p^I \frac{\partial}{\partial X^I} F_0(X)$$

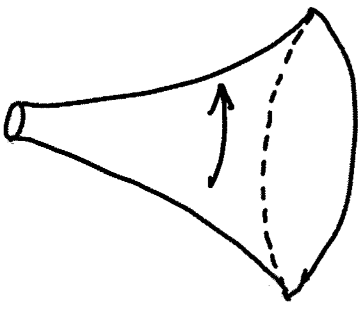
$$dW = 0 \Rightarrow \quad \mathrm{Re}\, X^I = p^I$$

$$\mathrm{Re}\, \partial_I F_0 = q^I$$

Classical Attractor



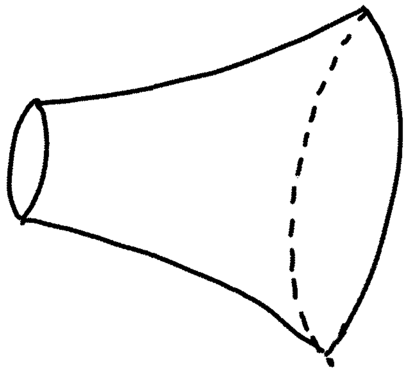
$$\Omega(p,q) = \mathrm{Tr}_{\mathcal{H}_{pq}} (-1)^F \lambda^2 e^{-\beta H}$$

$=$ 

$$= \int d\phi \, |\psi_{pq}(\phi)|^2$$

This suggests that the topologial string partition function is the Hartle-Hawking type wave-function for AdS_2 , Euclideanized and periodically identified.

$$\psi_{p\delta}(x) = x$$



This is similar in sprit to the earlier suggestion by de Boer, H. Verlinde, and V. Verlinde on the relation between the Weeler-de Witt equation and the renormalization group equation in the context of the AdS/CFT correspondence.

One of the new features is that our wave-function takes into account quantum corrections to all orders in the string perturbation theory.

In the following, I will show how the OSV conjecture works in an explicit example.

The example is based on the A-model, so we will take the mirror of the story.

Consider D branes in type IIA string theory wrapping 0, 2, 4, and 6 cycles of a Calabi-Yau 3-fold.

q_0 = # of D_0 branes

q_i = # of D_2 branes $\qquad i = 1, 2, \cdots, h^{1,1}$

p^i = # of D_4 branes

p^0 = # of D_6 branes

Classical attractor equations:
$$\mathrm{Re}\, X^I = p^I$$
$$\mathrm{Re}\, \partial_I F_0 = q_I$$

$\lambda = \dfrac{4\pi i}{X^0}$: topological string coupling

$t^i = \dfrac{X^i}{X^0}$: Kähler moduli

$$CY_3 : \mathcal{O}(-p) \oplus \mathcal{O}(p+2g-2) \to \Sigma_g$$

Two line bundles of degrees *-p* and *p+2g-2* over a genus *g* Riemann surface

The total space is a Calabi-Yau manifold.

$Z^{top}_{pert.}$ Topological string amplitudes on this CY was recently computed to all order in the perturbative expansion.

Bryan + Pandharipande/2004

Z_{BH} Consider *N* D4 branes on the total space of the degree *-p* bundle over the Riemann surface.

$$\mathcal{O}(-p) \to \Sigma_g$$

$$X^0 = \frac{4\pi i}{\lambda} \quad (\text{no } D_6 \text{ charge})$$

$$X^1 = \underbrace{(p+2g-2)N}_{D_4 \text{ charge}} + \frac{i}{\pi}\phi^1$$

We want to compute the partition function of the N=4 super Yang-Mills on this 4-manifold.

$\mathcal{N}=4$ SYM on $\mathcal{O}(-p) \to \Sigma_g$

$$S = \frac{i}{2\lambda} \int \mathrm{tr} \left(F \wedge F + 2\theta\, F \wedge k \right)$$

k : Kähler form on Σ_g

$\Downarrow$ Equivariant reduction on the fiber with respect to the $U(1)$ rotation.

2d YM with q-deformation

$$S_{qYM} = \frac{i}{\lambda} \int_{\Sigma_g} \mathrm{tr} \left(\Phi F + \theta \Phi + i \frac{p}{2} \Phi^2 \right)$$

q-deformed since Φ is periodic.

