The interacting boson model

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Introduction to the IBM
Practical applications of the IBM
Overview of nuclear models

- *Ab initio* methods: Description of nuclei starting from the bare nn & nnn interactions.
- Mean-field methods: Nuclear average potential with global parametrization (+ correlations).
- Nuclear shell model: Nuclear average potential + (residual) interaction between nucleons.
- Phenomenological models: Specific nuclei or properties with local parametrization, e.g. the interacting boson model.
**Ab initio** methods

- Many **ab initio** methods exist and give consistent results.
- Example: \( A=4 \)

<table>
<thead>
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<th>Method</th>
<th>( \langle T \rangle )</th>
<th>( \langle V \rangle )</th>
<th>( E_b )</th>
<th>( \sqrt{\langle r^2 \rangle} )</th>
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<tr>
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<td>-128.33(10)</td>
<td>-25.94(5)</td>
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</tbody>
</table>


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**Ab initio** calculations for light nuclei

- Systematic studies of light nuclei ($A \leq 12$) ⇒ evidence for three-body nucleon interactions.


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Tri-partite classification of nuclei

• Empirical evidence for seniority-type, vibrational- and rotational-like nuclei:

• Need for model of vibrational nuclei.


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The interacting boson model

- Spectrum generating algebra for the nucleus is U(6). All physical observables (hamiltonian, transition operators,…) are expressed in terms of $s$ and $d$ bosons.

- Justification from
  - Shell model: $s$ and $d$ bosons are associated with $S$ and $D$ fermion ($Cooper$) pairs.
  - Geometric model: for large boson number the IBM reduces to a liquid-drop hamiltonian.

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The IBM hamiltonian

- Rotational invariant hamiltonian with up to $N$-body interactions (usually up to 2):
  \[ \hat{H}_{\text{IBM}} = \varepsilon_s \hat{n}_s + \varepsilon_d \hat{n}_d + \sum_{l_1 l_2 l'_1 l'_2, L} \nu_{l_1 l_2 l'_1 l'_2, L} \left( b^+_{l_1} \times b^+_{l_2} \right)^{(L)} \cdot \left( \tilde{b}_{l'_1} \times \tilde{b}_{l'_2} \right)^{(L)} + \cdots \]

- For what choice of single-boson energies $\varepsilon$ and boson-boson interactions $\nu$ is the IBM hamiltonian solvable?

- This problem is equivalent to the enumeration of all algebras $G$ satisfying
  \[ \mathbb{U}(6) \supset G \supset \mathbb{SO}(3) \equiv \left\{ \hat{L}_\mu = \sqrt{10} \left( d^+ \times \tilde{d} \right)_\mu^{(1)} \right\} \]

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Dynamical symmetries of the IBM

- **U(6)** has the following subalgebras:

  \[ U(5) = \left\{ \left( d^+ \times \tilde{d} \right)_\mu^{(0)}, \left( d^+ \times \tilde{d} \right)_\mu^{(1)}, \left( d^+ \times \tilde{d} \right)_\mu^{(2)}, \left( d^+ \times \tilde{d} \right)_\mu^{(3)}, \left( d^+ \times \tilde{d} \right)_\mu^{(4)} \right\} \]

  \[ SU(3) = \left\{ \left( d^+ \times \tilde{d} \right)_\mu^{(1)}, \left( s^+ \times \tilde{d} + d^+ \times \tilde{s} \right)_\mu^{(2)} - \sqrt{7/4} \left( d^+ \times \tilde{d} \right)_\mu^{(2)} \right\} \]

  \[ SO(6) = \left\{ \left( d^+ \times \tilde{d} \right)_\mu^{(1)}, \left( s^+ \times \tilde{d} + d^+ \times \tilde{s} \right)_\mu^{(2)}, \left( d^+ \times \tilde{d} \right)_\mu^{(3)} \right\} \]

  \[ SO(5) = \left\{ \left( d^+ \times \tilde{d} \right)_\mu^{(1)}, \left( d^+ \times \tilde{d} \right)_\mu^{(3)} \right\} \]

- **Three solvable limits are found:**

  \[ U(6) \supset \begin{cases} U(5) \supset SO(5) \\ SU(3) \supset SO(3) \\ SO(6) \supset SO(5) \end{cases} \supset SO(3) \]
Dynamical symmetries of the IBM

- The general IBM hamiltonian is
  \[ \hat{H}_{\text{IBM}} = \epsilon_s \hat{n}_s + \epsilon_d \hat{n}_d + \sum_{l_1 l_2 l'_1 l'_2, L} \nu_{l_1 l_2 l'_1 l'_2} \left( b_{l_1}^+ \times b_{l_2}^+ \right)^{(L)} \cdot \left( \tilde{b}_{l'_1} \times \tilde{b}_{l'_2} \right)^{(L)} + \cdots \]

