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School and Workshop on Structure and Function of Complex Networks

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Homogeneous vs growing complex networks

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Homogeneous vs growing complex networks

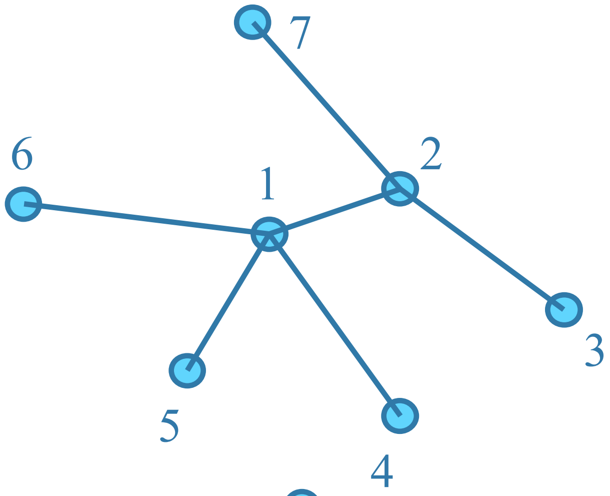
B. Waclaw (IF UJ, Poland)

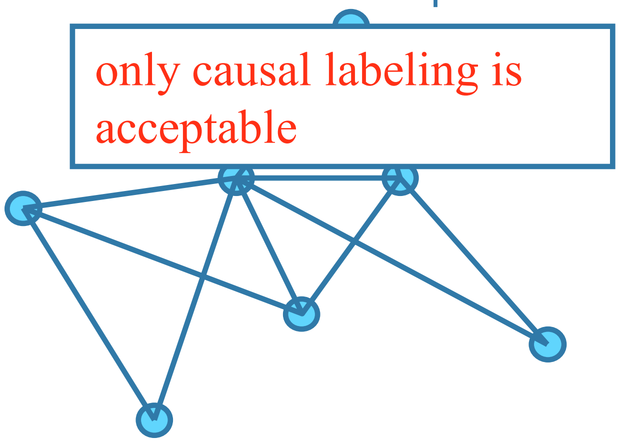
Burda, Dorogovtsev, Samukhin, Khang, Lässig, Newman, Snijders, Vicsek

and many others

Growing vs Homogeneous

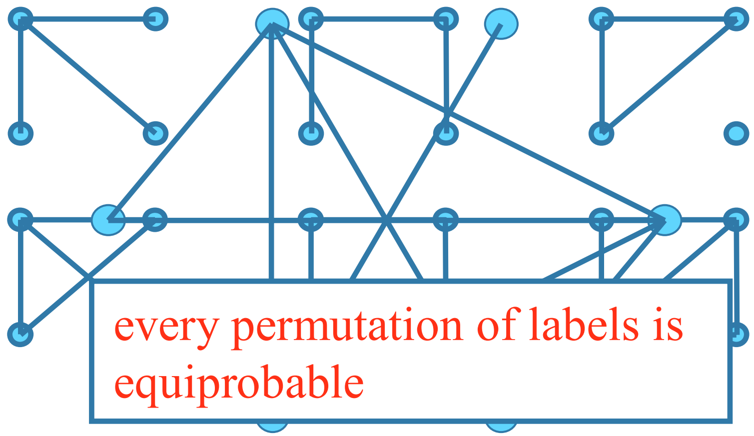
one adds new nodes to existing network:





only causal labeling is acceptable

every nodes are equivalent – the labeling has no meaning



every permutation of labels is equiprobable

the most known construction – the **Erdős-Rényi random graphs**

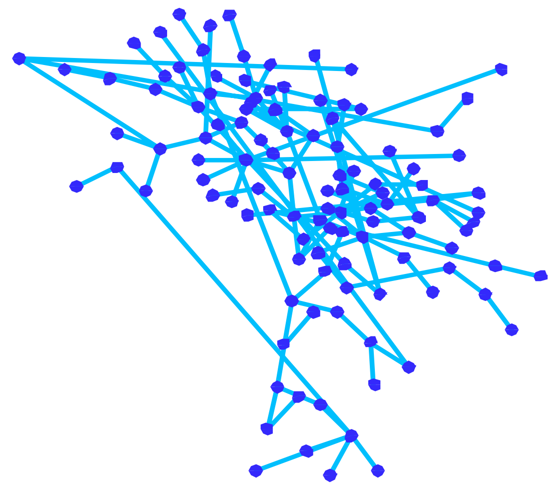
-start from *N* empty nodes,

-add *L* links at random.

-Equivalent with some kind of rearragement of initial network

Why homogeneous network?

- today many observed networks still grow
- but there is another process – rearrangement
- this can play important role in future



- better to examine the structural properties (growing – for dynamical)
- for randomizing
- average case of algorithms

Statistical ensemble

- it is convenient to define statistical ensemble of homogeneous networks: one can use classical statistical physics techniques
- partition function ***Z*** allows to calculate many quantities

Many possibilities, but the three most popular:

- **microcanonical**

  Degrees q_i of all nodes are fixed, graphs differ in such properties like number of triangles ($\rightarrow$ clustering), diameter, etc.

