The Abdus Salam
International Centre for Theoretical Physics

United Nations
Educational, Scientific
and Cultural Organization

International Atomic
Energy Agency

SMR. 1649 - 10

*SPRING SCHOOL ON SUPERSTRING THEORY*
*AND RELATED TOPICS*

*14 - 22 March 2005*

# Black Holes and Elementary String States

**A. SEN**
**Harish-Chandra Research Institute**
**Chhatnag Road, Jhunsi**
**211 019 Allahabad**
**INDIA**

**Please note: These are preliminary notes intended for internal distribution only.**

Strada Costiera 11, 34014 Trieste, Italy - Tel. +39 040 2240 111; Fax +39 040 224 163 - sci_info@ictp.it, www.ictp.it

# Black Holes and Elementary String States

Some references

A.S., hep-th/9504147, 9712150

Cardoso, de Wit, Mohaupt, hep-th/9812082, 9906094

Cardoso, de Wit, Kappeli, Mohaupt, hep-th/0009234, 0012232

Mohaupt, hep-th/0007195

Ooguri, Strominger, Vafa, hep-th/0405146

Dabholkar, hep-th/0409148

Dabholkar, Kallosh, Maloney, hep-th/0410076

A.S. hep-th/0411255, 0502126

Hubeny, Maloney, Rangamani, hep-th/0411272

Dabholkar, Denef, Moore, Pioline, hep-th/0502157

Motivation:

Quantization of a relativistic string gives an infinite tower of massive states.

The degeneracy of these states grow rapidly with mass.

Thus it seems natural to define a 'statistical entropy' associated with elementary string states:

$$S_{stat}(M, \vec{Q}) = \ln d(M, \vec{Q})$$

$d(M, \vec{Q}) =$ degeneracy of elementary string states with a given mass $M$ and charges $\vec{Q} = (Q_1, Q_2, \ldots)$.

Since string theory includes gravity, one might expect that a string of very large mass behaves like a black hole.

One can assign an 'entropy' to these black holes via the Bekenstein-Hawking formula:

$$S_{BH}(M, \vec{Q}) = A/(4G_N)$$

$A$: area of the event horizon

$G_N$: Newton's constant

Question: Do these two different ways calculating entropy of an elementary string agree?

't Hooft, Susskind

Is $\quad S_{stat}(M, \vec{Q}) = S_{BH}(M, \vec{Q})$?

In order to make a meaningful comparison we must ensure that the parameters $M$ and $\vec{Q}$ appearing in the arguments of $S_{stat}$ and $S_{BH}$ refer to the physical mass and charge.

Usually $S_{BH}$ is computed as a function of the physical mass (ADM mass).

But $S_{stat}$ is calculated as a function of the tree level mass of the elementary string state.

The physical mass is related to the tree level mass via a large but finite renormalization effect. Susskind

Due to this renormalization effect it is difficult to figure out how $S_{stat}$ depends on the physical mass of the black hole.

This makes the comparison of $S_{stat}$ and $S_{BH}$ difficult.

In supersymmetric theories, this problem in principle can be avoided by considering BPS states.

For these states the mass is determined in terms of the charges carried by the state.

Furthermore BPS states remain BPS as we vary the coupling constant.

As a result tree level result for $S_{stat}$, calculated as a function of the quantized charges, gives $S_{stat}$ as a function of the charges at arbitrary value of the coupling constant.

Consider heterotic string theory compactified on $T^5 \times S^1$.

Use $\alpha' = 16$ unit

$R$: radius of $S^1$ in string metric

$g$: string coupling constant

coordinate radius of $S^1 = \sqrt{\alpha'} = 4$

The spectrum of tree level heterotic string theory is generated by 24 sets of left-moving bosonic oscillators $\bar{\alpha}^I_{-n}$, 8 sets of right moving bosonic oscillators $\alpha^i_{-n}$ and 8 sets of right moving fermionic oscillators $\psi^i_{-n}$

$$1 \le I \le 24, \qquad 1 \le i \le 8, \qquad 1 \le n < \infty$$

A generic state:

$$\bar{\alpha}^{I_1}_{-n_1} \ldots \bar{\alpha}^{I_s}_{-n_s} \alpha^{i_1}_{-m_1} \ldots \alpha^{i_r}_{-m_r} \psi^{j_1}_{-p_1} \ldots \psi^{j_t}_{-p_t} |\vec{Q}\rangle$$

$\vec{Q}$: labels the momentum, winding number and various gauge charges carried by the state.

Define

$$N_L = \sum_{k=1}^{s} n_k, \qquad N_R = \sum_{k=1}^{r} m_k + \sum_{k=1}^{t} p_k$$

Consider an elementary string state wound $w$ times along $S^1$ and carrying $n$ units of momentum along $S^1$.

Suppose the string has level $N_L$ left-moving oscillator excitations and level $N_R$ right-moving oscillator excitations.