- An entirely equivalent form of \( H_{\text{IBM}} \) is
  \[ \hat{H}_{\text{IBM}} = \eta_0 \hat{C}_1[U(6)] + \eta_1 \hat{C}_1[U(5)] + \kappa'_0 \hat{C}_1[U(6)] \hat{C}_1[U(5)] \\
  + \kappa_0 \hat{C}_2[U(6)] + \kappa_1 \hat{C}_2[U(5)] + \kappa_2 \hat{C}_2[\text{SU}(3)] \\
  + \kappa_3 \hat{C}_2[\text{SO}(6)] + \kappa_4 \hat{C}_2[\text{SO}(5)] + \kappa_5 \hat{C}_2[\text{SO}(3)] \]

- The coefficients \( \eta \) and \( \kappa \) are certain combinations of the coefficients \( \epsilon \) and \( \nu \).
The solvable IBM hamiltonians

- *Excitation* spectrum of $H_{\text{IBM}}$ is determined by
  \[
  \hat{H}_{\text{IBM}} = E_0 + \eta_1 \hat{C}_1[U(5)] + \kappa_1 \hat{C}_2[U(5)] + \kappa_2 \hat{C}_2[SU(3)] + \kappa_3 \hat{C}_2[SO(6)] + \kappa_4 \hat{C}_2[SO(5)] + \kappa_5 \hat{C}_2[SO(3)]
  \]

- If certain coefficients are zero, $H_{\text{IBM}}$ can be written as a sum of commuting operators:
  \[
  \begin{align*}
  \hat{H}_{U(5)} &= \eta_1 \hat{C}_1[U(5)] + \kappa_1 \hat{C}_2[U(5)] + \kappa_4 \hat{C}_2[SO(5)] + \kappa_5 \hat{C}_2[SO(3)] \\
  \hat{H}_{SU(3)} &= \kappa_2 \hat{C}_2[SU(3)] + \kappa_5 \hat{C}_2[SO(3)] \\
  \hat{H}_{SO(6)} &= \kappa_3 \hat{C}_2[SO(6)] + \kappa_4 \hat{C}_2[SO(5)] + \kappa_5 \hat{C}_2[SO(3)]
  \end{align*}
  \]
The U(5) vibrational limit

- Anharmonic vibration spectrum associated with the quadrupole oscillations of a spherical surface.
- Conserved quantum numbers: $n_d$, $\nu$, $L$.

D. Brink et al., Phys. Lett. 19 (1965) 413
The SU(3) rotational limit

- Rotation-vibration spectrum of quadrupole oscillations of a spheroidal surface.
- Conserved quantum numbers: $(\lambda, \mu), L$.

The SO(6) $\gamma$-unstable limit

- Rotation-vibration spectrum of quadrupole oscillations of a $\gamma$-unstable spheroidal surface.
- Conserved quantum numbers: $\sigma$, $\upsilon$, $L$.

A. Arima & F. Iachello, Ann. Phys. (NY) 123 (1979) 468
L. Wilets & M. Jean, Phys. Rev. 102 (1956) 788

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Modes of nuclear vibration

• Nucleus is considered as a droplet of nuclear matter with an equilibrium shape. Vibrations are modes of excitation around that shape.

• Character of vibrations depends on symmetry of equilibrium shape. Two important cases in nuclei:
  – Spherical equilibrium shape
  – Spheroidal equilibrium shape
Vibrations about a spherical shape

- Vibrations are characterized by a multipole quantum number $\lambda$ in surface parametrization:

\[
R(\theta, \varphi) = R_0 \left( 1 + \sum_{\lambda} \sum_{\mu=-\lambda}^{+\lambda} \alpha_{\lambda\mu} Y_{\lambda\mu}^*(\theta, \varphi) \right)
\]

- $\lambda=0$: compression (high energy)
- $\lambda=1$: translation (not an intrinsic excitation)
- $\lambda=2$: quadrupole vibration
Vibrations about a spheroidal shape

• The vibration of a shape with axial symmetry is characterized by $a_{\lambda \nu}$.

