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SPRING SCHOOL ON SUPERSTRING THEORY AND RELATED TOPICS

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Classifying Supergravity Solutions and Applications

PART I

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CLASSIFYING SUPERGRAVITY SOLUTIONS AND APPLICATIONS

Jerome Gauntlett

Introduction

Supersymmetric solutions of SUGRA theories have played a very important tole in many developments in string/M-theory.

What are they? Consider bosonic solutions with $\psi = 0$:

$$\begin{aligned} E_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} - T_{\mu\nu} &= 0 \\ Matter Equations &= 0, \end{aligned}$$

and $\delta\psi = 0$:

$$\begin{aligned} \hat{\nabla}_\mu \epsilon &= 0, \\ M\epsilon &= 0. \end{aligned}$$

i.e. admitting a "Killing spinor" ϵ

$\hat{\nabla}_\mu \sim \nabla_\mu + (fluxes \times \Gamma)_\mu$ where ∇_μ is the Levi-Civita connection.

$M = M(fluxes)$ is a matrix, which gives algebraic constraints.

Can we classify such susy solutions?

(i) Matter =0

$$\begin{aligned} R_{\mu\nu} &= 0 \\ \nabla_\mu \epsilon &= 0 \end{aligned}$$

$\Rightarrow$ special holonomy.

Eulcidean case:
$SU(n)$ in $d = 2n$ - Calabi-Yau
$Sp(n)$ in $d = 4n$ - Hyper-Kahler
G_2 in $d = 7$
$Spin(7)$ in $d = 8$

Lorentzian: more possibilities.

(ii) Matter$\neq$0

$\hat{\nabla}$ is a connection on the Clifford bundle and not, in general on the spin bundle.

What should we do?

Motivation

1. Compactifications to $D = 4$: fluxes tend to stabilise moduli.

2. Black Holes: seek exotica such as black rings. Uniqueness theorems (assumed in black hole state counting calculations).

3. AdS/CFT: new examples; deeper understanding.

4. Surprises: previous construction of SUGRA solutions relied on guessing ansatz. (eg black rings, Gödel)

5. Mathematics

Want:
(i) Precise characterisation of geometry
(ii) Explicit solutions where possible.

Key Tool for classification: G-Structures
Gauntlett, Martelli, Pakis, Waldram

PLAN:

1. G-structures and classification.
Three examples, including minimal D=5 supergravity solutions.

2. Black Rings in D=5

3. New AdS/CFT examples and new Sasaki-Einstein manifolds.

G-Structures

Let M be an n-dimensional manifold.
$F(M)$ be the frame bundle: a principal $Gl(n)$ bundle.
A G-Structure is a principal G sub-bundle.

Equivalent to no-where vanishing tensors. e.g.

$$
\begin{aligned}
g_{ab} &\Rightarrow O(n) \quad (or\, O(p,q)) \\
g_{ab}, \epsilon_{a_1 \ldots a_n} &\Rightarrow SO(n)
\end{aligned}
$$

$n = 2m$

$$
\begin{aligned}
J_a{}^b, J^2 = -1 &\Rightarrow Gl(m, C) \\
g_{ab}, J_a{}^b, J^2 = -1 &\Rightarrow U(m) \\
g_{ab}, J_a{}^b, \Omega_{a_1 \ldots a_m} &\Rightarrow SU(m)
\end{aligned}
$$

Classify G-structures by their intrinsic torsion: it measures the deviation from special holonomy.

Intrinsic Torsion

Let η define a $G \subset SO(n)$ structure. Basic idea:

$$\nabla\eta \leftrightarrow \oplus_i W_i$$

W_i are G-modules which specify the type of G-structure.

