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# Structural Correlations and Critical Phenomena of Random Scale-Free Networks: Analytic Approaches

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# Structural correlations and critical phenomena of random scale-free networks: Analytic approaches

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# Plan

I. Static model of scale-free networks

II. Vertex correlation functions

III. Number of self-avoiding walks and circuits

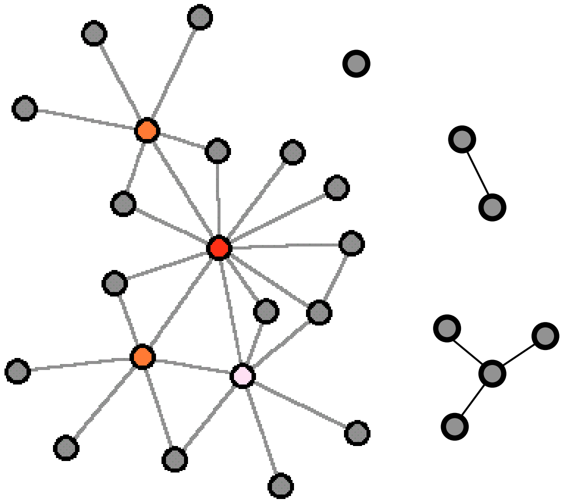
IV. Percolation transition

V. Critical phenomena of spin models defined on the static model

VI. Conclusion

# I. Static model of scale-free networks

## static model



$a_{i,j}$ = adjacency matrix element $(0,1)$

$$G = \{a_{i,j}\}$$

$$i, j = 1, \cdots, N \quad (N \text{ fixed})$$

1. Degree of a vertex $i$: $$k_i = \sum_{j=1}^{N} a_{i,j}$$

2. Degree distribution: $$P_D(k) \sim k^{-\lambda}$$

❋ We consider sparse, undirected, non-degenerate graphs only.

- Ensemble of graphs: $\langle O \rangle = \sum_G P(G)O(G)$
- Various equilibrium ensembles for SF networks
- Grandcanonical ensemble is for *G* with fluctuating number of link

➔ Static model: Goh et al PRL (2001), Lee et al NPB (2004), Pramana (2005, Statpys 22 proceedings), DH Kim et al PRE(2005 to appear)

Precursor of the fitness or hidden variable model [Caldarelli et al PRL (2002), Soederberg PRE (2002) , Boguna and Pastor-Satorras PRE (2003)]

## ❖ Construction of the static model

1. Each site is given a weight ("fitness")

$$P_i = i^{-\mu} \Big/ \sum_{j=1}^{N} j^{-\mu} \quad (i = 1, \cdots, N), \qquad \sum_i P_i = 1, \quad (0 < \mu < 1)$$

$\mu$ = Zipf exponent
= $1/(\lambda - 1)$

2. In each unit time, select one vertex $i$ with prob. $P_i$ and another vertex $j$ with prob. $P_j$.
3. If $i=j$ or $a_{ij}=1$ already, do nothing (fermionic constraint). Otherwise add a link, i.e., set $a_{ij}=1$.
4. Repeat steps 2,3 $NK$ times ($K$ = time = fugacity = $\langle L \rangle / N$).

**Comments**

- ✓ When $\mu$=0 $\Rightarrow$ ER case.
- ✓ Walker algorithm (+Robin Hood method) constructs networks in time $\mathcal{O}(N)$.
  $\Rightarrow$ $N = 10^7$ network in 1 min on a PC.
- ✓ Monte Carlo simulation with edge addition (deletion) prob. $f_{ij}$ $(1 - f_{ij})$ $\Rightarrow$ equivalent but inefficient.

- Prob$(a_{ij}=0)=(1-2P_iP_j)^{NK}=e^{-2KNP_iP_j}=1-f_{ij}$
- Prob$(a_{ij}=1)=f_{ij}=1-e^{-2KNP_iP_j}$

➔ Such algorithm realizes a “grandcanonical ensemble” of graphs $G=\{a_{ij}\}$ with weights

$$P(G)=\prod_{b\in G} f_{ij} \prod_{b\notin G}(1\text{-}f_{ij}) = \prod_{i>j} f_{ij}^{a_{ij}}(1-f_{ij})^{1-a_{ij}}$$

Each link is attached independently but with inhomegeous probability $f_{i,j}$ .

