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Complex Networks, Public Goods and Externalities

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Complex networks, public goods and externalities

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Many interesting phenomena combine:

Local interaction: as embedded in the social network

Local externalities: reflected by a payoff/profit function whose arguments are choices in the neighborhood

Examples:

- Investment in education: benefits to an agent of effort devoted to education (research) depends on efforts of people with whom he interacts.
- Information acquistion: returns to gathering information are shared locally if new knowledge is pooled with "neighbors".
- Technological adoption: payoffs of a technology depends on number of neighbors who use it (issue of compatibility)
- Crime and social pathologies: inducement to crime and other misbehavior is shaped locally by the behavior of friends, classmates, family.

Formally, let $\Gamma = \{ij \in 2^N : i \leftrightarrow j\}$

be the underlying social network

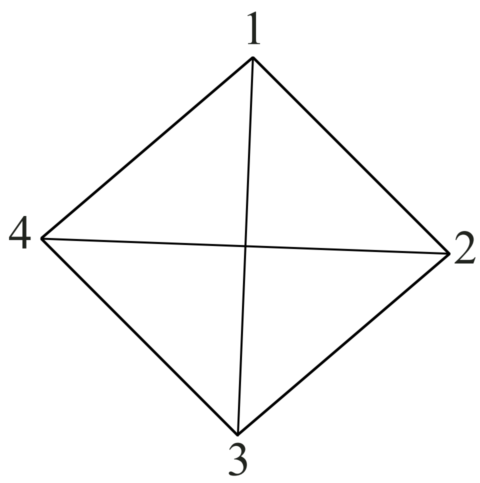
Paradigmatic payoff functions:

- complementary efforts (investment in education) $$\pi_i(e_i, \vec{e}_{-i}; \Gamma) = \phi\left(e_i \times \prod_{j \in N_i} e_j\right) - c(e_i)$$

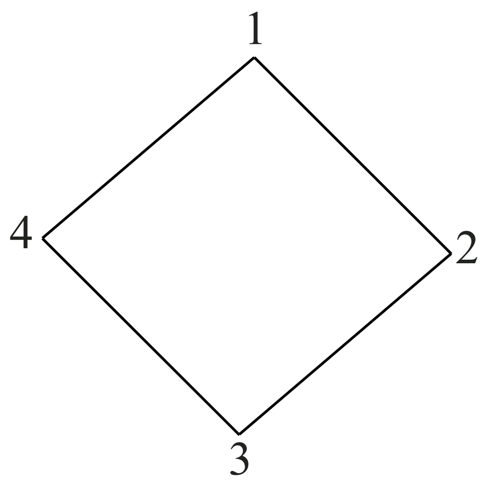
- substitutable efforts (public good, e.g. information) $$\pi_i(e_i, \vec{e}_{-i}; \Gamma) = \phi\left(e_i + \sum_{j \in N_i} e_j\right) - c(e_i)$$

where $\phi(\cdot)$ and $c(\cdot)$ are positive increasing functions

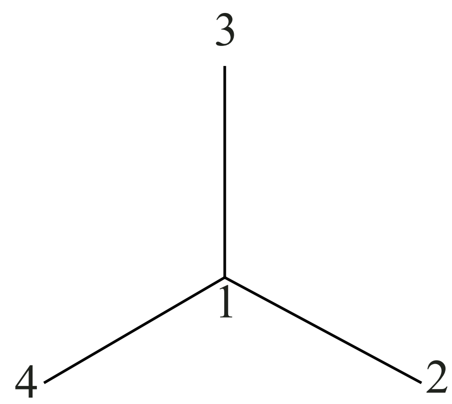
Assume $n = 4$ and three possible networks:



complete



cycle

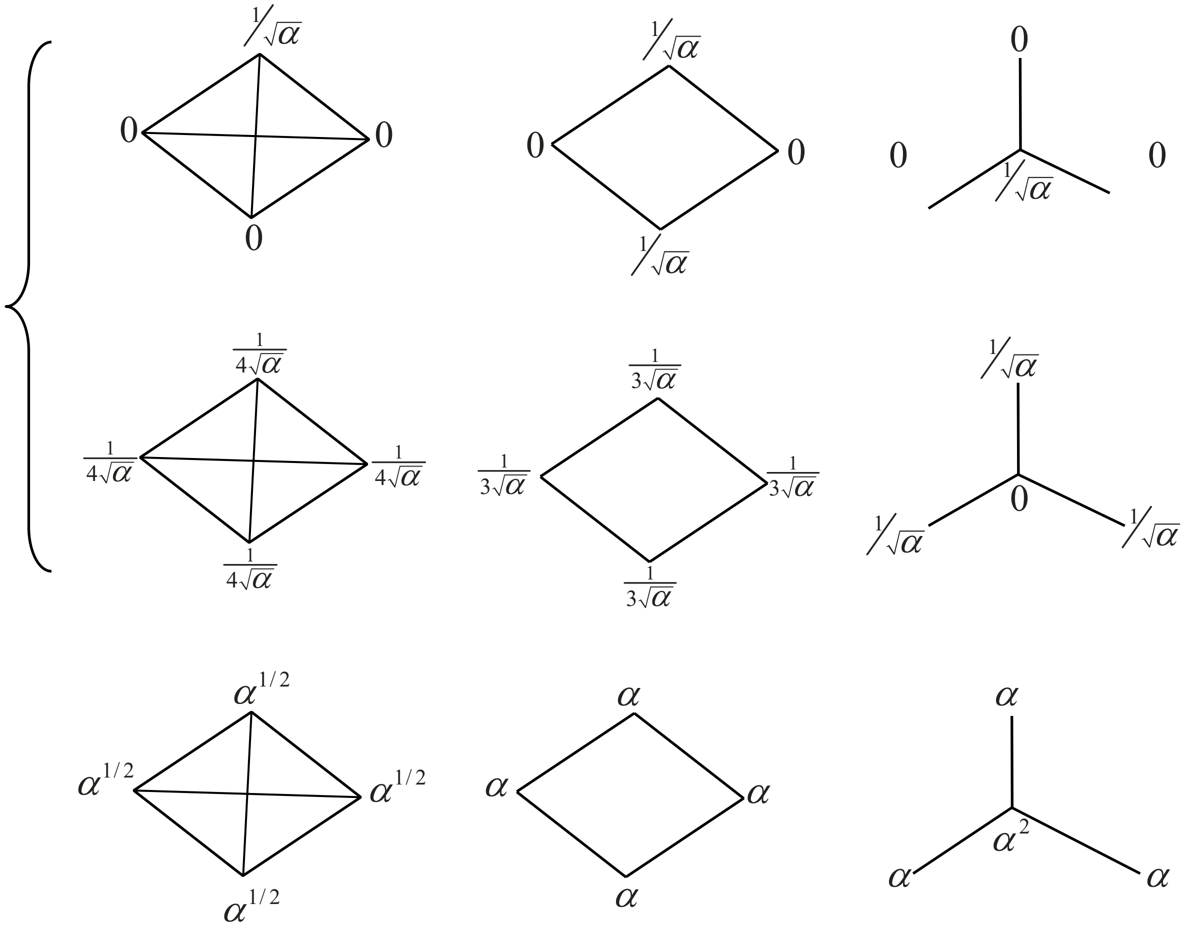


star

Nash equilibria: effort profiles

substitutable efforts

with $\phi(x) = 2\sqrt{x}$
$c(e_i) = \alpha e_i \ , \ \alpha > 0$



complementary efforts

with $\phi(x) = x$
$c(e_i) = \frac{1}{2}\alpha e_i^2 \ , \ \alpha > 0$

In general, Nash equilibria display high complexity or/and wide multiplicity!

Complexity/multiplicity "finely" depends on the overall network architecture

Is it reasonable to posit that players can tailor behavior to such precise knowledge of the network?

If network is complex, players might have only

- local information (e.g. her degree)
- overall aggregate information on statistical regularities

This suggests looking at

- games of incomplete information
- on an underlying random (complex) network

Then, we find sharper predictability and enhanced tractability!!

Model

Large (infinite) population N interacts via network Γ

Network Γ : realization of a statistical ensemble characterized by degree distribution $\boldsymbol{p} = \{p_\kappa\}_{\kappa=0}^{\infty}$

- Each agent $i \in N$ is informed only of his degree κ_i
- he has prior $\tilde{\boldsymbol{p}} = \{\tilde{p}_\kappa\}_{\kappa=0}^{\infty}$ with $\tilde{p}_\kappa = \dfrac{p_\kappa \kappa}{\sum_{\kappa'} \kappa' p_{\kappa'}}$

  for the degree of each of his κ_i neighbors
- he has to choose effort level $e_i \in \mathbb{R}_+$

Payoffs: complementary efforts

[Galeotti and Vega-Redondo (2005)]

Let $\hat{\boldsymbol{e}} = \{\hat{e}(\kappa)\}_{\kappa=0}^{\infty}$ be the strategy played by population