• Quadrupolar oscillations
  – $\nu=0$: along the axis of symmetry ($\beta$)
  – $\nu=\pm1$: spurious rotation
  – $\nu=\pm2$: perpendicular to axis of symmetry ($\gamma$)
Synopsis of IBM symmetries

• Three standard solutions: $U(5)$, $SU(3)$, $SO(6)$.
• Analytic solution for $U(5) \rightarrow SO(6)$ via $SU(1,1)$ Richardson-Gaudin integrability.
• Hidden symmetries because of parameter transformations: $SU_{\pm}(3)$ and $SO_{\pm}(6)$.
• Partial dynamical symmetries.
• Critical-point symmetries?
Classical limit of IBM

- For large boson number $N$, a coherent (or intrinsic) state is an approximate eigenstate,

$$\hat{H}_{\text{IBM}}|N;\alpha_\mu\rangle \approx E|N;\alpha_\mu\rangle, \quad |N;\alpha_\mu\rangle \propto \left(s^+ + \sum_\mu \alpha_\mu d^+\right)^N|0\rangle$$

- The real parameters $\alpha_\mu$ are related to the three Euler angles and shape variables $\beta$ and $\gamma$.

- Any IBM hamiltonian yields energy surface:

$$\langle N;\alpha_\mu|\hat{H}_{\text{IBM}}|N;\alpha_\mu\rangle = \langle N;\beta\gamma|\hat{H}_{\text{IBM}}|N;\beta\gamma\rangle \equiv V(\beta,\gamma)$$

Geometry of IBM

• A simplified, much used IBM hamiltonian:
  \[ \hat{H}_{\text{CQF}} = \varepsilon_d \hat{n}_d - \kappa \hat{Q}^x \cdot \hat{Q}^x, \quad \hat{Q}^x_\mu = s^+ \tilde{d}_\mu + d^+_\mu s + \chi (d^+ \times \tilde{d})_\mu^{(2)} \]

• \( H_{\text{CQF}} \) can acquire the three IBM symmetries.

• \( H_{\text{CQF}} \) has the following classical limit:

  \[
  V_{\text{CQF}}(\beta, \gamma) \equiv \langle N; \beta \gamma | \hat{H}_{\text{CQF}} | N; \beta \gamma \rangle \\
  = \varepsilon_d N \frac{\beta^2}{1 + \beta^2} - \kappa N \frac{5 + (1 + \chi^2)\beta^2}{1 + \beta^2} \\
  - \kappa \frac{N(N-1)}{1 + \beta^2} \left( \frac{2}{7} \chi^2 \beta^4 - 4 \sqrt{\frac{2}{7}} \chi \beta^3 \cos 3\gamma + 4 \beta^2 \right)
  \]
Phase diagram of IBM

$\varepsilon_d = 0, \kappa \neq 0$

$\chi = +\sqrt{\frac{7}{4}}$

$\varepsilon_d = 0, \kappa \neq 0$

$\chi = 0$

$\varepsilon_d \neq 0, \kappa = 0$

$\chi = 0$

$\varepsilon_d \neq 0, \kappa \neq 0$

$\chi = -\sqrt{\frac{7}{4}}$

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Microscopy of IBM

• In a boson mapping, fermion pairs are represented as bosons:
  \[ s^+ \Leftrightarrow S^+ = \sum_j \alpha_j (a_j^+ \times a_j^+)^{(0)}, \quad d^+_\mu \Leftrightarrow D^+_\mu = \sum_{jj'} \beta_{jj'} (a_j^+ \times a_j)^{(2)} \mu \]

• Mapping of operators (such as hamiltonian) should take account of Pauli effects.

• Two different methods by
  – requiring same commutation relations;
  – associating state vectors.
Extensions of the IBM

- Neutron and proton degrees freedom (IBM-2):
  - $F$-spin multiplets ($N_\nu+N_\pi=$constant).
  - Scissors excitations.

- Fermion degrees of freedom (IBFM):
  - Odd-mass nuclei.
  - Supersymmetry (doublets & quartets).

- Other boson degrees of freedom:
  - Isospin $T=0$ & $T=1$ pairs (IBM-3 & IBM-4).
  - Higher multipole ($g,...$) pairs.
Scissors excitations

- **Collective displacement modes between neutrons and protons:**
  - *Linear* displacement (giant dipole resonance): \( R_\nu - R_\pi \Rightarrow E1 \) excitation.
  - *Angular* displacement (scissors resonance): \( L_\nu - L_\pi \Rightarrow M1 \) excitation.

Supersymmetry

- A simultaneous description of even- and odd-mass nuclei (*doublets*) or of even-even, even-odd, odd-even and odd-odd nuclei (*quartets*).

