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**School and Workshop on
Structure and Function of Complex Networks**

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Emergence and Resilience of Social Networks: A General Model

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Emergence and resilience of social networks: a general model

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General Model

- N nodes
- links: adjacency matrix $a_{ij} = a_{ji}$
- nodes have 'attribute' x_i

dynamics:

- attribute x_i depends on neighbours of node i
- a_{ij} depends on attributes

The Model

- N nodes
- links: adjacency matrix $a_{ij} = a_{ji}$

- nodes have 'attribute' x_i, an integer from 1 to q
- nodes choose their attribute to align with neighbours' attributes

- nodes form links with nodes of same attribute AT RATE η
- links decay at rate λ ($\lambda = 1$)

Attribute dynamics

Coordination Game

node i gets payoff 1 from interaction with j if $x_i = x_j$, 0 otherwise

total payoff for node i: $Payoff = \Sigma_j \, \delta_{x_i,x_j}$
at rate $\nu \gg 1$, nodes change their states:

$x_i \longrightarrow x_{inew}$ with probability 1 if $\text{Payoff}_{\text{new}} > \text{Payoff}_{\text{old}}$
with probability $\exp(-\frac{1}{T}(\text{Payoff}_{\text{old}} - \text{Payoff}_{\text{new}})$ if $\text{Payoff}_{\text{new}} > \text{Payoff}_{\text{old}}$

Potts model.

What Happens ?

- 2 parameters:
- η - link formation rate
- T - noise, error rate, 'temperature'

- assume n large
- set $q = 10$, for $q > 2$, qualitative behaviour unchanged
- $\nu \gg 1$ i.e. coordination game played much faster than network rewiring

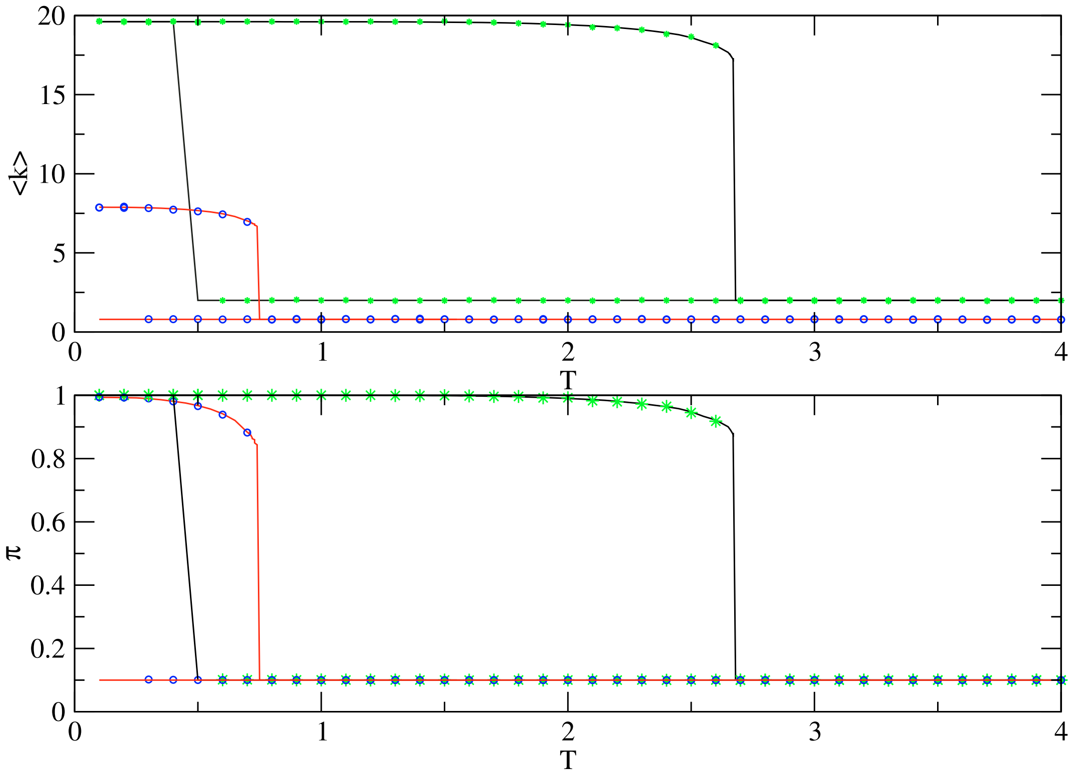
What Happens ?

low η: low connectivity. high η: high connectivity and Giant Component.
high T: no coordination. low T: coordination.