## Some basic properties:

$$P(k_i) = \text{ Poissonian with mean } <k_i> = 2KNP_i,$$

$$<k> \equiv \sum_{i=1}^{N} <k_i> / N = 2K,$$

$$<k^2> \equiv \sum_{i=1}^{N} <k_i^2> / N = <k> + <k>^2 \left( \sum_{i=1}^{N} NP_i^2 \right),$$

$$P_D(k) = \text{ Scale-free with } \lambda = 1 + \frac{1}{\mu}$$

Comments

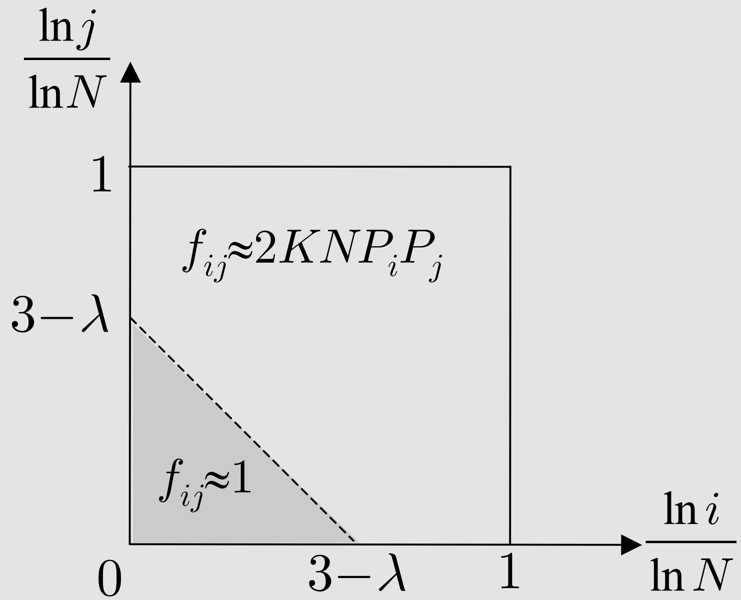
- ✓ Recall $f_{ij} = 1 - e^{-2KNP_iP_j}$

- ✓ When $\lambda > 3$ $(0 < \mu < 1/2)$,

$$f_{i,j} \approx 2KNP_iP_j = k_ik_j / <k> N$$

- ✓ When $2 < \lambda < 3$ $(1/2 < \mu < 1)$

$f_{ij}$ $\Rightarrow$




- ✓ Strictly uncorrelated in links, but vertex correlation enters when $2 < \lambda < 3$ due to the "fermionic constraint" (no self-loops and no multiple edges) unless one introduces an ad hoc cut-off in $k_{max} \sim N^{1/2}$.