- Example of $^{194}$Pt, $^{195}$Pt, $^{195}$Au & $^{196}$Au:

\[
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\hline
195_{\text{Pt}} t_{11/2}
\hline
\end{array}
\begin{array}{c}
\hline
196_{\text{Au}} t_{11/2}
\hline
\end{array}
\end{array}
\begin{array}{c}
\hline
194_{\text{Pt}} t_{11/2}
\hline
\end{array}
\begin{array}{c}
\hline
195_{\text{Au}} t_{11/2}
\hline
\end{array}
\end{array}
\end{array}
\end{array}
\]


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Example of $^{195}$Pt
Example of $^{196}_{79}\text{Au}_{117}$
Isospin invariant boson models

• Several versions of IBM depending on the fermion pairs that correspond to the bosons:
  – IBM-1: single type of pair.
  – IBM-2: $T=1$ nn ($M_T=-I$) and pp ($M_T=+I$) pairs.
  – IBM-3: full isospin $T=1$ triplet of nn ($M_T=-I$), np ($M_T=0$) and pp ($M_T=+I$) pairs.
  – IBM-4: full isospin $T=1$ triplet and $T=0$ np pair (with $S=1$).

• Schematic IBM-$k$ has only $S$ ($L=0$) pairs, full IBM-$k$ has $S$ ($L=0$) and $D$ ($L=2$) pairs.
Algebraic many-body models

• The integrability of quantum many-body (bosons and/or fermions) systems can be analyzed with algebraic methods.

• Two nuclear examples:
  – Pairing vs. quadrupole interaction in the nuclear shell model.
  – Spherical, deformed and $\gamma$-unstable nuclei with $s,d$-boson IBM.

\[
\begin{align*}
U(6) \subset & \left\{ 
\begin{array}{c}
U(5) \supset SO(5) \\
SU(3) \\
SO(6) \supset SO(5)
\end{array}
\right\} \supset SO(3)
\end{align*}
\]
Other fields of physics

• Molecular physics:
  – $U(4)$ vibron model with $s,p$-bosons.

\[ U(4) \supset \begin{cases} U(3) \\ SO(4) \end{cases} \supset SO(3) \]

  – Coupling of many $SU(2)$ algebras for polyatomic molecules.

• Similar applications in hadronic, atomic, solid-state, polymer physics, quantum dots…

F. Iachello, 1975 to now
The interacting boson model

P. Van Isacker, GANIL, France

Introduction to the IBM
Practical applications of the IBM
The IBM hamiltonian

- Rotational invariant hamiltonian with up to $N$-body interactions (usually up to 2):

$$\hat{H}_{\text{IBM}} = \varepsilon_s \hat{n}_s + \varepsilon_d \hat{n}_d + \sum_{l_1l_2l_1'l_2'} u^L_{l_1l_2l_1'l_2'} \left( b^{+}_{l_1} \times b^{+}_{l_2} \right)^{(L)} \cdot \left( \tilde{b}_{l_1'} \times \tilde{b}_{l_2'} \right)^{(L)} + \cdots$$

- Explicit forms of the hamiltonian: multipole expansion and “standard representation”.

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The IBM hamiltonian

- **Standard representation:**

\[
\hat{H} = c(1)\hat{N} + c(2)\hat{n}_d + c(3)\frac{1}{2}[[d^\dagger \times d^\dagger]^0] \times [\tilde{d} \times \tilde{d}]^0 \\
+ c(4)\sqrt{5}\frac{1}{2}[[d^\dagger \times d^\dagger]^2] \times [\tilde{d} \times \tilde{d}]^2 \\
+ c(5)\frac{3}{2}[[d^\dagger \times d^\dagger]^4] \times [\tilde{d} \times \tilde{d}]^4 \\
+ c(6)[[s^\dagger \times d^\dagger]^2] \times [\tilde{d} \times \tilde{d}]^2 + [d^\dagger \times d^\dagger]^2 \times [\tilde{s} \times \tilde{d}]^2 \\
+ c(7)[[s^\dagger \times s^\dagger]^0] \times [\tilde{d} \times \tilde{d}]^0 + [[d^\dagger \times d^\dagger]^0] \times [\tilde{s} \times \tilde{s}]^0 \\
+ c(8)\sqrt{5}[[s^\dagger \times d^\dagger]^2] \times [\tilde{s} \times \tilde{d}]^2 \\
+ c(9)[[s^\dagger \times s^\dagger]^0] \times [\tilde{s} \times \tilde{s}]^0.
\]

- **Multipole expansion:**

\[
\hat{H} = EPS\hat{n}_d + A(0)\hat{P}^\dagger \hat{P} + A(1)\hat{L} \cdot \hat{L} + A(2)\hat{Q}_\chi \cdot \hat{Q}_\chi + A(3)\hat{T}_3 \cdot \hat{T}_3 + A(4)\hat{T}_4 \cdot \hat{T}_4,
\]
The U(5) vibrational limit