- for ordering (coordination), need to have a Giant Component

- positive feedback:
- ordered state enhances link formation $\longrightarrow$ more connected network
- highly connected network enhances ordering

What Happens



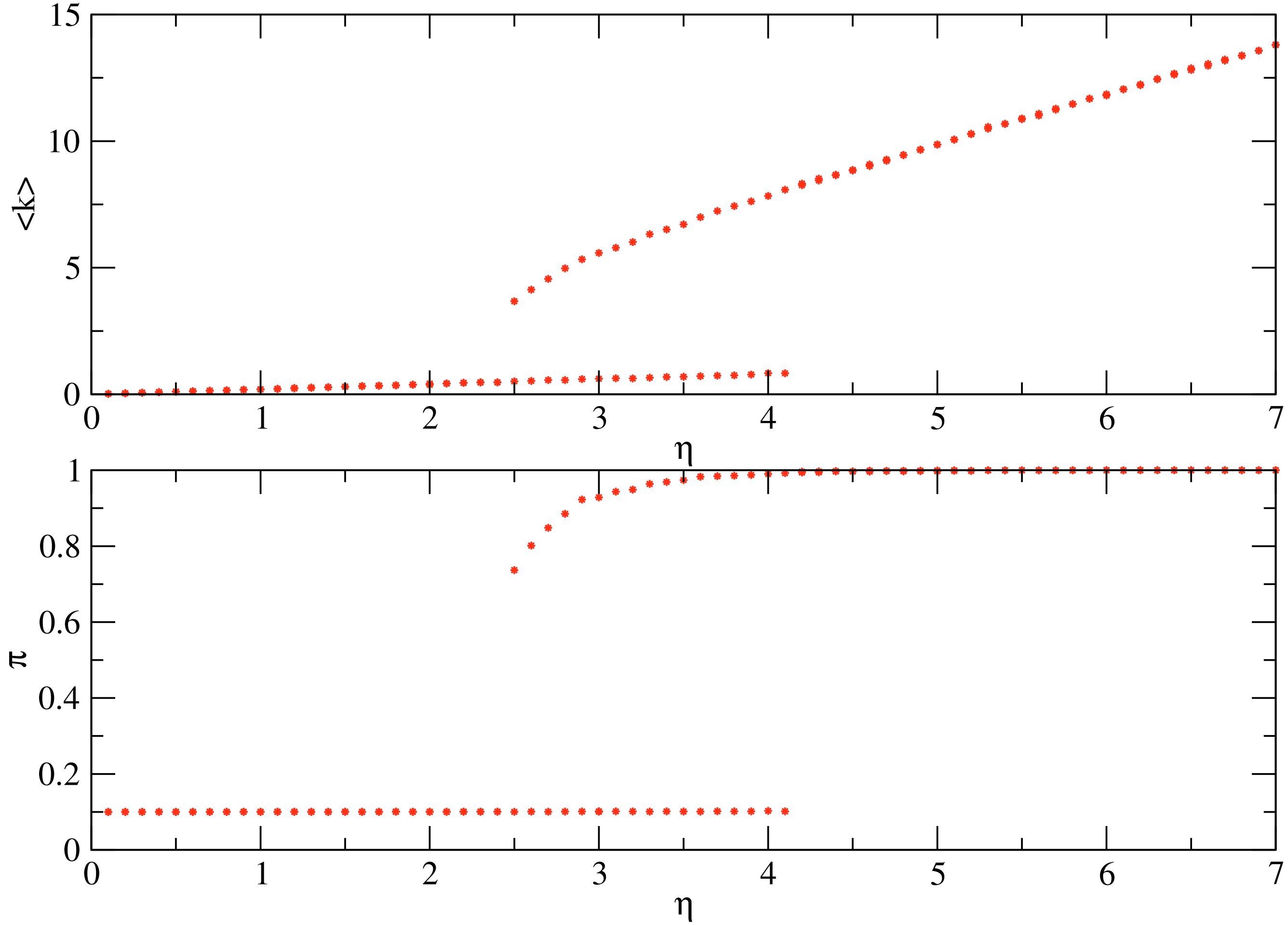
black and green: $\eta = 10$ theory and simulation.