$$k_{min} = <k_N> = 2K(1-\mu) = <k> (\lambda - 2)/(\lambda - 1)$$

$$k_{max} = <k_1> = k_{min} N^{\mu} \sim N^{1/(\lambda-1)}$$

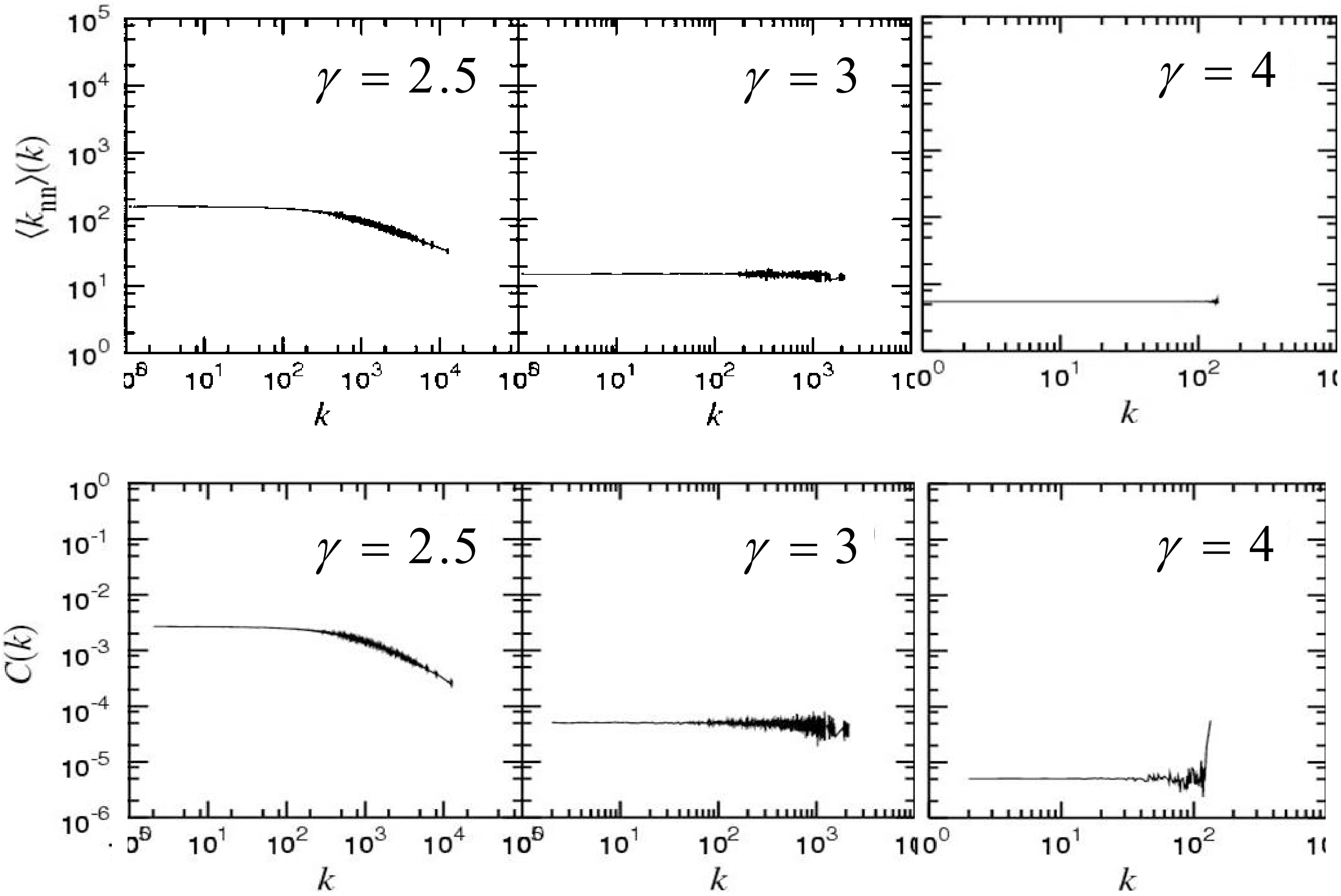
# II. Vertex correlation functions

$k_{\mathrm{nn}}(k) =$ average degree of neighbors of vertices with degree $k$.

$C(k) =$ clustering coefficient of vertices with degree $k$

Related work: Catanzaro and Pastor-Satoras, EPJ (2005)

# Vertex correlations



Our method of analytical evaluations:

$$k_{\mathrm{nn}}(i) \equiv \left\langle \frac{\sum_{j \neq i} a_{i,j} (\sum_{m \neq i,j} a_{j,m} + 1)}{k_i} \right\rangle \approx \frac{\sum_{j \neq i} f_{i,j} (\sum_{m \neq i,j} f_{j,m} + 1)}{< k_i >}$$

For a monotone decreasing function *F(x)* ,

$$\int_1^N F(x)\,dx + F(N) \leq \sum_{m=1}^N F(m) \leq \int_1^N F(x)\,dx + F(1)$$

Use this to approximate the first sum as

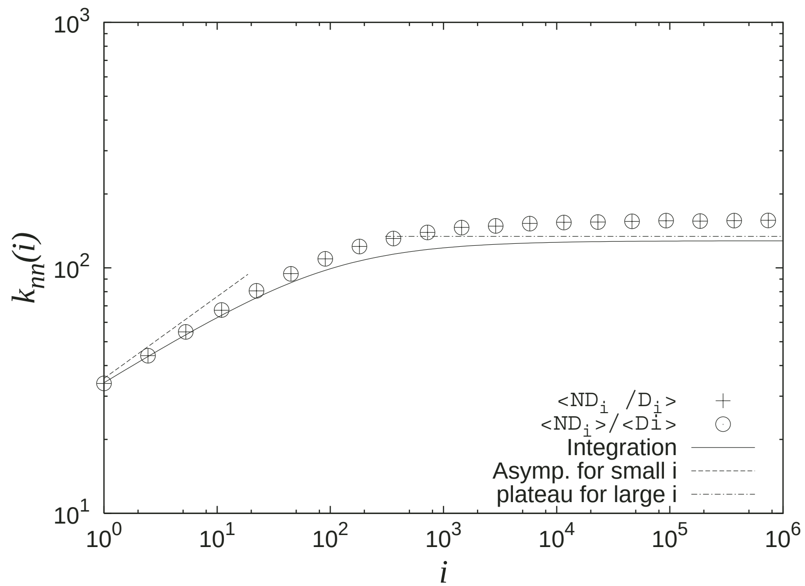
$$\sum_{m \neq i,j} f_{j,m} = (\lambda - 1)\Lambda^{(\lambda-1)} \int_\varepsilon^\Lambda (1 - e^{-xy}) y^{-\lambda}\,dy + O(1 - e^{-x\Lambda})$$