Denote $$\psi_{\kappa}(e_i, \hat{\boldsymbol{e}}) \equiv \mathbf{E}_p\left[\phi\left(e_i \times \prod_{j \in N_i} e_j\right) - c(e_i) \mid \kappa_i = \kappa, e_j = \hat{e}(\kappa_j)\right]$$

the expected payoff of an agent of degree κ

when population strategy is $\hat{\boldsymbol{e}} = \{\hat{e}(\kappa)\}_{\kappa=0}^{\infty}$

Simplifying assumptions

linear utility: $\phi(x) = x$

quadratic costs: $c(e_i) = \tfrac{1}{2}\alpha e_i^2 \ , \ \alpha > 0$

Then:

$$\psi_\kappa(e,\boldsymbol{e^*}) = \sum_{\substack{r=(r_1,r_2,\ldots) \\ r_1+r_2+\ldots=\kappa}} P_\kappa(r_1,r_2,\ldots)\left[e \times \underbrace{e^*(1) \times \ldots \times e^*(1)}_{r_1 \text{ times}} \times \underbrace{e^*(2) \times \ldots \times e^*(2)}_{r_2 \text{ times}} \times \ldots \right]$$

$$- \tfrac{1}{2}\alpha e^2$$

with multinomial probabilities:

$$P_\kappa(r_1,r_2,\ldots) \equiv \frac{\kappa!}{r_1!r_2!\ldots}(\tilde{p}_1)^{r_1} \times (\tilde{p}_2)^{r_2} \times \ldots$$

Bayes-Nash equilibrium

A strategy $\boldsymbol{e}^* = \{e^*(\kappa)\}_{\kappa=0}^{\infty}$ defines a Bayes-Nash Equilibrium (BNE) if

$$e^*(\kappa) \in \arg\max_{e \in \mathbb{R}_+} \psi_\kappa(e, \boldsymbol{e}^*) \qquad (\kappa = 0, 1, 2, \ldots)$$

FONC at BNE:

$$\left. \frac{\partial \psi_\kappa}{\partial e}(e, \boldsymbol{e}^*) \right|_{e=e^*(\kappa)} = 0 \qquad (\kappa = 0, 1, 2, \ldots)$$

FONC yield

$$\begin{cases} \alpha e^*(0) = 1 \\ \alpha e^*(1) = \sum_{\kappa'} \tilde{p}_{\kappa'} e^*(\kappa') \\ \alpha e^*(\kappa) = \left[\sum_{\kappa'} \tilde{p}_{\kappa'} e^*(\kappa')\right]^{\kappa} \qquad (\kappa = 2, 3, \ldots) \end{cases}$$

so that $e^*(\kappa) = \frac{1}{\alpha}\left(\alpha e^*(1)\right)^{\kappa} \qquad (\kappa = 1, 2, 3, \ldots)$

and therefore $\alpha e^*(1) = \frac{1}{\alpha} G_1(\alpha e^*(1))$

where $G_1(x) = \sum_{\kappa=1}^{\infty} \tilde{p}_{\kappa} x^{\kappa}$ is gen. fctn. of $\tilde{\boldsymbol{p}} = \{\tilde{p}_{\kappa}\}_{\kappa=0}^{\infty}$

Consequently, $\{e^*(\kappa)\}_{\kappa=1}^{\infty} \begin{matrix} \uparrow \\ \downarrow \end{matrix}$ if $\begin{matrix} \alpha e^*(1) > 1 \\ \alpha e^*(1) < 1 \end{matrix}$

Three paradigmatic scenarios

1. Poisson networks

$$p(\kappa) = \exp(-z)\frac{z^{\kappa}}{\kappa!} \quad \kappa = 0,1,2,\ldots$$

z average connectivity

2. Scale-free networks

$$p(\kappa) = \frac{1}{\Re(\gamma)}\kappa^{-\gamma} \quad \kappa = 1,2,\ldots$$

$\Re(\gamma) = \sum_{\kappa \geq 1} \kappa^{-\gamma}$ (Riemann zeta), $\gamma > 2$