red and blue: $\eta = 4$ theory and simulation.

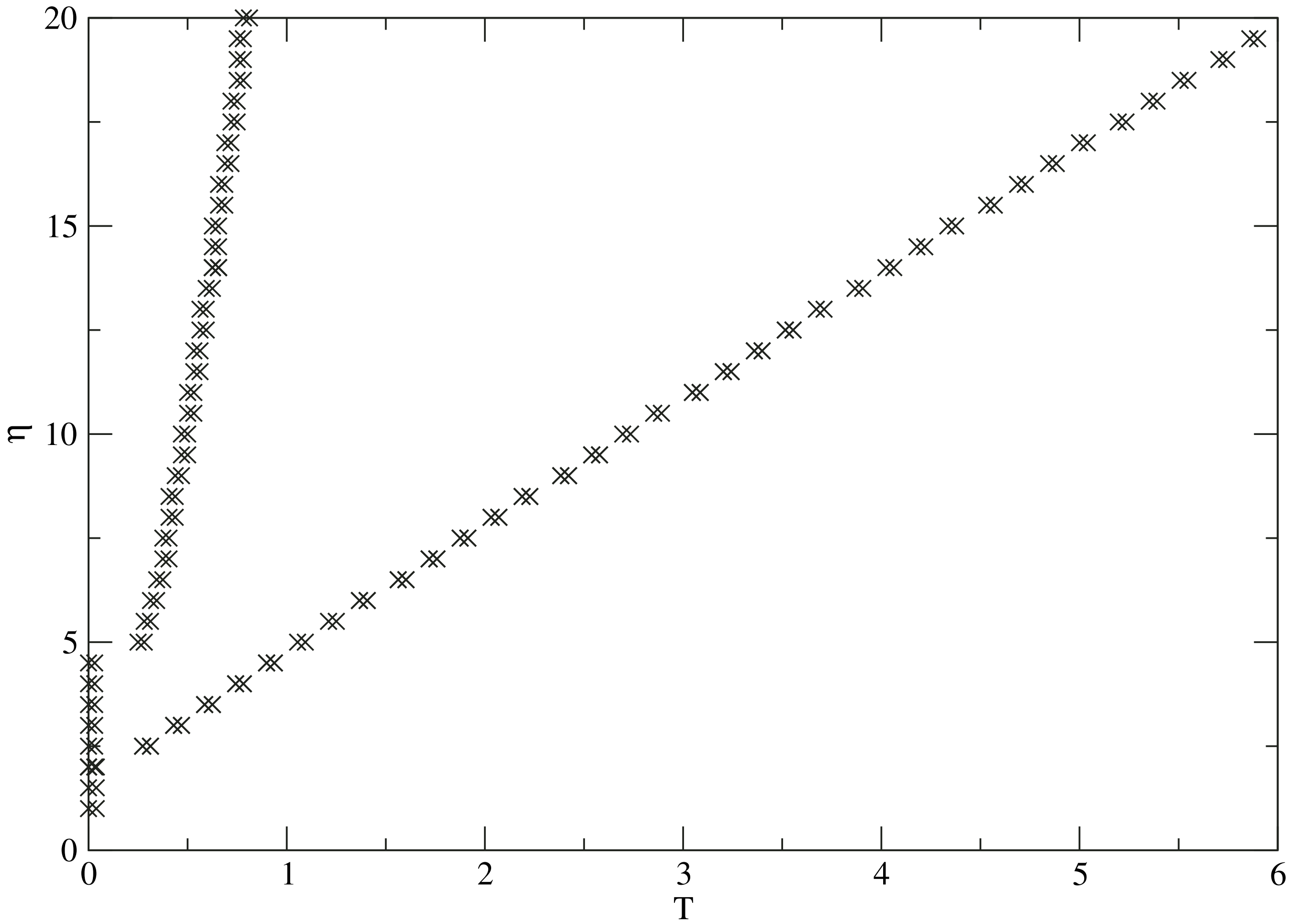
n=1000

What Happens

red: $T = 0.25$



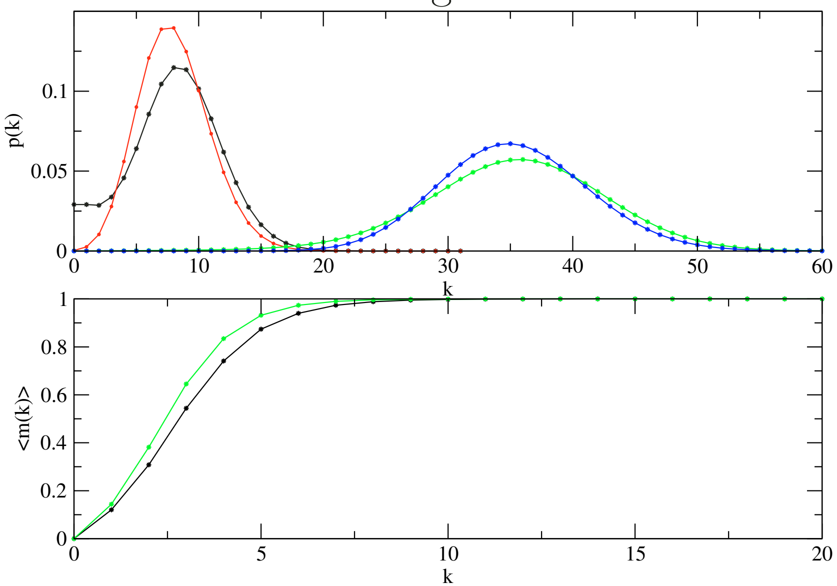
phase diagram. simulations



THEORY

need to know p(k)

need to know π, the probability 2 nodes are in the same state - by solving Potts model to find magnetisation



simulation results

so p(k) NOT poissonian

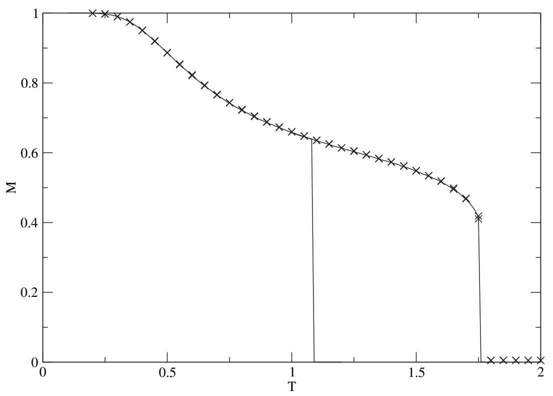
$< m_i >$ depends on k_i

Potts model on random graph

Dorogovtsev, Goltsev, Mendes. Eur. Phys. J. B. 2004
GE and M. Marsili J. Stat. Mech. 2005

from p(k), for a given T: find magnetisation

use local tree-like nature of the network



for Giant Component only !