- **canonical**

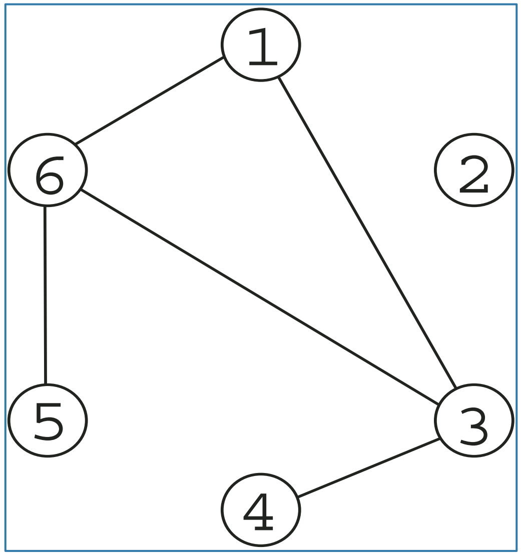
  Only number of nodes N and number of links L fixed, like in E-R graphs.

- **grandcanonical**

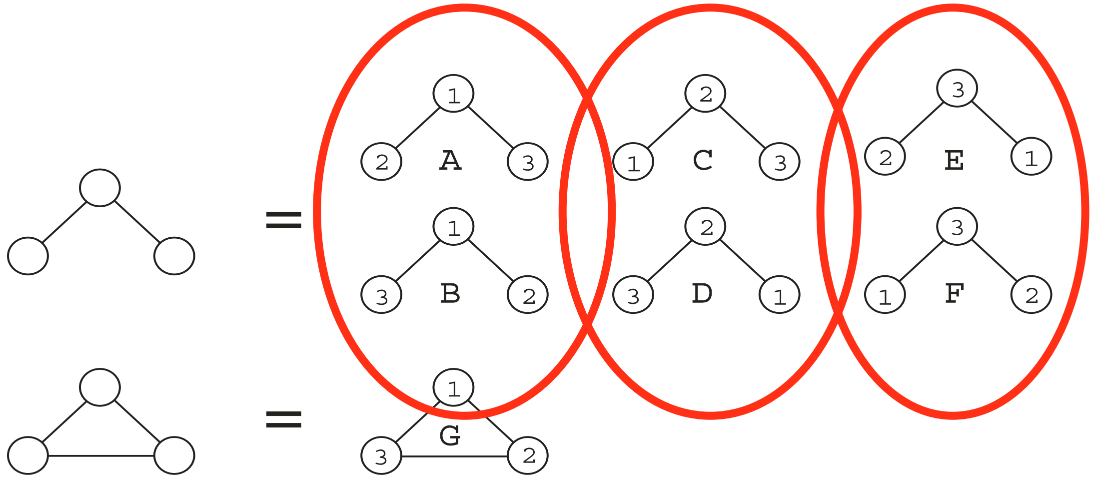
  N fixed, L may fluctuate, like in binomial graphs.

Starting point – ER graphs

- N nodes and L links chosen at random from $N(N-1)/2$ possible
- each labeled graph has the same weight ($1/N!$ for convenience)



To get E-R graphs we take off the labels:



three distinct labeled graph →weight 3/3!

one labeled graph → weight 1/3!

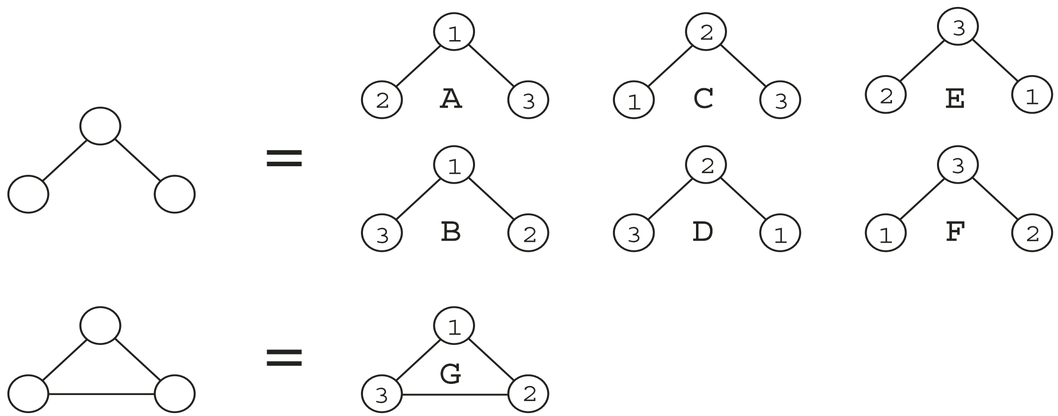
The partition function

$$Z(N,L) = \sum_{\alpha' \in lg(N,L)} \frac{1}{N!} = \sum_{\alpha \in g(N,L)} w(\alpha),$$

the sum over all labeled graphs (with equal weights)

the sum over distinct unlabeled graphs

where $w(\alpha)$ = (# of labeled graphs equiv. to α)/N!