$m$: Mass of the string measured in the canonical Einstein metric

$$\begin{aligned} m^2 &= g^2\left[\left(\frac{n}{R}+\frac{wR}{16}\right)^2+4\left(N_R-\frac{1}{2}\right)\right] \\ &= g^2\left[\left(\frac{n}{R}-\frac{wR}{16}\right)^2+4\left(N_L-1\right)\right] \end{aligned}$$

This formula is valid for bosonic states, but due to supersymmetry for every bosonic state there will be a fermionic state with the same mass and charge.

$$
\begin{aligned}
m^2 &= g^2\left[\left(\frac{n}{R}+\frac{wR}{16}\right)^2+\frac{1}{4}\left(N_R-\frac{1}{2}\right)\right] \\
&= g^2\left[\left(\frac{n}{R}-\frac{wR}{16}\right)^2+\frac{1}{4}\left(N_L-1\right)\right]
\end{aligned}
$$

$N_R = \frac{1}{2} \rightarrow$ BPS states

These states are invariant under half of the space-time supersymmetry transformations.

$\rightarrow N_L = 1 + nw$

Thus $nw \geq -1$ for BPS states.

The degeneracy $d_{nw}$ for these states can be calculated by knowing the number of different ways we can get level $N_L = nw + 1$ excitations.

Formula for $d_{nw}$:

$$\sum_{N=0}^{\infty} d_{N-1} q^N = 16 \prod_{n=1}^{\infty} (1 - q^n)^{-24}$$

For large $nw$:

$$d_{nw} \sim \exp(4\pi\sqrt{nw})$$

Dabholkar, Harvey

Goal:

1) Construct the BPS black hole solution carrying the same mass and charge quantum numbers as the elementary string state.

2) Calculate its Bekenstein-Hawking entropy and compare with $\ln d_{nw}$.

To carry out this goal we begin by writing down the low energy effective field theory describing heterotic string theory on $T^5 \times S^1$.

Relevant massless fields in ten dimension:

$G^{(10)}_{MN}$, $B^{(10)}_{MN}$ and $\Phi^{(10)}$, $\quad 0 \le M, N \le 9$

The dynamics of this theory is described by $N = 1$ supergravity theory in (9+1) dimensions.

$$\text{Action} \quad = \quad S\left(G^{(10)}_{MN}, B^{(10)}_{MN}, \Phi^{(10)}, \cdots\right)$$

$\cdots$ denote other bosonic and fermionic fields which will be set to zero in our analysis of classical solution describing the elementary string state.

$x^\mu$: non-compact directions ($0 \le \mu \le 3$)

$x^4$: coordinate along $S^1$

For constructing the black hole solution describing the elementary string described above, we need non-trivial:

$G^{(10)}_{\mu\nu}$, $B^{(10)}_{\mu\nu}$, $G^{(10)}_{4\mu}$, $G^{(10)}_{44}$, $B^{(10)}_{4\mu}$, $\Phi^{(10)}$

All other fields are set to zero.

Furthermore we take all fields to be independent of the compact directions.

Define 'four dimensional fields'

$$\Phi = \Phi^{(10)} - \frac{1}{2}\ln(G_{44}^{(10)}),$$

$$S = e^{-\Phi}, \qquad T = \sqrt{G_{44}^{(10)}},$$

$$G_{\mu\nu} = G_{\mu\nu}^{(10)} - (G_{44}^{(10)})^{-1} G_{4\mu}^{(10)} G_{4\nu}^{(10)},$$

$$g_{\mu\nu} = e^{-\Phi} G_{\mu\nu},$$

$$A_\mu^{(1)} = \frac{1}{2}(G_{44}^{(10)})^{-1} G_{4\mu}^{(10)},$$

$$A_\mu^{(2)} = \frac{1}{2} B_{4\mu}^{(10)},$$

$$B_{\mu\nu} = B_{\mu\nu}^{(10)} - 2(A_\mu^{(1)} A_\nu^{(2)} - A_\nu^{(1)} A_\mu^{(2)}).$$

$G_{\mu\nu}$: string metric

$g_{\mu\nu}$: canonical metric

The low energy effective action is given by:

$$\begin{aligned}\mathcal{S} \;=\; & \frac{1}{32\pi}\int d^4x\,\sqrt{-\det g}\,\Big[R_g - \frac{1}{2S^2}\,g^{\mu\nu}\,\partial_\mu S\partial_\nu S \\ & -\frac{1}{T^2}\,g^{\mu\nu}\,\partial_\mu T\partial_\nu T \\ & -\frac{1}{12}S^2 g^{\mu\mu'}g^{\nu\nu'}g^{\rho\rho'}H_{\mu\nu\rho}H_{\mu'\nu'\rho'} \\ & -ST^2\,g^{\mu\nu}\,g^{\mu'\nu'}\,F^{(1)}_{\mu\mu'}F^{(1)}_{\nu\nu'} \\ & -ST^{-2}\,g^{\mu\nu}\,g^{\mu'\nu'}\,F^{(2)}_{\mu\mu'}F^{(2)}_{\nu\nu'}\Big]\,,\end{aligned}$$

where

$$F^{(a)}_{\mu\nu} = \partial_\mu A^{(a)}_\nu - \partial_\nu A^{(a)}_\mu\,, \qquad a = 1,2\,,$$

$$\begin{aligned}H_{\mu\nu\rho} = & \Big[\partial_\mu B_{\nu\rho} + 2\left(A^{(1)}_\mu F^{(2)}_{\nu\rho} + A^{(2)}_\mu F^{(1)}_{\nu\rho}\right)\Big] \\ & +\text{cyclic permutations of } \mu,\ \nu,\ \rho\,.\end{aligned}$$

In this normalization convention the Newton's constant is given by

$$G_N = 2\,.$$

Note: The action is completely universal without any parameter.