In more detail:
$\exists\ \nabla'$ such that $\nabla'\eta = 0$. Define

$$\begin{aligned} T \equiv \nabla - \nabla' = \omega - \omega' \ &\in\ \Lambda^1 \otimes \Lambda^2 \\ &\cong\ \Lambda^1 \otimes so(n) \\ &\cong\ \Lambda^1 \otimes (g \oplus g^\perp) \end{aligned}$$

Then $\nabla\eta = (\nabla - \nabla')\eta \rightarrow$ element of $\Lambda^1 \otimes g^\perp$. This is the part of T that is independent of ∇' and is called the intrinsic torsion:

$$\begin{aligned} T^{(0)} \in \Lambda^1 \otimes g^\perp &= \oplus_i \mathcal{W}_i \\ T^{(0)} &= \oplus_i W_i \end{aligned}$$

The type of G-structure is classified by which $W_i \neq 0$. e.g.
(i) all $W_i = 0 \Leftrightarrow T^{(0)} = 0 \Leftrightarrow \nabla\eta = 0 \Leftrightarrow$ Special holonomy G
(ii) all $W_i \neq 0 \rightarrow$ most general kind of G-structure.

An example: $SU(3)$ structures in $d=6$. Specified by a real form J_{ab} and a complex form Ω_{abc}, satisfying

$$\begin{aligned} J \wedge \Omega &= 0 \\ \Omega \wedge \bar{\Omega} &= -i\frac{4}{3} J \wedge J \wedge J \end{aligned}$$

This defines a metric, and orientation and an almost complex structure.

The intrinsic torsion has 5 components. Decompose fundamental and adjoint of $SO(6)$ into $SU(3)$ reps: $\mathbf{6} = \mathbf{3} + \bar{\mathbf{3}}$, $\mathbf{15} = \mathbf{1} + \mathbf{3} + \bar{\mathbf{3}} + \mathbf{8}$. Hence $\Lambda^1 \otimes g^{\perp}$ gives the reps:

$$\begin{aligned} (\mathbf{3} + \bar{\mathbf{3}}) \quad \times \quad & (\mathbf{1} + \mathbf{3} + \bar{\mathbf{3}}) = (\mathbf{1} + \mathbf{1}) + (\mathbf{8} + \mathbf{8}) + \\ & (\mathbf{6} + \bar{\mathbf{6}}) + (\mathbf{3} + \bar{\mathbf{3}}) + (\mathbf{3} + \bar{\mathbf{3}}) \end{aligned}$$

corresponding to 5 $\mathcal{W}_i$.

Each $W_i \in \mathcal{W}_i$ can be expressed entirely in terms of dJ and $d\Omega$:

$$\begin{aligned} dJ &\to W_1, W_3, W_4 \\ d\Omega &\to W_1, W_2, W_5 \end{aligned}$$

e.g.

$$\begin{aligned} (W_4)_a &= J^{b_1 b_2}(dJ)_{a b_1 b_2} \\ (W_5)_a &= \Omega^{b_1 b_2 b_3}(d\Omega)_{a b_1 b_2 b_3} \end{aligned}$$

Examples:

$$\begin{aligned} W_1 = W_2 = 0 \quad &\rightarrow \quad complex \\ W_1 = W_2 = W_3 = W_4 = 0 \quad &\rightarrow \quad Kahler \\ W_i = 0 \quad &\rightarrow \quad Calabi - Yau \\ &\Leftrightarrow \quad dJ = d\Omega = 0 \end{aligned}$$

There exists (more than) 32 different $SU(3)$ structures.

$\boxed{\text{To classify supergravity solutions}}$

1. Observe that the isotropy group G of the Killing spinor ϵ defines a G-structure. Explicitly, the tensors defining the G-structure can be constructed as bi-linears:

$$T_{(k)} \sim \bar{\epsilon}\Gamma_{(k)}\epsilon$$

The algebraic conditions staisfied by the tensors can be obtained e.g. by using Fierz identities.

2. $\hat{\nabla}_\mu \epsilon = \nabla_\mu + (fluxes \times \Gamma)_\mu \epsilon = 0$ restricts the intrinsic torsion and determines some of the flux. $M\epsilon = 0$ places additional conditions on the flux and intrinsic torsion. Note that some of the flux components can drop out completely. HARD WORK.