Example: **binomial graphs**

- we start from N empty nodes
- add a link with probability equal to p
- the weight of graph is:

$$P(L) \propto \binom{N(N-1)/2}{L} p^L (1-p)^{N(N-1)/2-L}$$

p – probability of presence of link

the weight of graph with N nodes and L links

$$Z(N,\mu) = \sum_L \sum_{\alpha \in lg(N,L)} \frac{1}{N!} P(L(\alpha)) = (1-p)^{\binom{N}{2}} \sum_L \left(\frac{p}{1-p}\right)^L \sum_{\alpha \in lg(N,L)} \frac{1}{N!}$$

$$\propto \sum_L \exp(-\mu L)\, Z(N,L) \propto \sum_L \exp(-\mu L + S(N,L)),$$

$$\mu = \ln \frac{1-p}{p}$$

"chemical potential"

Z for Erdős-Rényi graphs (can be calculated very easy)

Therefore we have:

$$Z(N,\mu) = \sum_{L=0}^{\binom{N}{2}} e^{-\mu L} \frac{1}{N!} \binom{\binom{N}{2}}{L} = \frac{1}{N!}(1+e^{-\mu})^{\binom{N}{2}}$$

one can calculate many quantities:

$$\langle L \rangle = -\partial_\mu \ln Z(N,\mu)$$

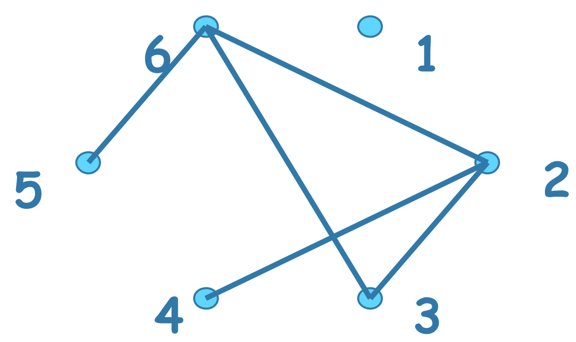
$$\langle L^2 \rangle - \langle L \rangle^2 = \partial_\mu^2 \ln Z(N,\mu)$$

$$\langle L \rangle = p\frac{N(N-1)}{2} = \frac{1}{1+e^{\mu}} \frac{N(N-1)}{2}$$

$$\langle L^2 \rangle - \langle L \rangle^2 = \binom{N}{2} \frac{e^{-\mu}}{(1+e^{-\mu})^2}$$

etc...

Weighten graphs



W = $p(0)p(3)p(2)p(1)p(1)p(3)$

- additional functional weight $W(\alpha)$:

$$Z(N,L) = \sum_{\alpha' \in lg(N,L)} (1/N!)\, W(\alpha') = \sum_{\alpha \in g(N,L)} w(\alpha) W(\alpha)$$

- the simplest non-trivial choice:

$$W(\alpha) = \prod_{i=1}^{N} p(q_i)$$

$p(q)$ is an arbitrary function of node's degree

What $p(q)$ should we take?

→ such that the resulting network has interesting properties.

- the degree distribution $\pi(q) \propto q^{-\gamma}$,
- in the limit of $N \to \infty$:

$$\pi(q) = \frac{p(q)}{q!} \exp(-Aq - B)$$

(q-1)! for tree graphs

- taking $p(q) \propto q! q^{-\gamma}$ and appropriate form of $p(q)$ for small q's, one can set A=0 and then $\pi(q) \propto q^{-\gamma}$,
- finite size corrections for $N < \infty$