Expectation values of $S$ and $T$ determine the string coupling constant and the radius of $S^1$.

$$\langle S \rangle = \frac{1}{g^2}, \qquad \langle T \rangle = R\sqrt{\alpha'} = \frac{R}{4}$$

We now want to construct an extremal black hole solution satisfying the following properties:

- It should have the same mass and charge as the elementary string state carrying quantum numbers $(n, w)$.

- It should be a solution of the classical field equations.

- It should be invariant under half of the space-time supersymmetry transformations.

- Asymptotically $S \to \frac{1}{g^2}$, $T \to \frac{R}{4}$.

The extremal black hole solution:

$$\begin{aligned}
ds_c^2 &\equiv g_{\mu\nu}dx^\mu dx^\nu \\
&= -(F(\rho))^{-1/2}\rho dt^2 \\
&\quad +(F(\rho))^{1/2}\,\rho^{-1}\left(d\rho^2+\rho^2 d\Omega_2^2\right)\,, \\
F(\rho) &= (\rho+gwR/2)(\rho+8gnR^{-1})\,, \\
d\Omega_2^2 &\equiv d\theta^2+\sin^2\theta\, d\phi^2\,, \\
S &= g^{-2}\,(F(\rho))^{1/2}\,\rho^{-1}\,, \\
T &= \frac{1}{4}R\sqrt{(\rho+8gnR^{-1})/(\rho+gwR/2)}\,, \\
F^{(1)}_{\rho t} &= \frac{16g^2R^{-2}n}{(\rho+8gnR^{-1})^2}\,, \\
F^{(2)}_{\rho t} &= \frac{1}{16}\frac{g^2wR^2}{(\rho+gwR/2)^2}\,, \\
H_{\mu\nu\rho} &= 0\,, \\
ds^2_{string} &\equiv G_{\mu\nu}dx^\mu dx^\nu = S^{-1}ds_c^2 \\
&= -g^2\,\rho^2\,(F(\rho))^{-1}dt^2+g^2\,d\vec{x}^2\,.
\end{aligned}$$

'Horizon' is at $\rho=0$.

Area of the horizon $=0$

Naively this would imply that the Bekenstein-Hawking entropy of the black hole is zero!

However let us not give up immediately and study the solution in some detail.

We shall be interested in the limit of large $n$ and $w$ at fixed $\rho$.

In this limit

$$\rho << gwR/2, \quad 8gnR^{-1}$$

Define:

$$r = g\,\rho, \qquad \tau = g^{-1}\,t/\sqrt{nw}$$

In this coordinate system the solution for large $n, w$ takes the form:

$$\begin{aligned}
ds^2_{string} &= -\frac{r^2}{4}\,d\tau^2 + dr^2 + r^2\,d\Omega_2^2 \\
S &= \frac{2\sqrt{nw}}{r}, \\
T &= \sqrt{\frac{n}{w}}, \\
F^{(1)}_{r\tau} &= \frac{1}{4}\sqrt{\frac{w}{n}} \\
, \\
F^{(2)}_{r\tau} &= \frac{1}{4}\sqrt{\frac{n}{w}}.
\end{aligned}$$

Curvatures in string metric are small for $r >> 1$ but of order 1 for $r \sim 1$.

$\rightarrow$ higher derivative terms become important for $r \sim 1$.

We see that $S \sim \sqrt{nw}$ for $r \sim 1$

$S =$ Inverse string coupling$^2$

$\rightarrow$ string coupling $\sim (nw)^{-1/4}$ for $r \sim 1$.

Thus for large $nw$ we can ignore string loop corrections to the effective action. AS

The relevant corrections come from higher derivative terms in the effective action appearing at string tree level.

In order to study the effect of these higher derivative corrections, we shall now try to simplify the solution further by using some exact symmetries of the string tree level effective action.

The effective action at string tree level has an exact symmetry:

$$G_{44}^{(10)} \to e^{2\beta} G_{44}^{(10)}, \qquad G_{4\mu}^{(10)} \to e^{\beta} G_{4\mu}^{(10)},$$
$$B_{4\mu}^{(10)} \to e^{\beta} B_{4\mu}^{(10)}$$

This corresponds to changing the radius of $S^1$.

In terms of four dimensional fields this becomes:

$$T \to e^{\beta} T, \qquad A_\mu^{(1)} \to e^{-\beta} A_\mu^{(1)},$$
$$A_\mu^{(2)} \to e^{\beta} A_\mu^{(2)}$$

Two solutions related by this transformation has the same entropy.

Choosing $e^{\beta} = \sqrt{w/n}$ we can map the original solution to the 'checked solution':

$$\begin{aligned}
\check{ds}^2_{string} &= -\frac{r^2}{4} d\tau^2 + dr^2 + r^2 d\Omega_2^2 \\
\check{S} &= \frac{2\sqrt{nw}}{r}, \\
\check{T} &= 1, \\
\check{F}^{(1)}_{r\tau} &= \frac{1}{4}, \\
\check{F}^{(2)}_{r\tau} &= \frac{1}{4}.
\end{aligned}$$

As before, the form of the solution is expected to change near $r \sim 1$ by higher derivative corrections.