Find magnetisation

Hamiltonian

$$\mathcal{H} = -J \sum_{<i,j>} \delta_{\sigma_i,\sigma_j} - \mathrm{b} \sum_i \delta_{\sigma_i,1} \tag{1}$$

Partition function

$$Z = \sum_{\{\sigma\}} \exp\left(K \sum_{<i,j>} \delta_{\sigma_i,\sigma_j} + H \sum_i \delta_{\sigma_i,1} \right) \tag{2}$$

Magnetisation of site i

$$m_i = \frac{q \langle \delta_{\sigma_i,1} \rangle - 1}{q-1} \tag{3}$$

Solve iteratively

$$g_{1,i}(\sigma_0) = \sum_{\{\sigma_l\}} e^{\sum_{<l,m>} K\delta_{\sigma_l,\sigma_m} + K\delta_{\sigma_0,\sigma_i} + H\sum_l \delta_{\sigma_l,1}} \tag{4}$$

$$Z = \sum_{\sigma_0} e^{H\delta_{\sigma_0,1}} \prod_{i=1}^{k_0} g_{1,i}(\sigma_0) \tag{5}$$

Recursion relation for $g_{r,i}(\sigma_{r-1})$

$$g_{r,i}(\sigma_{r-1}) = \sum_{\sigma_r} e^{K\delta_{\sigma_r,\sigma_{r-1}} + H\delta_{\sigma_r,1}} \prod_{j=1}^{k_{r,i}-1} g_{r+1,j}(\sigma_r) \tag{6}$$

$k_{r,i}$ is the degree of node (r, i).

$$x_{r,i} = \frac{g_{r,i}(\alpha)}{g_{r,i}(1)}, \qquad \alpha > 1 \tag{7}$$

$$x_{r,i} = \frac{e^H + (q - 2 + e^K) \prod_{j=1}^{k_{r,i}-1} x_{r+1,j}}{e^{H+K} + (q-1) \prod_{j=1}^{k_{r,i}-1} x_{r+1,j}}. \tag{8}$$

Change of variables $h_{r,j} = -\ln(x_{r,j})$
Distribution density of $h_{r,i}$ with $k_{r,i} = k$

$$\rho_r(h|k) = \int_{-\infty}^{\infty} \prod_{j=1}^{k-1} dh_j \tilde{\rho}_{r+1}(h_j) \delta \left[h - Y \left(\sum_{j=1}^{k-1} h_j \right) \right] \tag{9}$$

$$Y(s) = \ln \left[\frac{e^{H+K} + (q-1)e^{-s}}{e^H + (q-2+e^K)e^{-s}} \right] \tag{10}$$

Distribution of h on the neighbours of a node

$$\tilde{\rho}_r(h) = \sum_{k=1}^{\infty} \tilde{P}(k) \rho_r(h|k) \tag{11}$$

Distribution of fields at the neighbour of a node

$$\tilde{\rho}_r(h) = \sum_{k=1}^{\infty} \tilde{P}(k) \int_{-\infty}^{\infty} \prod_{j=1}^{k-1} dh_j \tilde{\rho}_{r+1}(h_j) \delta \left[h - Y \left(\sum_{j=1}^{k-1} h_j \right) \right] \tag{12}$$

Find magnetisation: population dynamics

population h_i, $i = 1, \ldots, M$ of $M \gg 1$ values of h. Evolve the population by iteration with the following procedure:

- Draw at random k from the distribution $\tilde{P}(k)$
- Draw $k-1$ values of h at random from the population $\{h_i\}$ and sum them to get h_{sum}.
- replace a random member of the population by

$$h_{new} = Y\left(h_{sum}\right) \tag{13}$$

Iterate until convergence. The magnetisation is then found by averaging the local magnetisations on many nodes,

- draw k from $P(k)$
- draw k values of h from $\tilde{\rho}(h)$ and sum them to get h_{sum} and insert this into

$$\langle m_0 \rangle = \frac{\exp(H) - \exp(-h_{sum})}{\exp(H) + (q-1)\exp(-h_{sum})} \tag{14}$$

Iterate many times and average the results to get $M(T)$.