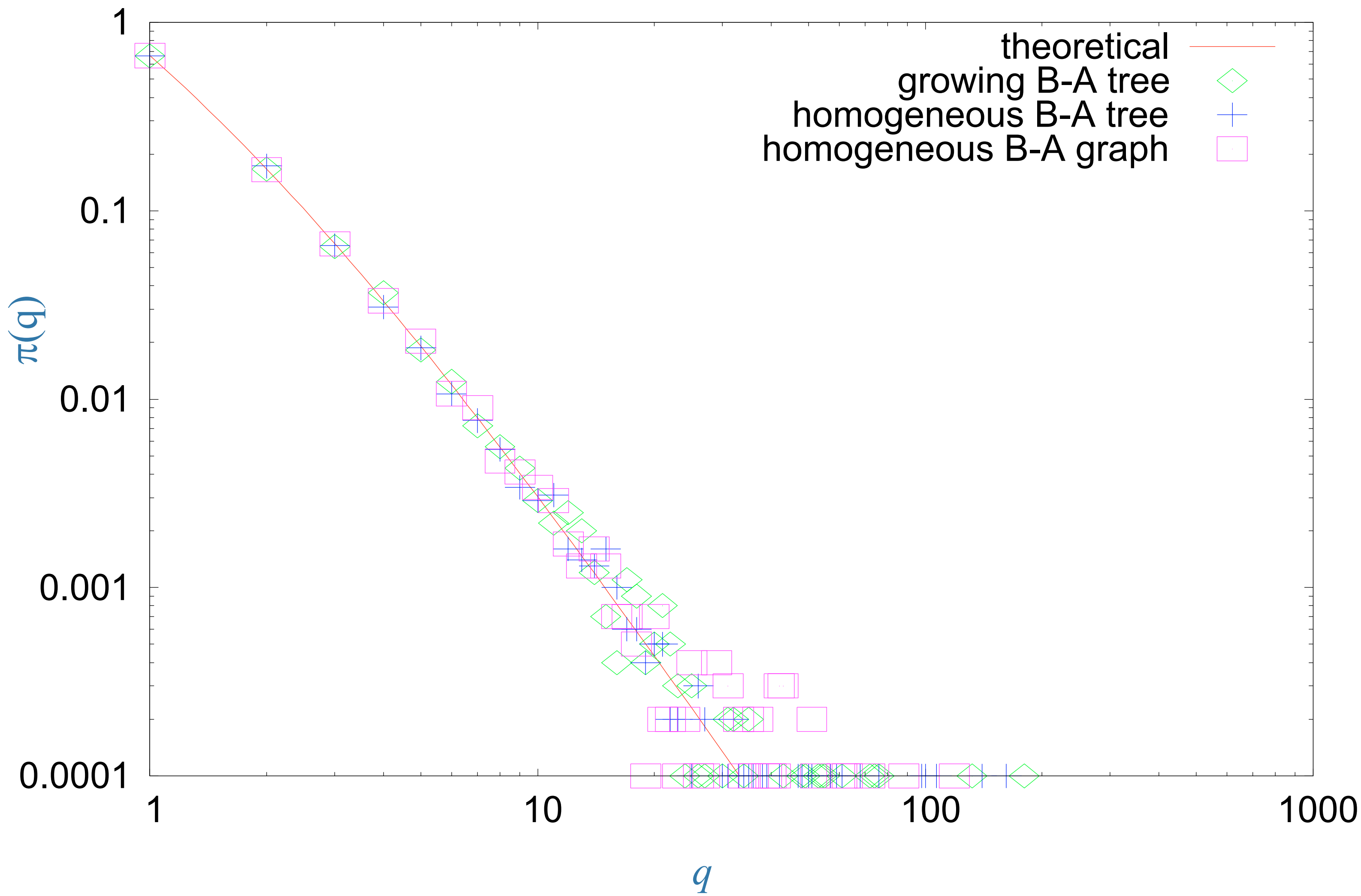
- **the Barabasi-Albert model with m = 1 (tree graphs):**