3. Geometric networks

$$p(\kappa) = (1-\gamma)\gamma^{\kappa} \quad \kappa = 0,1,2,\ldots$$

$$0 < \gamma < 1$$

Poisson degree distributions

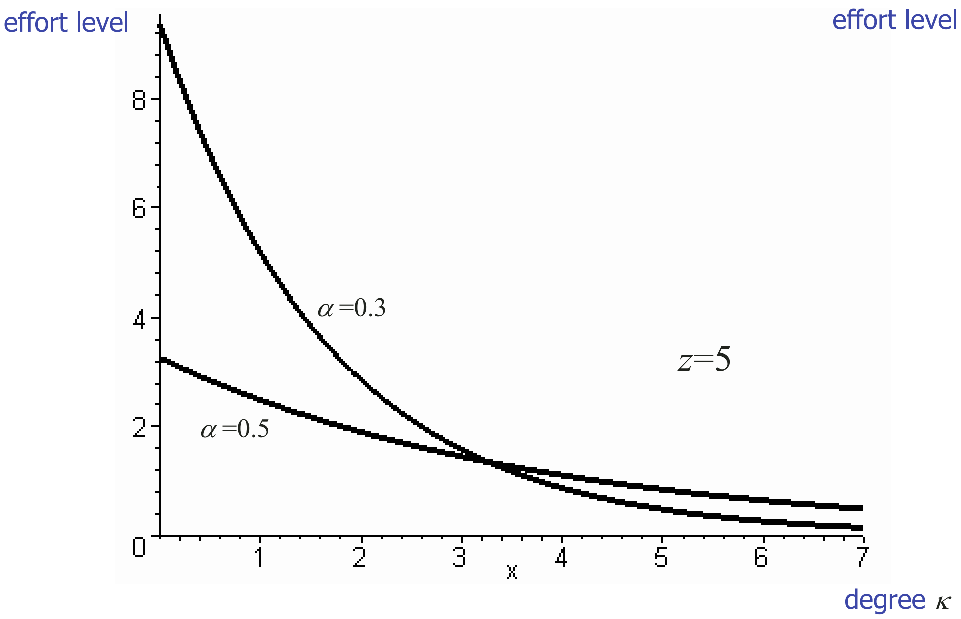
The equilibrium strategy $\boldsymbol{e}^* = \{e^*(\kappa)\}_{\kappa=0}^{\infty}$ satisfies:

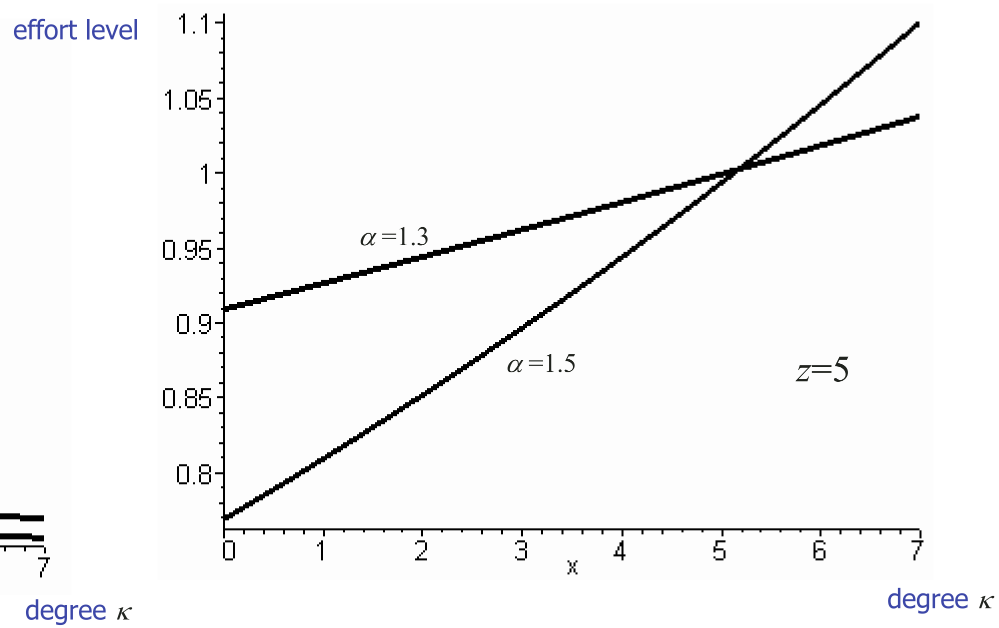
$$e^*(\kappa) = \tfrac{1}{\alpha}\left(\frac{\ln\alpha + z}{z}\right)^{\kappa} \qquad (\kappa = 0, 1, 2, \ldots)$$