($e^{\Phi} = e^{\oint A}$ is a good variable.)

$$q = e^{-\lambda}$$

2d q-YM partition function

$$q = e^{-\lambda}$$

$$Z_{BH} = S^{2-2g} \sum_{R} (\dim_q R)^{2-2g} \, q^{\frac{p}{2} C_2} e^{i\theta C_1}$$

$$R = [R_1, \cdots, R_N] \ : \ \text{representation of } U(N)$$

$$C_2 = \sum_i R_i (R_i - 2i + 1) + N R_i$$

$$C_1 = \sum_i R_i$$

$$\dim_q R = \prod_{i<j} \frac{[R_i - R_j - i + j]_q}{[i-j]_q}$$

$$\left(\text{where } [m]_q = q^{m/2} - q^{-m/2} \right)$$

$$S = q^{\rho^2} \prod_{i<j} [i-j]_q \ , \quad \vec{\rho} \ : \ \text{Weyl vector}$$

The chemical potentials are given by

$$\phi^0 = \frac{4\pi^2}{\lambda} \ , \quad \phi^1 = \frac{2\pi\theta}{\lambda}$$

The above expansion is not in the form expected for Z_{BH}

Z_{BH} can be brought into the form expected from the black hole state counting as well as from the instanton expansion of the N=4 SYM:

$$Z_{BH} = \sum_{q} \Omega(p,q)\, e^{-q\cdot\phi}$$

by the S-duality transformation, $\lambda \to \frac{1}{\lambda}$.

The resulting expression reproduces mathematical facts about cohomologies on the moduli space of U(N) instantons on the 4-manifold: $\mathcal{O}(-p) \to \Sigma_g$

For $\mathcal{O}(-2) \to \mathbb{P}^1$,
the cohomologies of the moduli space of of U(N) instantons make representations of the affine SU(2) Lie algebra of level N. (Nakajima)

Thus, Z_{BH} should be expressed in terms of the SU(2) affine Lie algebra characters and indeed it does.

Z_{BH} also reproduce the blow-up formula in the case of $\mathcal{O}(-1) \to \mathbb{P}^1$. (Yoshioka)

Large N Limit

$$Z_{BH} = S^{2-2g} \sum_{R} (\dim_q R)^{2-2g} \; q^{\frac{p}{2} C_2} e^{i\theta C_1}$$

$$\sim \sum_{m=-\infty}^{\infty} \sum_{R_i} Z_{R_1 \cdots R_{|2g-2|}}(\lambda ; t + p\lambda m)$$

$$\times \bar{Z}_{R_1 \cdots R_{|2g-2|}}(\lambda ; \bar{t} - p\lambda m)$$

$$Z_{R_1 \cdots R_{2g-2}}(\lambda ; t) \qquad g>1 \text{ case}$$

$$= C(t) \sum_{R} q^{\frac{1}{2}(p+2g-2)\kappa_R} e^{-t|R|}$$

$$\times \frac{W_{RR_1} \cdots W_{RR_{2g-2}}}{(W_{R0})^{4g-4}}$$

where

$$W_{R_1 R_2}(q) = \lim_{N \to \infty} S_{R_1 R_2}(q, N)$$

modular S-matrix of the WZW model

The chiral components:

$$Z(U_1, \cdots, U_{2g-2})$$

$$= \sum_{R_1, \cdots, R_{2g-2}} Z_{R_1 \cdots R_{2g-2}} \times \mathrm{Tr}_{R_1} U_1 \cdots \mathrm{Tr}_{R_{2g-2}} U_{2g-2}$$

are topological string amplitudes for the Calabi-Yau with (2g-2) D branes wrapping Lagrangian 3-cycles.

$U_1, \cdots, U_{2g-2}$ are holonomies on the D branes.

These D branes correspond to the Omega points in the ordinary (undeformed) 2d YM.

Summary

The mixed ensemble of BPS black holes gives a non-perturbative definition of topological string theory.

This is a large N duality relating the worldvolume theory of D3 branes to topological string theory.

The topological string partition function has a mathematical interpretation as a quantization of the third cohomology of a Calabi-Yau 3-fold.

The relation to the black hole entropy shows that it has a physical interpretation as the Hartle-Hawking wave-function, including all string loop corrections.

We studied the case when the Calabi-Yau manifold is the total space of the rank 2 vector bundle. The worldvolume theory of D4 branes wrapping one of the line bundles is related to the q-deformed Yang-Milles theory on the base Riemann surface.

The large N limit of the gauge theory partition function is holomorphically factorized, and the chiral blocks are interepreted in terms of the perturbative topological string theory.

Fin