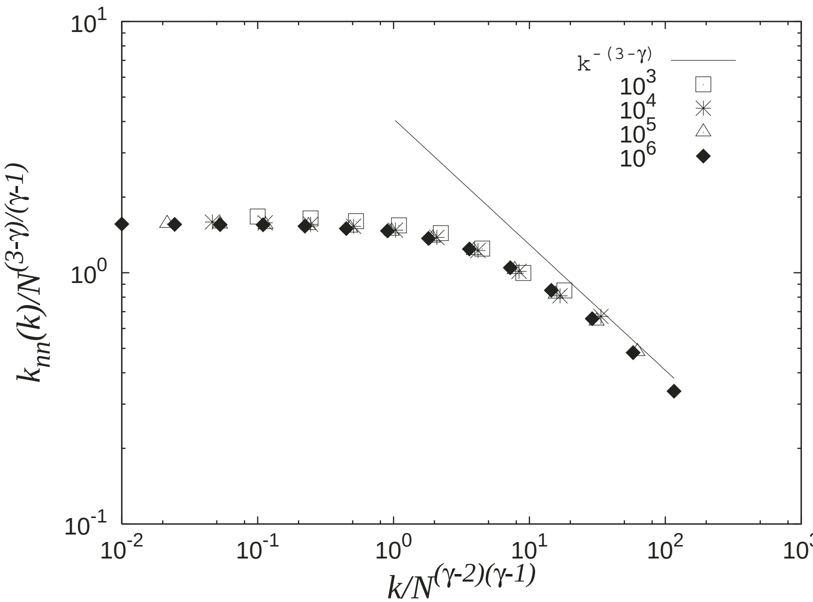
$$(\Lambda = N^\mu \varepsilon,\ \ \varepsilon \sim N^{-1/2},\ \ x = \sqrt{< k > N} P_j \sim < k_i > / \sqrt{N})$$

Similarly for the second sum.

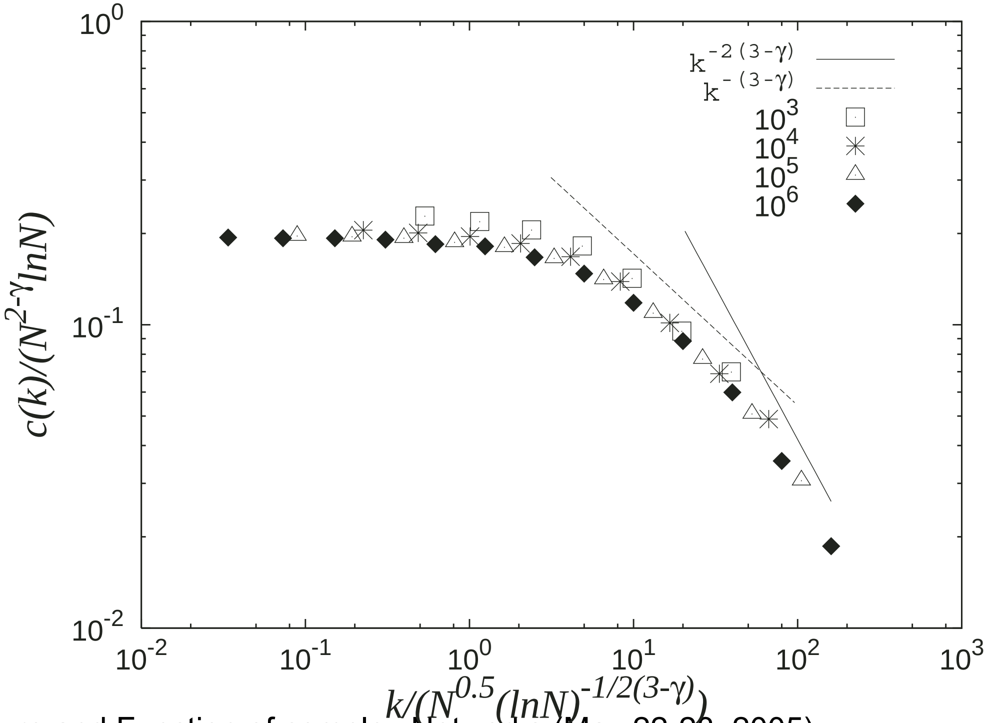
## Result (1) $k_{nn}(k)$ for $2<\lambda<3$

$$k_{nn}(k) \sim \begin{cases} N^{\frac{3-\lambda}{\lambda-1}} & \text{for } k < k_c \sim N^{\frac{\lambda-2}{\lambda-1}} \\ N^{3-\lambda}\, k^{-(3-\lambda)} & \text{for } k > k_c \sim N^{\frac{\lambda-2}{\lambda-1}} \end{cases}$$



$$C(k) \sim \begin{cases} N^{2-\lambda} \ln N & \text{for } k < k_C \\ N^{(-2\lambda^2+8\lambda-7)/(\lambda-1)}\, k^{-(3-\lambda)} & \text{for } k_C < k < N^{1/2} \\ N^{5-2\lambda}\, k^{-2(3-\lambda)} & \text{for } N^{1/2} < k < k_{\max} \sim N^{1/(\lambda-1)} \end{cases}$$