3. Equations of motion. Consider $[\hat{\nabla}_m, \hat{\nabla}_n]\epsilon = 0$. Impose matter equations of motion and Bianchi identities $\Rightarrow E_{mn}\epsilon = 0$. Need to impose at most one component of $E_{mn} = 0$, and only in Lorentzian case.

An Example: Heterotic compactified on M_6. Set Yang-Mills fields to zero for simplicity. Susy $\Rightarrow$

$$\begin{aligned} \left(\nabla_m + \frac{1}{8} H_{mnp}\Gamma^{np}\right)\epsilon &= 0, \\ \left(\Gamma^m \partial_m \Phi + \frac{1}{12} H_{mnp}\Gamma^{mnp}\right)\epsilon &= 0, \end{aligned}$$

where ϵ is chiral $D = 6$ spinor.

$\star$ $\epsilon \longrightarrow SU(3)$ structure:

$$\begin{aligned} J_{mn} &= -i\epsilon^\dagger \Gamma_{mn}\epsilon \\ \Omega_{mnp} &= \epsilon^T \Gamma_{mnp}\epsilon \end{aligned}$$

$\star$ Analyse:

$$\begin{aligned} \mathrm{d}(\mathrm{e}^{-2\Phi}\Omega) &= 0, \\ \mathrm{d}(\mathrm{e}^{-2\Phi} J \wedge J) &= 0, \\ \mathrm{e}^{2\Phi}\mathrm{d}(\mathrm{e}^{-2\Phi} J) &= -*H \end{aligned}$$

Have $W_1 = W_2 = 0$, $\Rightarrow$ complex. $W_4 = -(1/2)W_5 = 2d\Phi$. H restricted by structure.

$\star$ H e.o.m. is automatically satisfied. Integrability of susy $\Rightarrow$ just need to impose $dH = 0$.

Neat reformulation [Gauntlett, Martelli, Waldram][Cardoso, Curio, Dall'Agata, Lust, Manousselis, Zoupanos] of results of Strominger and Hull. The key point is that the G-structure approach generalises and is systematic.

Can apply the programme in 3 broad ways:

1. Classify the most general supergravity solutions in D=10/11 supergrvaity

2. Lower-Dimensional Supergravities

$D = 4, 5, 6, 7$: can be much more explicit.

3. Special classes of Solutions

Compactifications from $D = 11, 10$ to e.g. $\mathbb{M}_4$ or to AdS_5.

Minimal $D = 5$ supergravity

The Model

⋆ Fields: $g_{\mu\nu}$, A_μ, $\psi = 0$

⋆ Equations of motion:

$$
\begin{aligned}
R_{\alpha\beta} + 2(F_{\alpha\gamma}F_\beta{}^\gamma - \frac{1}{6}g_{\alpha\beta}F^2) &= 0 \\
d * F + \frac{2}{\sqrt{3}}F \wedge F &= 0 \\
dF &= 0
\end{aligned}
$$

⋆ Supersymmetry

$$
\left[D_\alpha + \frac{1}{4\sqrt{3}}\left(\gamma_\alpha{}^{\beta\gamma} - 4\delta_\alpha^\beta\gamma^\gamma\right)F_{\beta\gamma}\right]\epsilon = 0.
$$

Classification [Gauntlett, Gutowski, Hull, Pakis, Reall]

1. From the Killing spinor we can construct

$$f, K_\mu, X^i_{\mu\nu}, i = 1,2,3$$

2. Algebraic conditions $\to$ two cases, depending whether K is time-like or null:

$\star$ Timelike: $SU(2)$ structure in D=5.
$K \sim e^0$, $ds^2 = -e^0 e^0 + e^a e^a$ and $X^i \sim X^i_{ab} e^a \wedge e^b$

$\star$ Null: R^3 structure:
$K = e^+$, $ds^2 = 2e^+ e^- + e^i e^i$ and $X^i \sim e^+ \wedge e^i$

$$\begin{aligned} e^+ &\to e^+ \\ e^- &\to e^- - (1/2)p^2 - p^i e^i \\ e^i &\to e^i + p^i e^+ \end{aligned}$$

preserves structure.