Under another transformation

$$S \to K\,S, \quad T \to T, \quad G_{\mu\nu} \to G_{\mu\nu}, \quad F^{(a)}_{\mu\nu} \to F^{(a)}_{\mu\nu}$$

the tree level effective action gets multiplied by $K$.

This leaves the equations of motion unchanged.

This transformation also multiplies the entropy associated with a solution by a factor of $K$.

Choosing $K = 1/\sqrt{nw}$ we can map the checked solution to the 'hatted solution':

$$
\begin{aligned}
\widehat{ds}^2_{string} &= -\frac{r^2}{4}d\tau^2 + dr^2 + r^2 d\Omega_2^2 \\
\widehat{S} &= \frac{2}{r}, \\
\widehat{T} &= 1, \\
\widehat{F}^{(1)}_{r\tau} &= \frac{1}{4}, \\
\widehat{F}^{(2)}_{r\tau} &= \frac{1}{4},
\end{aligned}
$$

Note that this solution is completely universal, independent of any external parameter.

Thus its modification near $r \sim 1$ by higher derivative corrections will also be completely universal.

Modified form of the hatted solution by higher derivative terms:

$$
\begin{aligned}
\widehat{ds}^2_{string} &= -\frac{f_1(r)}{f_3(r)}\,d\tau^2 + \frac{f_2(r)}{f_3(r)}\,(dr^2 + r^2 d\Omega_2^2)\\
\widehat{S} &= f_3(r)\,,\\
\widehat{T} &= f_4(r)\,,\\
\widehat{F}^{(1)}_{r\tau} &= f_5(r)\,,\\
\widehat{F}^{(2)}_{r\tau} &= f_6(r)\,,\\
\widehat{ds}^2_c &= -f_1(r)\,d\tau^2 + f_2(r)\,(dr^2 + r^2 d\Omega_2^2)
\end{aligned}
$$

$f_i(r)$: universal functions

The entropy computed from this modified solution is also going to be a purely numerical constant $a$.

Naively,

$$
a = \frac{A_{horizon}}{4G_N} = \frac{1}{8}\,4\pi\,\lim_{r\to 0}(r^2 f_2(r))\,.
$$

We can now make inverse transformations to go back to the checked and then to the original solution.

The original solution:

$$
\begin{aligned}
ds^2_{string} &= -\frac{f_1(r)}{f_3(r)}\, d\tau^2 + \frac{f_2(r)}{f_3(r)}\,(dr^2 + r^2 d\Omega_2^2) \\
S &= \sqrt{nw}\, f_3(r)\,, \\
T &= \sqrt{\frac{n}{w}}\, f_4(r)\,, \\
F^{(1)}_{r\tau} &= \sqrt{\frac{w}{n}}\, f_5(r)\,, \\
F^{(2)}_{r\tau} &= \sqrt{\frac{n}{w}}\, f_6(r)\,.
\end{aligned}
$$

It is also easy to calculate the entropies associated with the checked and the original solutions in terms of $a$.

The entropy associated with the checked solution

$= \sqrt{nw} \times$ the entropy associated with the hatted solution

$= a\sqrt{nw}$

The entropy $S_{BH}$ associated with the original solution

$=$ entropy associated with the checked solution

$= a\sqrt{nw}$

$$S_{BH} = a\sqrt{nw}$$

On the other hand

$$S_{stat} \equiv \ln d_{nw} \simeq 4\pi\sqrt{nw}$$

for large $nw$.

Thus we see that $S_{BH}$ and $S_{stat}$ has same dependence on $n$, $w$, $g$ and $R$. (AS)

Q. Can we compute $a$?

## A brief history of subsequent developments

1. Strominger and Vafa computed the statistical entropy $S_{stat}$ of BPS black holes in five dimensions carrying three different types of charges by describing them as configurations of D-branes.

The corresponding black hole solutions have finite area event horizon and hence finite Bekenstein-Hawking entropy $S_{BH}$ even without taking into account $\alpha'$ corrections.

In the limit of large charges

$$S_{BH} = S_{stat}$$

2. This result was generalized to many other examples including black holes in four dimensional heterotic string theories carrying both electric and magnetic charges, in the limit where all charges are large.

3. For a special class of these four dimensional black holes, Maldacena, Strominger, Witten computed the subleading (in 1/charges) corrections to $S_{stat}$.

4. For these black holes, subleading corrections to $S_{BH}$ were computed by Cardoso, de Wit, Mohaupt + Kapelli by taking into account a special class of higher derivative terms in the action.

It was found that including these subleading corrections we get

$$S_{BH} = S_{stat}$$

In computing the subleading corrections to $S_{BH}$ we had to take into account modification of the Bekenstein-Hawking formula due to Wald.

5. Given the expression for the entropy of the black hole as a function of electric and magnetic charges, we can now set the magnetic charges to zero to compute entropy of black holes carrying electric charges $(n, w)$.

In the leading approximation the answer vanishes.

However the full expression including the subleading corrections do not vanish.