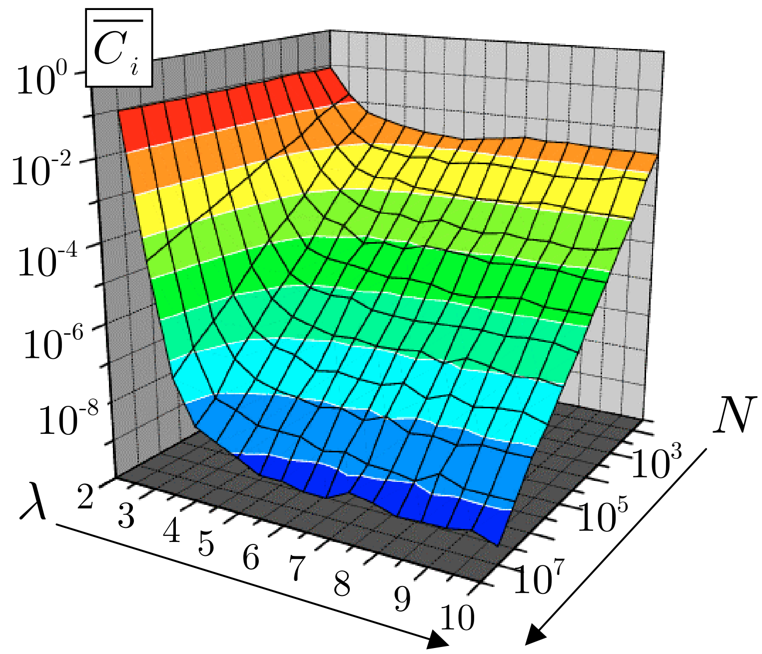


## Finite size effect of the clustering coefficient:

$$(\text{mean \# of triples}) \sim \sum_{i,j,k} f_{ij} f_{ik} \sim \begin{cases} K^2 N & \lambda > 3 \\ K^2 N^{2/(\lambda-1)} & 2 < \lambda < 3 \end{cases}$$

$$(\text{mean \# of triangles}) \sim \sum_{i,j,k} f_{ij} f_{jk} f_{ki} \sim \begin{cases} K^3 & \lambda > 3 \\ K^{\frac{3}{2}(\lambda-1)} N^{\frac{3}{2}(3-\lambda)} & 2 < \lambda < 3 \end{cases}$$

$$C = \overline{C(i)} \sim \frac{\ln N}{N^{\lambda-2}}$$



# III. Number of self-avoiding walks and circuits

- ✓ The number of self-avoiding walks and circuits (self-avoiding loops) are of basic interest in graph theory.
- ✓ Some related works are: Bianconi and Capocci, PRL (2003), Herrero, cond-mat (2004), Bianconi and Marsili, cond-mat (2005) etc.
- ✓ Issue: How does the vertex correlation work on the statistics for $2<\lambda<3$ ?

## Number of SAWs and Circuits

The number of *L*-step self-avoiding walks on a graph is

$$\sum{}' \left\langle a_{i,j} a_{j,k} \cdots a_{l,m} \right\rangle = \sum{}' f_{i,j} f_{j,k} \cdots f_{l,m}$$

where the sum is over distinct ordered set of (*L*+1) vertices,

$\{i, j, \cdots, m\}$. We consider finite *L* only.

The number of circuits or self-avoiding loops of size *L* on a graph is

$$\sum{}' \left\langle a_{i,j} a_{j,k} \cdots a_{l,i} \right\rangle = \sum{}' f_{i,j} f_{j,k} \cdots f_{l,i}$$

Strategy for 2<$\lambda$ <3: For upper bounds, we use

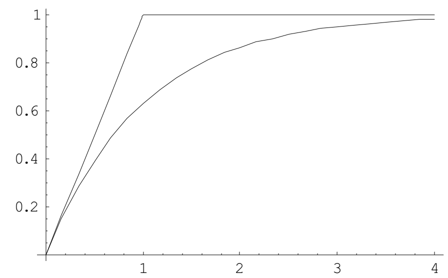
$$\sum_{j(\neq i,\cdots,m)} f_{i,j} \leq \sum_{j=1}^{N} f_{i,j} \leq \int_1^N f_{i,x}\,dx + f_{i,1}$$

$$= (\lambda - 1)\Lambda^{\lambda-1}\int_\varepsilon^\Lambda (1-e^{-xy})\,y^{-\lambda}\,dy + (1-e^{-x\Lambda})$$

$$(\Lambda = N^{\mu}\varepsilon, \quad \varepsilon \sim N^{-1/2}, \quad x = \sqrt{<k> N} P_i)$$

and

$$1-e^{-x} \leq L_0(x) \equiv \begin{cases} x & \text{if} \quad 0 \leq x \leq 1 \\ 1 & \text{if} \quad 1 \leq x \end{cases}$$




repeatedly. Similarly for lower bounds with $(1-e^{-x}) \geq (1-e^{-1})G_0(x)$

The leading powers of *N* in both bounds are the same.