3. Analyse Killing spinor equations plus equations of motion

1. Timelike case: $K = \partial_t$

$$
\begin{aligned}
ds^2 &= -f^2(dt+\omega)^2 + f^{-1}ds^2_{HK}(M_4) \\
F &= \frac{\sqrt{3}}{2}d[f(dt+\omega)] - \frac{1}{\sqrt{3}}G^+
\end{aligned}
$$

where ds^2_{HK} is an arbitrary hyper-Kähler metric and ω is a one-form on M_4, satisfying

$$
\begin{aligned}
G^+ &\equiv \frac{1}{2}f(d\omega + *d\omega) \\
dG^+ &= 0 \\
\Delta f^{-1} &= \frac{4}{9}(G^+)^2
\end{aligned}
$$

Preserve 1/2 susy

Can find many solutions.

2. Null case: $K = \partial_v$

$$\begin{aligned} ds^2 &= H^{-1}\left(\mathcal{F}du^2 + 2dudv\right) - H^2(d\mathrm{x} + \mathbf{a}du)^2, \\ F &= \ldots \end{aligned}$$

$H = H(u, \mathrm{x})$, $\mathcal{F} = \mathcal{F}(u, \mathrm{x})$, $\mathbf{a} = (u, \mathrm{x})$.

Satisfy 2nd order ODEs: general solution involves 3 u depenendent harmonic functions.

Plane fronted waves

Preserve 1/2 susy

Can find many explicit solutions.

3. Maximal Susy
Null and timelike: $AdS_3 \times S^2$, Plane wave
Timelike and timelike: $AdS_2 \times S^3$, Gödel, BMPV

Can generalise to:
$D = 4, 5, 6, 7$
gauged SUGRA
add matter fields

$\boxed{D = 11 \text{ SUGRA}}$ [Gauntlett, Pakis]

$\star$ Fields: $g_{\mu\nu}$, $C_{\mu\nu\rho}$, $\psi_\mu = 0$

1. From the Killing spinor we can construct

$$K_\mu, \Omega_{\mu\nu}, \Sigma_{\mu_1 \dots \mu_5}$$

2. Algebraic conditions $\rightarrow$ two cases, depending whether K is time-like or null:

Timelike: $SU(5)$ structure in D=11.
$K \sim e^0$, $ds^2 = -e^0 e^0 + e^a e^a$

Null: $(Spin(7) \times \mathbb{R}^8) \times \mathbb{R}$ structure:
$V = e^+$and $ds^2 = 2e^+ e^- + e^i e^i + e^9 e^9$ $i = 1, \dots, 8$.

3. Analyse Killing spinor equations plus equations of motion

Timelike case:

$$\begin{aligned} ds^2 &= -\Delta^2(dt+\omega)^2 + \Delta^{-1}ds^2(M_{10}) \\ F &= \ldots \end{aligned}$$

M_{10} has an $SU(5)$ structure with very weakly constrained intrinsic torsion $W_5 \sim dlog(\Delta)$.

This is to be expected, since this is the most general timelike susy solution preserving just 1/32 susy.

Similar story for null case.

Recent progress in refining classification for the case of preservation of more than one susy Mac Conamhna, Cariglia, Gran, Gutowski, Papadopoulos,Roest. Maximal case already done Figueroa-O'Farril, Papadopoulos.

Summary

G-Structures are simple and effective tools to classify SUGRA solutions.

Three main applications:

1. Compactifications to e.g. $\mathbb{M}_4$, AdS_5
Many cases considered.
Is there an analogue of the Calabi-Yau theorem?

2. Classification of low-dimensional SUGRA theories.
Powerful way of finding new solutions. Constructs solutions in quite different ways than previous ansatze.

3. Classification of most general solutions.
Being refined. More than 1/2 susy?