Result for heterotic string wound on $S^1$:

$$S_{BH} = 4\pi\sqrt{nw} \qquad \rightarrow \qquad a = 4\pi$$

$\rightarrow$ Exact agreement with $S_{stat}$. Dabholkar

Instead of reviewing the detailed analysis we shall now give a brief outline of the steps which are involved in the computation of $a$.

(Lopes Cardoso, de Wit, Kappeli, Mohaupt; Dabholkar; AS; Hubeny, Maloney, Rangamani)

Tree level heterotic string effective action contains a term

$$\frac{1}{16\pi}\int d^4x\,\sqrt{-\det g}\,S\,R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$$

Supersymmetrization of this term gives many other terms.

These constitute a special class of higher derivative terms which are ‘holomorphic’.

The analysis leading to the computation of $a$ takes into account only these higher derivative corrections to the effective action.

Given this modified action we proceed as follows in order to determine the modified solution describing the heterotic string configuration carrying charges $(n, w)$.

1. First we note that the modified solution describing the heterotic string configuration under study must satisfy the modified field equations derived from the new action.

2. We can also use the fact that we are trying to describe a BPS state that is invariant under a certain set of space-time supersymmetry transformations.

As a result the field configuration describing this state must also be invariant under these space-time supersymmetry transformations.

These give constraints on the field configurations.

3. Boundary condition: At large distance where higher derivative corrections are negligible, the solution must approach the leading order solution found earlier.

Now recall the general form of the 'hatted solution':

$$\begin{aligned}
\widehat{ds}^2_{string} &= -\frac{f_1(r)}{f_3(r)}\,d\tau^2 + \frac{f_2(r)}{f_3(r)}\,(dr^2 + r^2 d\Omega_2^2) \\
\widehat{S} &= f_3(r)\,, \\
\widehat{T} &= f_4(r)\,, \\
\widehat{F}^{(1)}_{r\tau} &= f_5(r)\,, \\
\widehat{F}^{(2)}_{r\tau} &= f_6(r)\,.
\end{aligned}$$

Substitute these into the field equations / BPS conditions.

This gives constraints on $f_1, \ldots f_6$.

For large $r$ the solution must match the solution of the low energy effective field theory found earlier.

This gives, for large $r$,

$$f_1(r) \simeq \frac{r}{2}, \qquad f_2(r) \simeq \frac{2}{r}$$

$$f_3(r) \simeq \frac{2}{r}, \qquad f_4(r) \simeq 1$$

$$f_5(r) \simeq \frac{1}{4}, \qquad f_6(r) \simeq \frac{1}{4}$$

Define $h(r) = \ln(f_1(r))$.

Then the constraints on $f_1, \dots f_6$ may be expressed as:

$$\begin{aligned}
f_1(r) &= e^{h(r)}\,, \\
f_2(r) &= e^{-h(r)}\,, \\
f_3(r) &= \frac{2}{r}\frac{1}{\sqrt{1+4\left(h'(r)\right)^2}}\,, \\
f_4(r) &= \frac{1}{\sqrt{1+4\left(h'(r)\right)^2}}\,, \\
f_5(r) &= \frac{1}{2}\partial_r\left(e^{h(r)}\sqrt{1+4\left(h'(r)\right)^2}\right)\,, \\
f_6(r) &= \frac{1}{2}\partial_r\left(e^{h(r)}\sqrt{1+4\left(h'(r)\right)^2}\right)\,.
\end{aligned}$$

$h$ satisfies the differential equation:

$$\begin{aligned}
& h'\left(1+4\left(h'\right)^2\right) + r\,h'' \\
= \; & \frac{r^2}{8}e^{-h}\left(1+4\left(h'\right)^2\right)^{3/2} - \frac{r}{4}\left(1+4\left(h'\right)^2\right)\,.
\end{aligned}$$

At large $r$ the equation for $h$ admits a solution:

$$h = \ln \frac{r}{2}$$

$f_1, \ldots f_6$ calculated from this gives us back the supergravity results.

For small $r$ the equation for $h$ admits a solution:

$$h = 2 \ln \frac{r}{2}$$

Thus $f_2(r) = e^{-h} = 4/r^2$

This gives the naive entropy associated with the hatted solution:

$$a = \frac{\pi}{2} \lim_{r \to 0} (r^2 f_2(r)) = 2\pi$$

$\rightarrow$ finite area of the event horizon but wrong answer!

However due to higher derivative terms in the action the Bekenstein-Hawking formula itself gets modified (Wald)

Wald's formula for spherically symmetric black holes:

$$S_{BH} = 2\pi \int_H d\theta \, d\phi \, \frac{\partial \mathcal{L}}{\partial R_{\mu\nu\rho\sigma}} \, \epsilon^{\mu\nu} \, \epsilon^{\rho\sigma} \sqrt{\det h}$$

$H$: the horizon

$\mathcal{L}$: Lagrangian density

$\epsilon^{rt} = -\epsilon^{tr} = \sqrt{-g^{rr} g^{tt}}$

$\epsilon^{\mu\nu} = 0$ otherwise

$h_{\alpha\beta}$: Metric along the horizon

Using this formula we get:

$$a = 4\pi$$

in exact agreement with the statistical entropy.

However our analysis is not complete yet.