Note: The “surface terms” are of the same order as the “bulk terms”.

- ✓ For $\lambda > 3$, straightforward in the static model
- ✓ For $2 < \lambda \le 3$, the leading order terms in *N* are obtained.

## Result(3): Number of *L*-step self-avoiding walks

$$\sim N < k > \left( \frac{< k^2 >}{< k >} - 1 \right)^{(L-1)} \qquad (\lambda > 3)$$

$$\sim N < k > \left( 0.25 < k > \ln N \right)^{(L-1)} \qquad (\lambda = 3)$$

$$\sim N^{1+\left(\frac{3-\lambda}{2}\right)(L-1)} N^{\frac{(3-\lambda)^2}{2(\lambda-1)}} \qquad (2 < \lambda < 3,\ L = \text{even})$$

$$\sim N^{1+\left(\frac{3-\lambda}{2}\right)(L-1)} \left( \ln N \right)^{(L-1)/2} \qquad (2 < \lambda < 3,\ L = \text{odd})$$

## Results(4): Number of circuits of size $L \ (L \geq 3)$

$$\frac{1}{2L}\left(\frac{<k^2>}{<k>}-1\right)^L \qquad (\lambda > 3)$$

$$\sim \left(0.25<k>\ln N\right)^L \qquad (\lambda = 3)$$

$$\sim N^{(3-\lambda)L/2}\ln N \qquad (2<\lambda<3,\ L=\text{even})$$

$$\sim N^{(3-\lambda)L/2} \qquad (2<\lambda<3,\ L=\text{odd})$$

# IV. Percolation transition

Lee, Goh, Kahng and Kim, NPB (2004)

- ✓ The static model graph weight $P(G)=\prod_{b\in G} f_{ij} \prod_{b\notin G}(1\text{-}f_{ij}) = \prod_{i>j} f_{ij}^{a_{ij}}(1-f_{ij})^{1-a_{ij}}$ can be represented by a Potts Hamiltonian,

$$\mathcal{H} = -2KN\sum_{i>j} P_i P_j [\delta(\sigma_i,\sigma_j)-1], \quad (\sigma_i = 1,\cdots,q)$$

$$\begin{aligned}
\mathrm{e}^{-\mathcal{H}} &= \prod_{i>j}[\mathrm{e}^{-2KNP_iP_j} + (1-\mathrm{e}^{-2KNP_iP_j})\delta(\sigma_i,\sigma_j)] \\
&= \prod_{i>j}[(1-f_{ij}) + f_{ij}\delta(\sigma_i,\sigma_j)] \\
&= \sum_G \prod_{b\notin G}(1-f_{ij}) \prod_{b\in G}[f_{ij}\delta(\sigma_i,\sigma_j)] \\
&= \sum_G P(G) \prod_{b\in G}\delta(\sigma_i,\sigma_j)
\end{aligned}$$

# Percolation transition

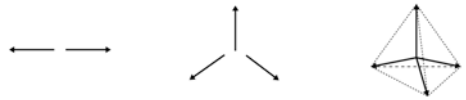
Partition function: $\mathcal{Z} = \left\langle q^{\#\text{ of clusters}} \right\rangle$