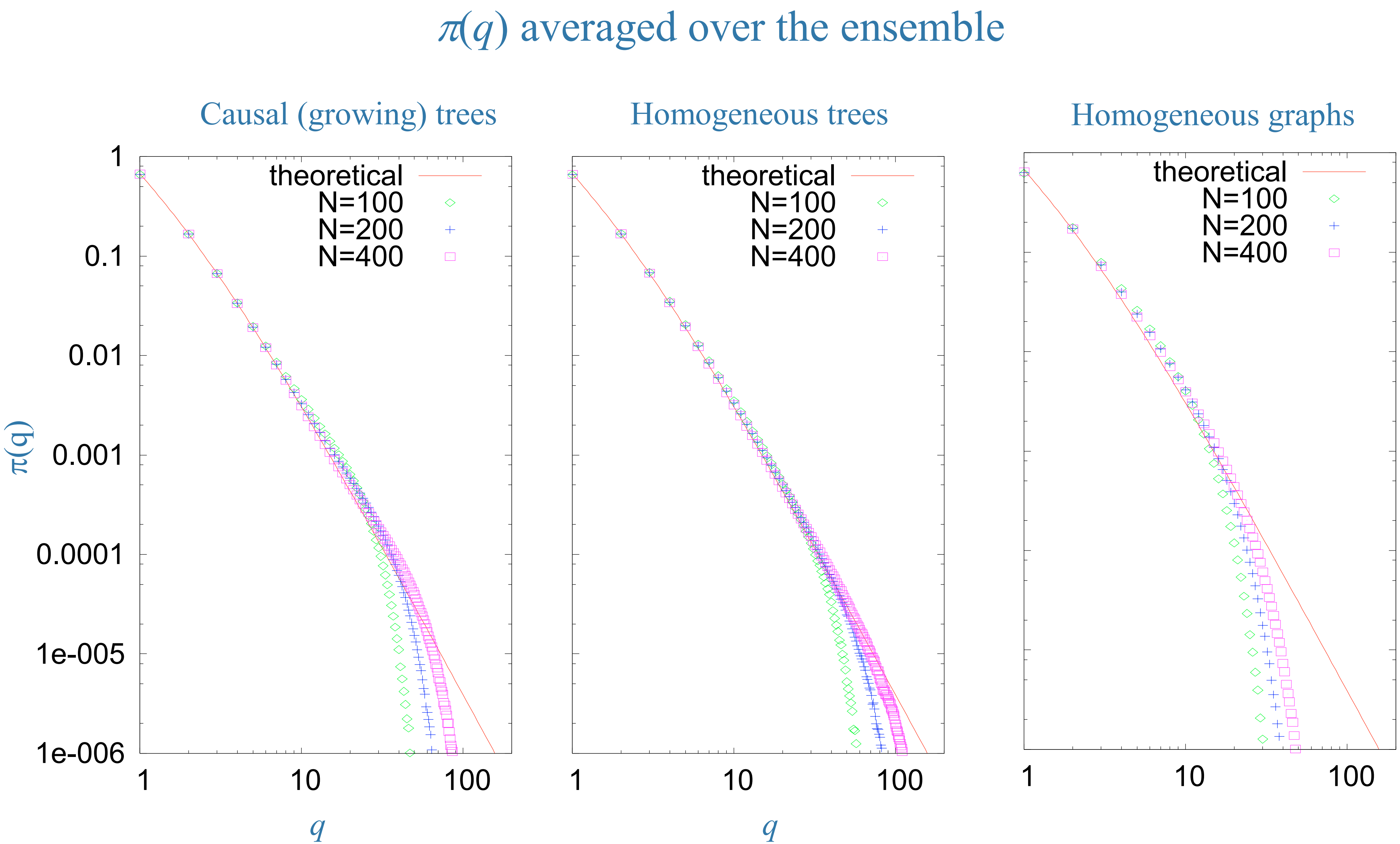
$$\pi(q) = \frac{4}{q(q+1)(q+2)}$$

- take $p(q) = (q-1)! \frac{4}{q(q+1)(q+2)}$
- if $L = N$, then $\langle q \rangle = 2$ and $A = B = 0 \Rightarrow$ degree distribution is B-A (but we have different graphs from those of B-A!)

A single network with N=10000 nodes and B-A degree distribution

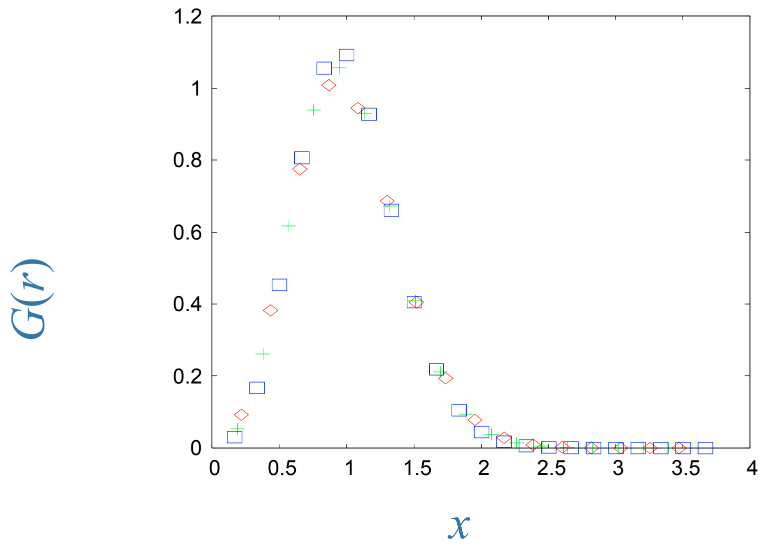
$$\pi(q) = \frac{4}{q(q+1)(q+2)}$$

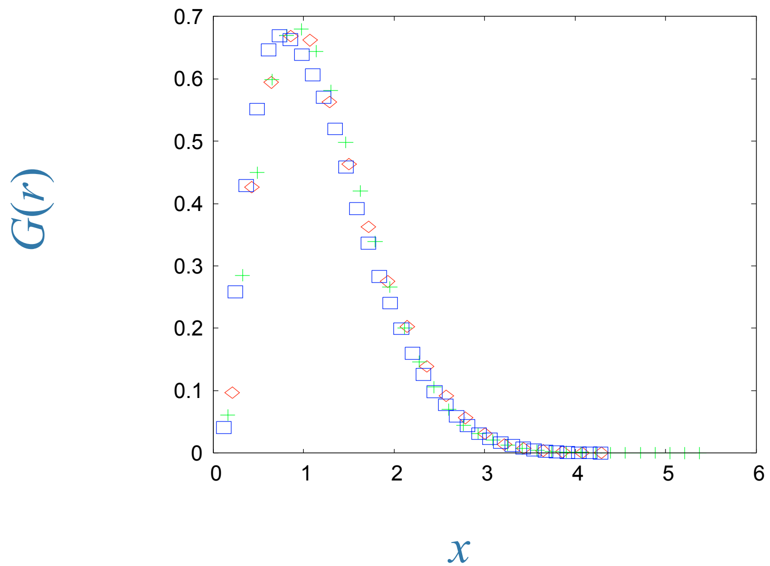




π(q) averaged over the ensemble
Causal (growing) trees
Homogeneous trees
Homogeneous graphs
theoretical
N=100
N=200
N=400
π(q)
q
1
0.1
0.01
0.001
0.0001
1e-005
1e-006
1
10
100

Node – node distance distribution $G(r)$





Growing (causal) networks are shorter than homogeneous

Causal BA graphs rescaling:

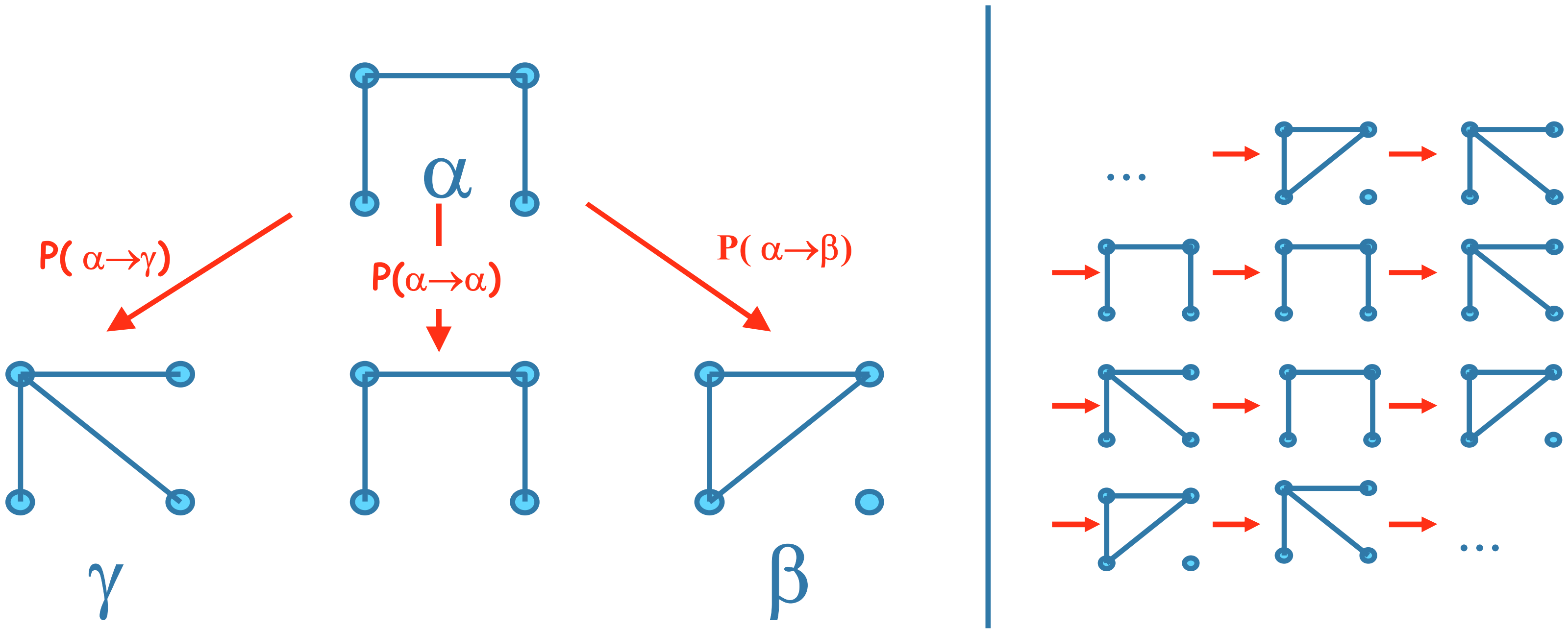
$$x = \frac{r}{\log N}$$

Homogeneous BA graphs rescaling:

$$x = \frac{r}{\sqrt{\frac{N}{\log N}}}$$

Can we calculate anything?

- many results possible for tree graphs,
- for simple graphs – more difficult but still possible
- numerical MC simulations possible – Markov process:



A program for these simulations – almost ready.

Analytically: E-R graphs + more triangles

$$Z(N,\mu) = \sum_A \exp[-\mu L(A) + S(A)] = \sum_A \exp\left[-\frac{\mu}{2} TrA^2 + gTrA^3\right]$$

sum over all possible

adjacency matrices

similar to matrix models

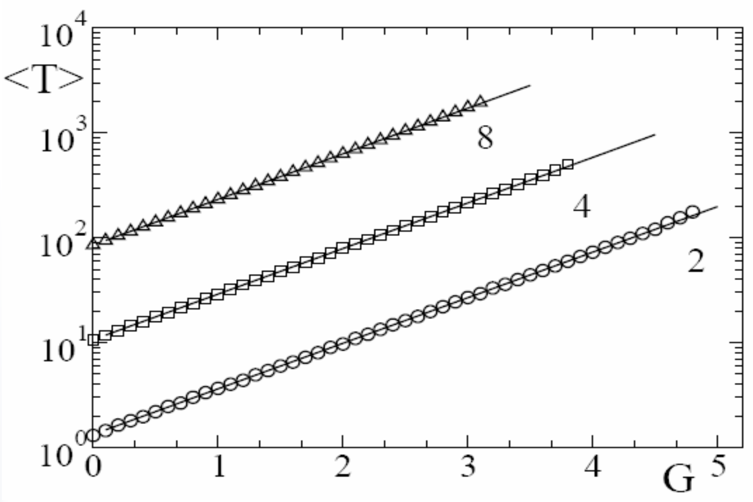
$$Z_{\mathrm{matrix}} = \int \mathrm{d}M \ \exp\left(-\frac{1}{2}Tr(M^2) + gTr(M^3)\right)$$

$$= Z_0 \sum \frac{g}{n!} \left\langle \left[Tr\left(A^3\right)\right]^n \right\rangle_{E-R}$$