$h(r)$ satisfies a second order differential equation.

It admits a solution $h = \ln(r/2)$ for large $r$ that gives the correct asymptotic behaviour.

It admits a solution $h = 2\ln(r/2)$ at small $r$ that gives the correct entropy.

However a second order differential equation has two integration constants.

Thus there is no guarantee *a priori* that a solution that has the small $r$ behaviour $h = 2\ln(r/2)$ will approach the asymptotic form $h = \ln(r/2)$ at large $r$.

Study small fluctuations about the solution $h = \ln(r/2)$ at large $r$.

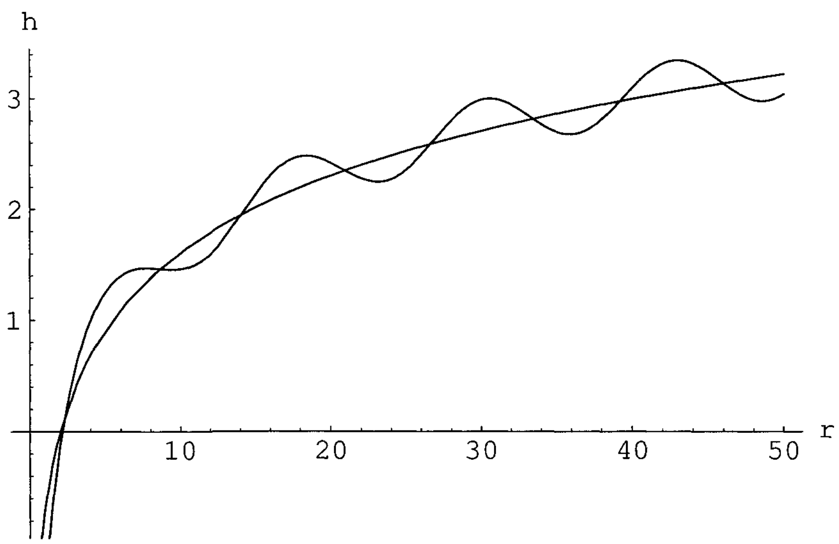
Result:

$$h \simeq \ln\frac{r}{2} + A\cos\left(\frac{r}{2} + B\right) + \mathcal{O}(A^2)$$

$A$, $B$: integration constants

Thus for a generic initial condition we expect the solution to oscillate about $h = \ln(r/2)$.

Numerical results show that this is exactly what happens. (AS; Hubeny, Maloney, Rangamani)



In order to interprete this result we need to analyse the origin of the oscillatory solutions around $h = \ln(r/2)$ for large $r$.

The $f_i$'s computed from $h = \ln(r/2)$ represent a flat (locally) background for large $r$.

(All field strengths fall off to zero as $r \to \infty$.)

Thus for small $A$

$$h \simeq \ln\frac{r}{2} + A\cos\left(\frac{r}{2} + B\right)$$

represents solution of the linearized equations of motion for various fields around flat background.

The $r$ dependence of the oscillatory part indicates as if we have fields of $\text{mass}^2 = -\frac{1}{4}$.

How is this possible?

Origin of the negative mass$^2$ modes:

In the presence of higher derivative terms in the action, typically there are additional solutions of the equations of motion even at the linearized level.

Example: Take a scalar field $\psi$ with action:

$$\frac{1}{2}\int d^4x\, \psi\, \partial_\mu \partial^\mu \left(1 - M^{-2}\, \partial^\mu \partial_\mu\right) \psi\,.$$

The equations of motion for $\psi$ has solutions:

$$\psi = A e^{ik.x}$$

with

$$k^2 = 0 \qquad \text{or} \quad k^2 = -M^2$$

Similarly, in the presence of higher derivative terms, the equations of motion of the string effective action will also have these additional oscillatory solutions even at the linearized level.

$\rightarrow$ responsible for the oscillations seen in our analysis.

Quantization of these additional solutions will give rise to additional states in the spectrum which are not present in the string spectrum.

Solution (Zwiebach):

We must try to remove these higher derivative terms by field redefinition.

In the scalar field example we take:

$$\widetilde{\psi} = \left(1 - M^{-2}\partial_\mu\partial^\mu\right)^{1/2} \psi$$

This gives the standard kinetic term for $\widetilde{\psi}$.

This maps $\psi = A\, e^{ik.x}$ with $k^2 = -M^2$ to

$$\widetilde{\psi} = 0$$

The generalization of this construction to gauge field, metric etc. will remove the higher derivative terms from the action at the quadratic level and map the additional oscillatory solutions to zero.

For example, for the metric, this will require defining a new metric

$$\widetilde{g}_{\mu\nu} = g_{\mu\nu} + aR_{\mu\nu} + b\,R\,g_{\mu\nu} + \ldots$$

The coefficients $a$, $b$, ... have to be chosen appropriately to remove higher derivative terms from the quadratic term in the action.

These new fields are the correct ones to be used in describing string theory.

Once we use these right field variables, the oscillatory part of the solution will get mapped to zero.

As a result our solution should approach the correct asymptotic form at large $r$.

Can we explicitly carry out this field redefinition and verify this?

This requires reformulating the supergravity action in terms of a new set of variables.

This has not been done yet.