Order parameter $\xrightarrow[q\to 1]{}$ giant cluster size $(S{=}mN)$

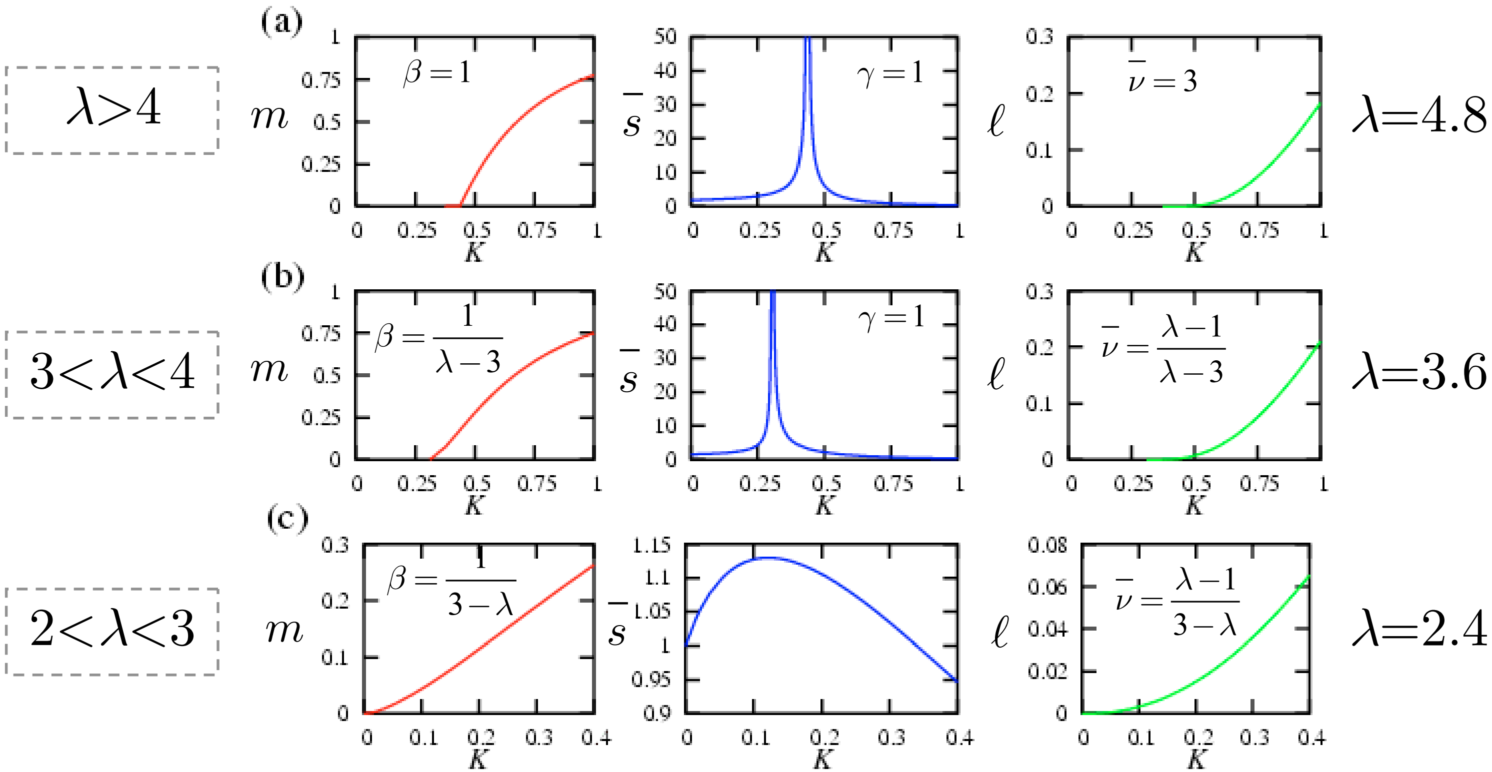
Susceptibility $\xrightarrow[\text{q}\to 1]{}$ mean cluster size $\bar{s} = \sum_s{}' \, s^2 n(s)/N$

$$\left.\frac{\partial}{\partial q}\mathcal{Z}\right|_{q=1} + \langle L \rangle - N =$$ mean number of independent loops $(\ell N)$

# Exact analytic evaluation of the Potts free energy:

1. Vector spin representation ↔   $\mathcal{H} \sim \left(\sum_i P_i \vec{s}_i\right)^2$
2. Integral representation of the partition function
3. Saddle-point analysis

- Percolation transition at $K_c = (2N\sum_i P_i^2)^{-1} \leftrightarrow \langle k^2\rangle / \langle k\rangle = 2$
- Explicit evaluations of thermodynamic quantities.



# V. Critical phenomena of spin models defined on the static model

# Spin models on SM

Spin models defined on the static model network can be analyzed by the replica method.

$$H[G] = -\sum_{i<j} a_{i,j} J_{i,j} \overrightarrow{S_i} \cdot \overrightarrow{S_j} \quad \text{with quenched bond disorder } \{a_{i,j}\}$$

$$\text{Free energy in replica method} = \lim_{n \to 0} \langle Z^n - 1 \rangle / n$$