expansion around

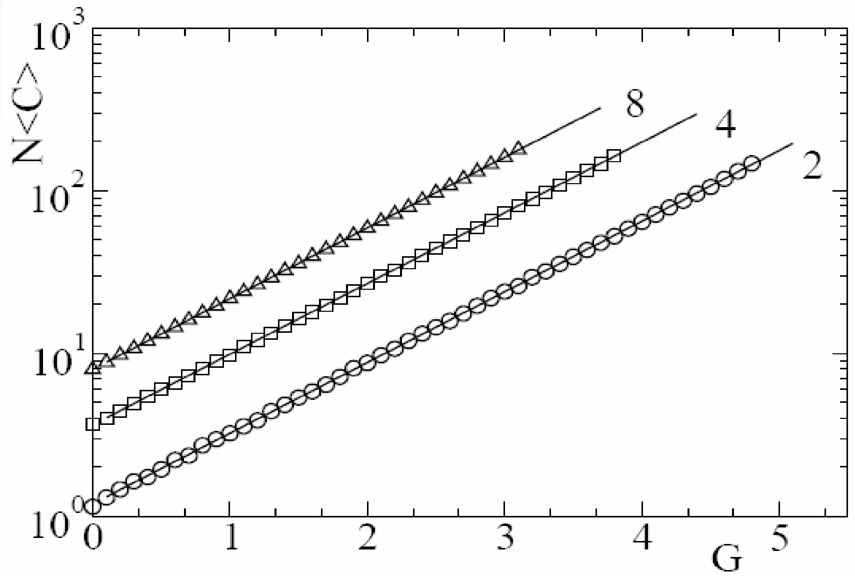
the E-R model

Some results



The number of triangles as a function of coupling constant G=6 g.

Plots for different average degree α = 2,4,8.



$N \times$ clustering coefficient

For E-R: $C = \alpha / N$

To summarize:

- for the same $\pi(q)$ homogeneous networks may have different properties than growing,
- statistical ensembles approach may be useful for static models,
- easy way of simulating (many newtorks – by changing only the weight)

For future works:

- ‘mixed model’: growing and rewiring,
- add triangles to increase clustering coefficient,