Presumably when we use the correct field variables, the second order differential equation for $h$ will be replaced by an ordinary equation with unique solution.

Generalization to other heterotic string compactifications.

Heterotic on $K_5 \times S^1$.

$K_5$: any manifold / orbifold, possibly with background gauge fields etc., that preserves at least $N = 2$ supersymmetry.

($N = 2$ supersymmetry is needed to get the BPS states.)

Consider a heterotic string wrapped on $S^1$ with winding number $w$ and carrying $n$ units of momentum along $S^1$.

In the limit of large $nw$, the statistical entropy associated with this state is still given by:

$$S_{stat} \simeq 4\pi\sqrt{nw}$$

(This is controlled by the central charge of the conformal field theory describing the heterotic string compactification.)

$\rightarrow$ does not depend on the choice of $K_5$.

The classical solution describing this heterotic string involves background fields along $S^1$ and the non-compact directions.

The tree level effective field theory involving these fields is independent of the choice of $K_5$ to all orders in $\alpha'$.

As a result the classical solution describing the black hole solution does not depend on the choice of $K_5$.

$\rightarrow$ we get the same entropy of the black hole:

$$S_{BH} = 4\pi\sqrt{nw}$$

$\rightarrow$ The agreement between $S_{stat}$ and $S_{BH}$ continues to hold.

AS

**Finite charge corrections:**

One of the advantages of working with the elementary string states is that we know their degeneracy very precisely.

The degeneracy $d_{nw}$ of BPS states carrying charge quantum numbers $(n, w)$ is determined from the formula

$$\sum_{N=0}^{\infty} d_{N-1} q^N = 16 \prod_{n=1}^{\infty} (1 - q^n)^{-24}$$

For large $nw$ this gives:

$$d_{nw} \sim \exp(4\pi\sqrt{nw})$$

However we can calculate the corrections to this formula.

$$S_{stat} = \ln(d_{nw}) = 4\pi\sqrt{nw} - \frac{27}{2}\ln(\sqrt{nw}) + \mathcal{O}(1)$$

Question: Can we reproduce these corrections by keeping track of non-leading contribution to $S_{BH}$?

Note: The field $S$ is of order $\sqrt{nw}$ near the horizon.

$\rightarrow$ string coupling $\sim S^{-1/2} \sim (nw)^{-1/4}$ near the horizon.

$\rightarrow$ in the limit of large $nw$ we can ignore the string loop corrections to the effective action and focus on the tree level contribution.

However this is no longer the case if we want to study the non-leading corrections to the entropy (in inverse powers of $nw$).

We need to take into account quantum corrections to the string effective action, and then repeat the whole analysis.

There is however an ambiguity in carrying out this comparison.

The proof of equivalence of different statistical ensembles is valid only in the limit of infinite size (charges).

The definition of various thermodynamic quantities is idependent of the ensemble we use for large charges, but depends on the ensemble when we consider non-leading corrections.

It is not *a priori* clear which definition of statistical entropy we should compare with $S_{BH}$.
(Ooguri, Strominger, Vafa)

Strategy:

1. Try to find a definition of statistical entropy that agrees with the black hole entropy for toroidal compactification.

2. Then check if this works for other compactifications.

In order to make this guess it is useful to examine the formula for black hole entropy after including quantum corrections to the special higher derivative terms which we have been analysing.

We consider black holes carrying general 28 dimensional electric charge vector $Q$

(6 winding $w_i$, 6 momentum $n_i$ and 16 gauge charges $\vec{Q}_g$)

Define

$$N \equiv \frac{1}{2}Q^2 \equiv \frac{1}{2}\left(\sum_{i=1}^{6} n_i w_i - \vec{Q}_g^2\right)$$

The entropy $S_{BH}$ of a black hole carrying electric charge vector $Q$ is given by the formula

$$S_{BH} \simeq \frac{\pi N}{S_0} + 4\pi\, S_0 - 12\, \ln(2S_0) + \mathcal{O}(e^{-2\pi S_0})\,,$$

where $S_0$ is the value of $S$ at the horizon and is given by

$$-\frac{\pi\, N}{S_0^2} + 4\pi - \frac{12}{S_0} + \mathcal{O}(e^{-2\pi S_0}) = 0\,.$$

Cardoso, de Wit, Mohaupt

Define

$$\mu \equiv \pi/S_0$$

Then $S_{BH}(N)$ can be regarded as the Legendre transform of a function

$$\mathcal{F}_{BH}(\mu) \equiv \frac{4\pi^2}{\mu} - 12 \ln \frac{2\pi}{\mu} + \mathcal{O}(e^{-2\pi^2/\mu})$$

$$S_{BH}(N) = \mathcal{F}_{BH}(\mu) + \mu N$$

with $\mu$ determined from

$$\frac{\partial \mathcal{F}_{BH}}{\partial \mu} + N = 0$$

$\mathcal{F}_{BH}(\mu)$ is simpler than $S_{BH}(N)$.

This suggests that $\mathcal{F}_{BH}(\mu)$ may be the quantity that has a more direct relation with its statistical counterpart.

On the other hand, the statistical counterpart of $\mathcal{F}_{BH}(\mu)$ is realized naturally in a kind of mixed ensemble rather than in the microcanonical ensemble.