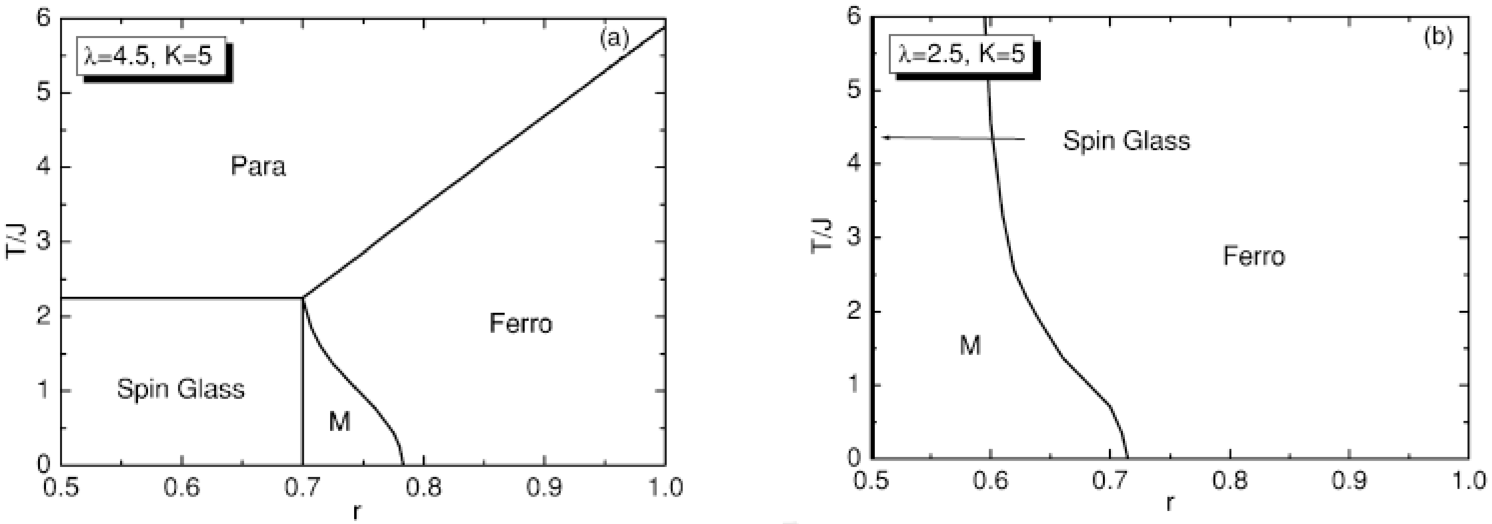
$$\langle Z^n \rangle = \mathrm{tr} \prod_{i<j} \left[ (1 - f_{i,j}) + f_{i,j} \exp(\beta J_{i,j} \sum_{\alpha} \overrightarrow{S^{\alpha}{}_i} \cdot \overrightarrow{S^{\alpha}{}_j}) \right] \equiv \exp[-H_{\text{eff}}]$$

$$-H_{\text{eff}} = \sum_{i<j} \langle k \rangle N P_i P_j \left( \exp(\beta J_{i,j} \sum_{\alpha} \overrightarrow{S^{\alpha}{}_i} \cdot \overrightarrow{S^{\alpha}{}_j}) - 1 \right) + o(N)$$

When $J_{i,j}$ are also quenched random variables, extra averages on each $J_{i,j}$ should be done.

We applied this formalism to the Ising spin-glass [Kim et al PRE (2005)]

$$P(J_{i,j}) = r\delta(J_{i,j} - J) + (1-r)\delta(J_{i,j} + J)$$




Phase diagrams in *T-r* plane for $\lambda > 3.0$ and $\lambda < 3.0$

Critical behavior of the spin-glass order parameter in the replica symmetric solution:

$$q \equiv \sum_{i=1}^{N} P_i \left\langle S_i^{\alpha} S_i^{\beta} \right\rangle \sim \begin{cases} (T_C - T) & (\lambda > 4) \\ (T_C - T) / \ln(T_C - T)^{-1} & (\lambda = 4) \\ (T_C - T)^{1/(\lambda-3)} & (3 < \lambda < 4) \\ T^2 \exp(-2T^2 / < k >) & (\lambda = 3) \\ T^{-2(\lambda-2)/(3-\lambda)} & (2 < \lambda < 3) \end{cases}$$

$$Q \equiv \frac{1}{N} \sum_{i=1}^{N} \left\langle S_i^{\alpha} S_i^{\beta} \right\rangle \sim \frac{q}{< k > T^2} \sim T^{-2/(3-\lambda)} \quad \text{for } 2 < \lambda < 3$$

To be compared with the ferromagnetic behavior for $2<\lambda<3$;

$$m \equiv \sum_{i=1}^{N} P_i < S_i > \sim T^{-(\lambda-2)/(3-\lambda)}, \quad M \equiv \frac{1}{N} \sum_{i=1}^{N} < S_i > \sim T^{-1/(3-\lambda)}$$

# VI. Conclusion

1. The static model of scale-free network allows detailed analytical calculation of various graph properties and free-energy of statistical models defined on such network.
2. The constraint that there is no self-loops and multiple links introduces local vertex correlations when $\lambda$, the degree exponent, is less than 3.
3. Two node and three node correlation functions, and the number of self-avoiding walks and circuits are obtained for $2<\lambda<3$. The walk statistics depend on the even-odd parity.
4. Kasteleyn construction of the Potts model is utilized to calculate thermodynamic quantities related to the percolation transition such as the mean number of independent loops.
5. The replica method is used to obtain the critical behavior of the spin-glass order parameters in the replica symmetry solution.

## ✓ Walker algorithm

Efficient method for selecting intergers $1, 2, \cdots, N$ with probabilities $P_1, P_2, \cdots, P_N$. $\left(\sum_i P_i = 1\right)$

$N=3$