which is well defined as long as $\alpha \geq 1/\exp z$

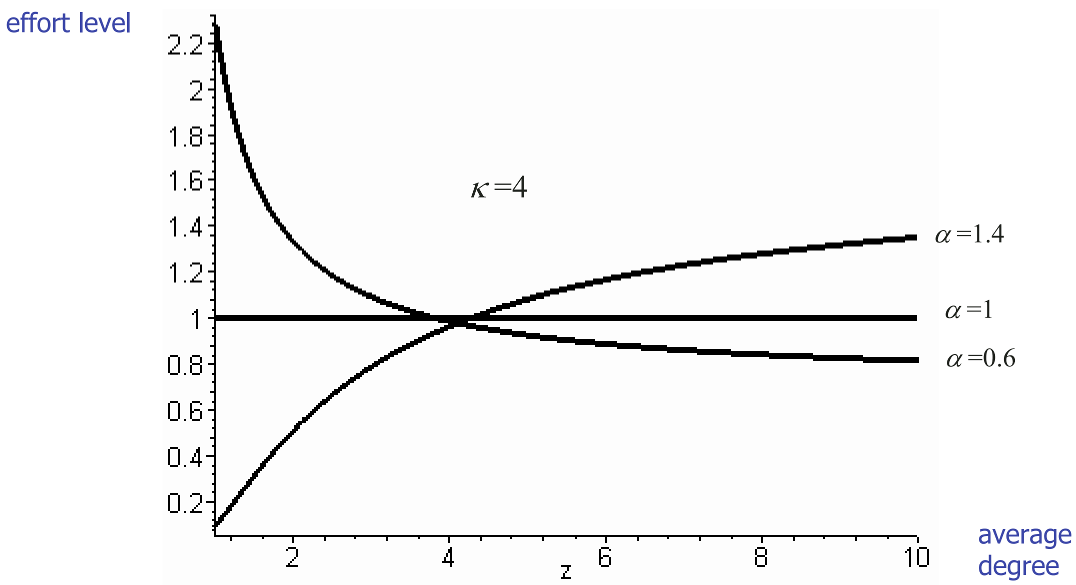
Otherwise, "snowball effect" incompatible with equilibrium displaying positive efforts,

and unique trivial equilibrium $\tilde{e}(\kappa) = 0 \qquad (\kappa = 0, 1, 2, \ldots)$





comparative equilibrium analysis



Scale-free degree distributions

There exists one non-trivial equilibrium strategy

$$\boldsymbol{e}^* = \{e^*(\kappa)\}_{\kappa=0}^{\infty}$$

if, and only if, $1 \geq \alpha > 1/\Re(\gamma-1)$

(Recall $\gamma > 2$ and, therefore, $\Re(\gamma - 1) < +\infty$)

If $\alpha < 1$, this equilibrium has:

$$e^*(0) > e^*(1) > e^*(2) > \ldots$$

Geometric degree distributions

There are two non-trivial equilibrium strategies:

1) $\mathbf{e}^H = \{e^H(\kappa)\}_{\kappa=0}^{\infty}$ s.t. $e^H(\kappa) = \alpha^{\kappa-1}\left(\frac{\alpha + \sqrt{\alpha}(1-\gamma)}{\gamma\alpha^2}\right) \quad (\kappa = 0,1,2,\ldots)$

with $e^H(0) < e^H(1) < e^H(2) < \ldots$

2) If $\alpha \geq (1-\gamma)^2$, $\mathbf{e}^L = \{e^L(\kappa)\}_{\kappa=0}^{\infty}$ s.t. $e^L(\kappa) = \alpha^{\kappa-1}\left(\frac{\alpha - \sqrt{\alpha}(1-\gamma)}{\gamma\alpha^2}\right) \quad (\kappa = 0,1,2,\ldots)$

with $e^L(0) > e^L(1) > e^L(2) > \ldots$

In a sense, combination of conclusions obtained for Poisson and scale-free degree distributions

Summary and conclusions

Illustration of how

- Network complexity
- Strategic behavior

} can be integrated in social network analysis

Topological features of the network have key implications

Approach may be generalized/extended to other contexts

E.g. Galeotti, Goyal, Vega-Redondo (2005)

Focus on general random networks & aggregative payoffs:

$$\pi_i(e_i, \vec{e}_{-i}; \Gamma) = \psi\left(e_i, \sum_{j \in N_i} e_j\right)$$

Consider two cases:

Strategic complements $\frac{\partial^2}{\partial x \partial y}\psi(x,y) \geq 0$

Strategic substitutes $\frac{\partial^2}{\partial x \partial y}\psi(x,y) \leq 0$

Preliminary conclusions:
- complements ⟹ Efforts increase in degree
- substitutes ⟹ Efforts decrease in degree

To understand: How specific features of network topology affect details of "strategic" behavior