Some references

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- J. Park and M.E.J. Newman, „Statistical mechanics of networks", Phys. Rev. E **70,** 066117 (2004), cond-mat/0405556.
- L. Bogacz, Z. Burda, and B. Waclaw, „Homogeneous complex networks", cond-mat/0502124.
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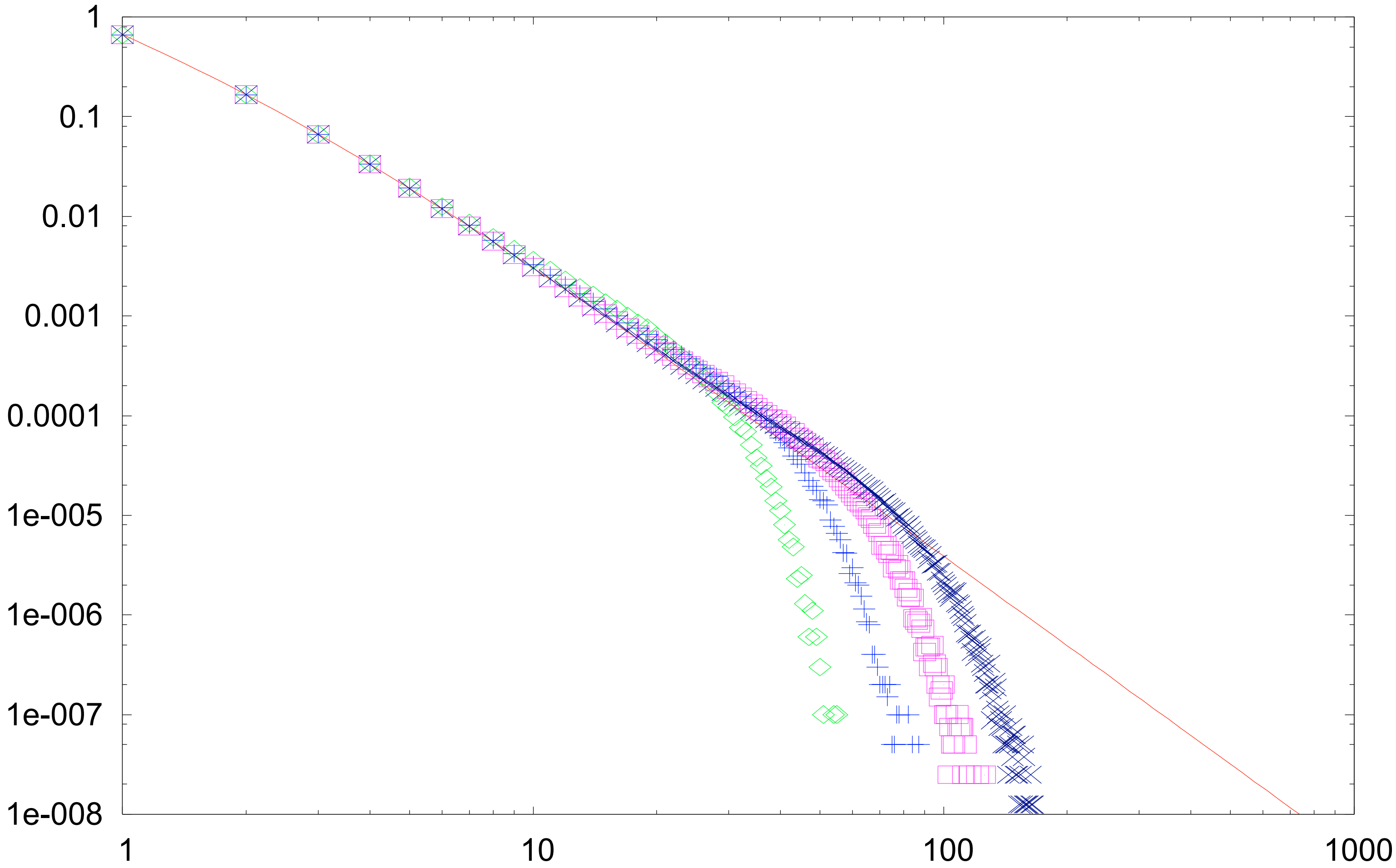
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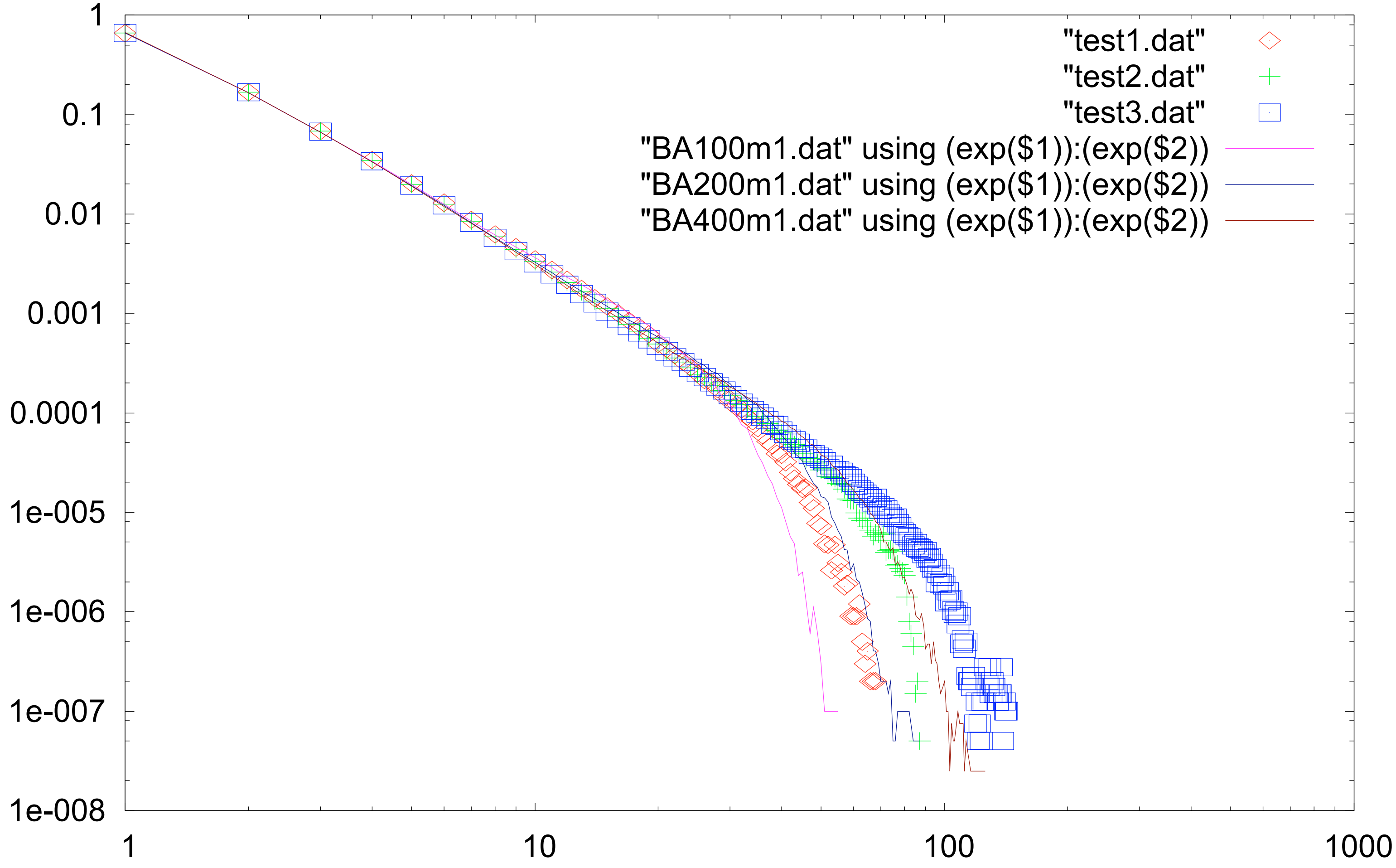

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