Find p(k) given magnetisation

$$\dot{p}(k) = \lambda(k+1)p(k+1) + 2\eta p(k-1)\pi(k-1) - 2\eta p(k)\pi(k) - \lambda k p(k)$$

$$\pi(k) = \frac{1}{q} + \frac{q-1}{q} < m(k) >< m > (1 - u^k)$$

$u =$ Prob(link does NOT lead to the Giant Component)

$$u = \Sigma_{k=1}^{\infty} \frac{kp(k)}{<k>} u^{k-1}$$

Find p(k) given magnetisation

so, find iteratively:

$$p(1) = 2\eta p(0)\pi(0)$$

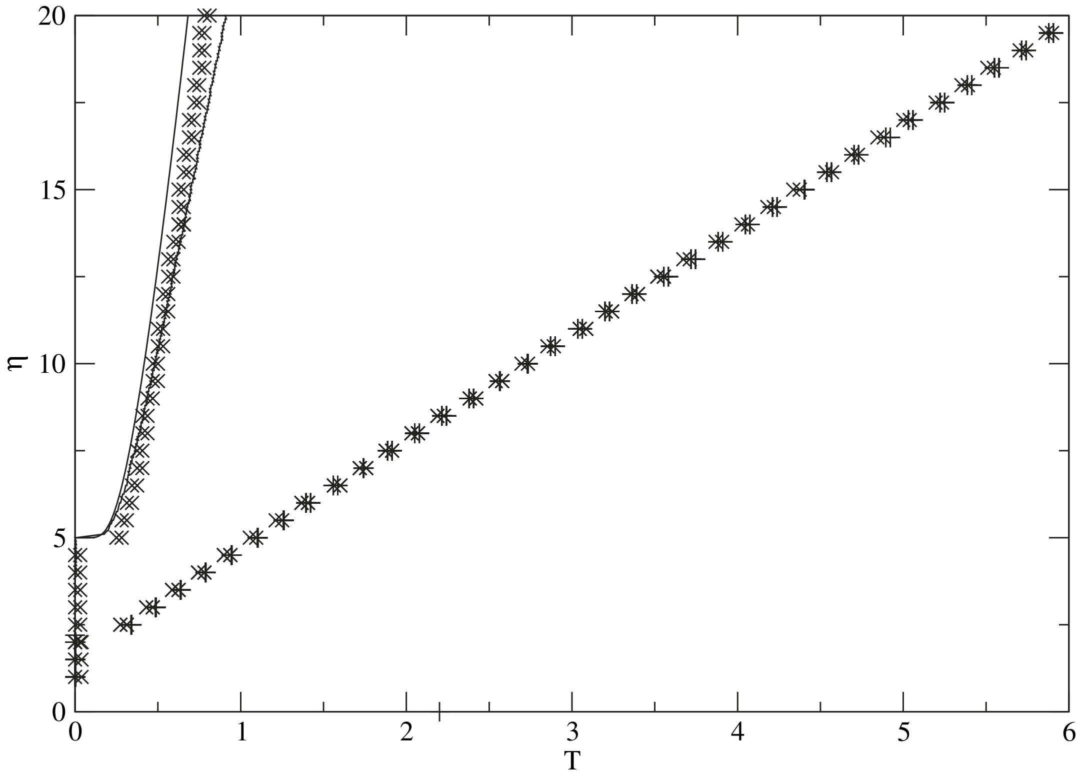
$$p(k) = \frac{1}{k}(2\eta p(k-1)\pi(k-1) + (k-1)p(k-1) - 2\eta p(k-2)\pi(k-2))$$

so:

- given $p(k)$, find $m(k)$ for Giant Component, and then $\pi(k)$
- given $\pi(k)$, find $p(k)$

iterate to find stable solution.
start from:
high connected, ordered state
low connected, disordered state

phase diagram. simulations and theory



ok, so theory is pretty good

fails near transition - this is particularly noticeable for T close to zero

this is because p(k) is not (quite) a full description of the network

Conclusions

Simple(ish) model gives:

- sharp transitions
- 'resilience' - equilibrium coexistence

due to interplay between network dynamics and attribute dynamics

- theory in good agreement with simulations

see physics/0504124