- **U(5) Hamiltonian:**

\[
\hat{H}_{\text{U}(5)} = \varepsilon \hat{n}_d + \sum_{L=0,2,4} c^L \frac{1}{2} (d^+ \times d^+)^{(L)} \cdot (\tilde{d} \times \tilde{d})^{(L)}
\]

- **Energy eigenvalues:**

\[
E(n_d, \nu, L) = \varepsilon n_d + \kappa_1 n_d (n_d + 4) + \kappa_4 \nu (\nu + 3) + \kappa_5 L (L + 1)
\]

with

\[
\kappa_1 = \frac{1}{12} c_0
\]

\[
\kappa_4 = -\frac{1}{10} c_0 + \frac{1}{7} c_2 - \frac{3}{70} c_4
\]

\[
\kappa_5 = -\frac{1}{14} c_2 + \frac{1}{14} c_4
\]
The U(5) vibrational limit

- Conserved quantum numbers: $n_d$, $\nu$, $L$. 

![Diagram showing energy levels and quantum numbers for U(5) vibrational limit]
The SU(3) rotational limit

• **SU(3) Hamiltonian:**

\[ \hat{H}_{\text{SU}(3)} = a \hat{Q}_\lambda \cdot \hat{Q}_\lambda + b \hat{L} \cdot \hat{L} \]

• **Energy eigenvalues:**

\[ E(\lambda, \mu, L) = \kappa_2 \left( \lambda^2 + \mu^2 + 3\lambda + 3\mu + \lambda\mu \right) + \kappa_5 L(L+1) \]

with

\[ \kappa_2 = \frac{1}{2} a \]

\[ \kappa_5 = b - \frac{3}{8} a \]
The SU(3) rotational limit

- Conserved quantum numbers: \((\lambda, \mu), L\).
The SO(6) $\gamma$-unstable limit

- **SO(6) Hamiltonian:**

\[
\hat{H}_{\text{SO}(6)} = a\hat{P}^+ \cdot \hat{P} + b\hat{T}_3 \cdot \hat{T}_3 + c\hat{L} \cdot \hat{L}
\]

- **Energy eigenvalues:**

\[
E(\sigma, \nu, L) = \kappa_3 [N(N + 4) - \sigma(\sigma + 4)] + \kappa_4 \nu(\nu + 3) + \kappa_5 L(L + 1)
\]

with

\[
\kappa_3 = \frac{1}{4} a
\]

\[
\kappa_4 = \frac{1}{2} b
\]

\[
\kappa_5 = -\frac{1}{10} b + c
\]
The SO(6) $\gamma$-unstable limit

- Conserved quantum numbers: $\sigma$, $v$, $L$. 
Configuration mixing in shell model

- Example of platinum isotopes (Z=78, 82<N<126):
  - Regular configuration: 4 proton holes in 50-82 shell.
  - Deformed configuration: 6 proton holes in 50-82 shell and 2 protons in the 82-126 shell.
  - Neutrons always in 82-126 shell.

Configuration mixing in IBM

- **Example of platinum isotopes (Z=78, 82<N<126):**
  - Regular configuration: $N_\pi = 2$ proton bosons.
  - Deformed configuration: $N_\pi = 4$ proton bosons.
  - Always $N_\nu$ neutron bosons.

- **IBM-1: configurations with $N$ and $N+2$ bosons.**


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Example: Coexistence in $^{186}$Pb

- Observation: triplet of differently shaped $0^+$ states in $^{186}$Pb.
- Mean-field theory predicts three minima.
- IBM calculation for Pb isotopes yields
  - spectroscopy;
  - geometry.

Lead isotopes in the IBM

- **Hamiltonian for three configurations:**

\[
H = H_{0p-0h} + H_{2p-2h} + H_{4p-4h} + H_{\text{mix}}^{02} + H_{\text{mix}}^{24}
\]

\[
H_{ip-ih} = \varepsilon_i n_d + \kappa_i Q_i \cdot Q_i, \quad Q_i = \left(s^+ \tilde{d} + d^+ \tilde{s}\right)^{(2)} + \chi_i \left(d^+ \tilde{d}\right)^{(2)}
\]

\[
H_{\text{mix}}^{ii} = \omega_0^{ii} \left(s^+ s^+ + \tilde{s} \tilde{s}\right) + \omega_2^{ii} \left(\tilde{d} \cdot d^+ + \tilde{d} \cdot \tilde{d}\right)
\]

- Single parameter set for all Pb isotopes.
- Parameters for 2p-2h and 4p-4h configurations obtained from \(I\)-spin considerations.

Spectroscopy of lead isotopes

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