$d_N \equiv$ degeneracy of elementary string states with charge $Q$ $\qquad (N \equiv Q^2/2)$

Define

$$e^{\mathcal{F}(\mu)} = \sum_N d_N e^{-\mu N}$$

$\widetilde{S}_{stat}(N)$: Legendre transform of $\mathcal{F}(\mu)$

$\widetilde{S}_{stat}(N) \simeq \ln d_N$ for large $N$

Conjecture: AS

$$\widetilde{S}_{stat}(N) = S_{BH}(N) + constant + \mathcal{O}(e^{-\pi\sqrt{N}})$$

Equivalently

$$\mathcal{F}(\mu) = \mathcal{F}_{BH}(\mu) + \mathcal{O}(e^{-\pi^2/\mu})$$

For toroidal compactification:

$$e^{\mathcal{F}(\mu)} = \sum_N d_N e^{-\mu N} = e^{\mu} \prod_{n=1}^{\infty} \frac{1}{(1-e^{-n\mu})^{24}}$$

For small $\mu$ this gives

$$\mathcal{F}(\mu) = \frac{4\pi^2}{\mu} - 12 \ln \frac{2\pi}{\mu} + \ln 16 + \mathcal{O}(e^{-4\pi^2/\mu})$$

Compare with

$$\mathcal{F}_{BH}(\mu) \equiv \frac{4\pi^2}{\mu} - 12 \ln \frac{2\pi}{\mu} + \text{constant} + \mathcal{O}(e^{-2\pi^2/\mu})$$

$\rightarrow$ agreement up to non-perturbative terms.

Does it work for other $N = 4$ supersymmetric heterotic string compactifications?

Examine this in the context of CHL compactification

Chaudhuri, Hockney, Lykken; Chaudhuri, Polchinski, ...

1. Take heterotic string on a $T^6$.

2. Take an orbifold of the theory by an abelian group whose elements act as a shift + rotation on the Narain lattice.

In order to preserve $N = 4$ supersymmetry we need to ensure that the rotation group does not act on the right-handed component of the lattice.

This procedure gives an $N = 4$ supersymmetric theory with reduced rank $(= 28 - k)$ gauge group.

We now follow the same procedure to calculate $\mathcal{F}_{BH}(\mu)$ and $\mathcal{F}(\mu)$ and compare.

Result:

$$\mathcal{F}(\mu) = \frac{4\pi^2}{\mu} - \frac{24-k}{2} \ln \frac{2\pi}{\mu} + \text{constant'} + \text{n.p.}$$

Compare with

$$\mathcal{F}_{BH}(\mu) = \frac{4\pi^2}{\mu} - \frac{24-k}{2} \ln \frac{2\pi}{\mu} + \text{constant} + \text{n.p.}$$

$\rightarrow$ agreement up to non-perturbative terms.

## Some open problems

1. We have seen that after appropriate symmetry transformations, the near horizon limit of the classical black solution representing an elementary string is independent of any external parameter or the choice of compactification.

Thus string propagation in this background is described by a universal conformal field theory.

A detailed analysis of this CFT is likely to generate new insight into the black holes that they describe.

2. One could try to carry out a similar analysis for heterotic string compactified on $T^n \times S^1$ for other values of $n$.

This requires studying entropy of higher dimensional black holes.

The argument showing that the $S_{BH}$ has the form $a\sqrt{nw}$ can be generalized to higher dimensions. (Peet)

Can we compute $a$ by taking into account the higher derivative corrections?

This might be possible if we can find supersymmetrization of the curvature$^2$ term in (9+1) dimensions.

We could then compactify this theory on $T^n$ and study black hole solutions describing elementary string states.

3. The analysis described here takes care of only part of the higher derivative corrections which come from supersymmetrizing the curvature square terms.

These terms are somewhat special in the sense that they come from holomorphic corrections to the generalized prepotential.

However since at $r \sim 1$ the curvature is of order 1, other higher derivative terms will also be important.

Is there some kind of non-renormalization theorem that tells us that only the holomorphic corrections affect the value of $a$?

4. Generalization to type II compactification

The scaling argument can be generalized to type II theory on $T^5 \times S^1$

$\longrightarrow$ the black hole entropy for fundamental string wrapped on $S^1$ with winding number $w$ and $n$ units of momentum has

$$S_{BH} = a' \sqrt{nw}$$

$a'$ is some universal constant

On the other hand, counting of degeneracy of elementary string states give

$$S_{stat} = 2\sqrt{2}\,\pi \sqrt{nw}$$

Q. Can we calculate $a'$ by the same method as in the case of heterotic string?

Unfortunately tree level type II theories have no curvature$^2$ corrections to the effective action.

Thus a computation similar to the one for heterotic string gives

$$a' = 0$$

Thus here if we want to reproduce the statistical entropy we must take into account other higher derivative corrections.

Q. What is the basic difference between heterotic and type II?

Most likely this method of computing black hole entropy gives some sort of ln(index) rather than ln(degeneracy).

This is not surprising in view of the fact that in our analysis we have taken into account only a very special class of terms (holomorphic) terms.

For heterotic string index may be of order degeneracy whereas for type II the index may vanish.

What exactly is the index that is being computed by our method?